Outline

1. Returns to Scale

2. Two-step Cost Minimization

3. Cost Minimization: Example

4. Geometry of Cost Curves
1 Returns to Scale

• Nicholson, Ch. 7, pp. 190–193 [OLD: Ch. 11, pp. 275–278]

• Effect of increase in labor: $f_L'$

• Increase of all inputs: $f(tz)$ with $t$ scalar, $t > 1$

• How much does output increase?
  
  – Decreasing returns to scale: for all $z$ and $t > 1$,
    
    $f(tz) < tf(z)$
− Constant returns to scale: for all $z$ and $t > 1$,
\[ f(tz) = tf(z) \]

− Increasing returns to scale: for all $z$ and $t > 1$,
\[ f(tz) > tf(z) \]
• Example: \( y = f(K, L) = AK^\alpha L^\beta \)

• Marginal product of labor: \( f'_L = \)

• Decreasing marginal product of labor: \( f''_L = \)

• \( MRTS = \)

• Convex isoquant?

• Returns to scale: \( f(tK, tL) = A(tK)^\alpha (tL)^\beta = t^{\alpha+\beta} AK^\alpha L^\beta = t^{\alpha+\beta} f(K, L) \)
2 Two-step Cost minimization

- Nicholson, pp. 212–220 [OLD, Ch. 12, pp. 298–307]

- Objective of firm: Produce output that generates maximal profit.

- Decompose problem in two:
  
  - Given production level $y$, choose cost-minimizing combinations of inputs
  
  - Choose optimal level of $y$.

- *First step.* Cost-Minimizing choice of inputs
• Two-input case: Labor, Capital

• Input prices:
  
  – Wage $w$ is price of $L$
  
  – Interest rate $r$ is rental price of capital $K$

• Expenditure on inputs: $wL + rK$

• Firm objective function:

\[
\begin{align*}
\min_{L,K} & \quad wL + rK \\
\text{s.t.} & \quad f(L, K) \geq y
\end{align*}
\]
Equality in constraint holds if:

1. $w > 0$, $r > 0$;
2. $f$ strictly increasing in at least $L$ or $K$.

Counterexample if ass. 1 is not satisfied

Counterexample if ass. 2 is not satisfied
• Compare with expenditure minimization for consumers

• First order conditions:

\[ w - \lambda f'_L = 0 \]

and

\[ r - \lambda f'_K = 0 \]

• Rewrite as

\[ \frac{f'_L(L^*, K^*)}{f'_K(L^*, K^*)} = \frac{w}{r} \]

• MRTS (slope of isoquant) equals ratio of input prices
• Graphical interpretation
• Derived demand for inputs:

\[ L = L^* (w, r, y) \]

\[ K = K^* (w, r, y) \]

• Value function at optimum is **cost function**:

\[ c (w, r, y) = wL^* (r, w, y) + rK^* (r, w, y) \]
• **Second step.** Given cost function, choose optimal quantity of $y$ as well

• Price of output is $p$.

• Firm’s objective:
  \[
  \max p y - c(w, r, y)
  \]

• First order condition:
  \[
  p - c'_y(w, r, y) = 0
  \]

• Price equals marginal cost – very important!
• Second order condition:

\[-c''_{y,y}(w, r, y^*) < 0\]

• For maximum, need increasing marginal cost curve.
3 Cost Minimization: Example

• Continue example above: \( y = f(L, K) = AK^\alpha L^\beta \)

• Cost minimization:

\[
\min wL + rK \\
\text{s.t.} AK^\alpha L^\beta = y
\]

• Solutions:

  – Optimal amount of labor:

\[
L^*(r, w, y) = \left( \frac{y}{A} \right)^{\frac{1}{\alpha+\beta}} \left( \frac{w \alpha}{r \beta} \right)^{-\frac{\alpha}{\alpha+\beta}}
\]

  – Optimal amount of capital:

\[
K^*(r, w, y) = w \frac{\alpha}{r \beta} \left( \frac{y}{A} \right)^{\frac{1}{\alpha+\beta}} \left( \frac{w \alpha}{r \beta} \right)^{-\frac{\alpha}{\alpha+\beta}} = \]

\[
= \left( \frac{y}{A} \right)^{\frac{1}{\alpha+\beta}} \left( \frac{w \alpha}{r \beta} \right)^{\frac{\beta}{\alpha+\beta}}
\]
• Check various comparative statics:

  – \( \partial L^*/\partial A < 0 \) (technological progress and unemployment)

  – \( \partial L^*/\partial y > 0 \) (more workers needed to produce more output)

  – \( \partial L^*/\partial w < 0, \partial L^*/\partial r > 0 \) (substitute away from more expensive inputs)

• Parallel comparative statics for \( K^* \)
• Cost function

\[ c(w, r, y) = wL^*(r, w, y) + rK^*(r, w, y) = \]
\[
= \left( \frac{y}{A} \right)^{\frac{1}{\alpha + \beta}} \left[ w \left( \frac{w \alpha}{r \beta} \right)^{-\frac{\alpha}{\alpha + \beta}} + r \left( \frac{w \alpha}{r \beta} \right)^{\frac{\beta}{\alpha + \beta}} \right]
\]

• Define \( B := w \left( \frac{w \alpha}{r \beta} \right)^{-\frac{\alpha}{\alpha + \beta}} + r \left( \frac{w \alpha}{r \beta} \right)^{\frac{\beta}{\alpha + \beta}} \)

• Cost-minimizing output choice:

\[
\max p y - B \left( \frac{y}{A} \right)^{\frac{1}{\alpha + \beta}}
\]
• First order condition:

\[ p - \frac{1}{\alpha + \beta} B \left( \frac{y}{A} \right) \frac{1-(\alpha+\beta)}{\alpha+\beta} = 0 \]

• Second order condition:

\[ -\frac{1}{\alpha + \beta} \frac{1-(\alpha + \beta)}{\alpha + \beta} \frac{B}{A^2} \left( \frac{y}{A} \right) \frac{1-2(\alpha+\beta)}{\alpha+\beta} \]

• When is the second order condition satisfied?
• Solution:

- \( \alpha + \beta = 1 \) (CRS):
  
  * S.o.c. equal to 0

  * Solution depends on \( p \)

  * For \( p > \frac{1}{\alpha+\beta} \frac{B}{A} \), produce \( y^* \rightarrow \infty \)

  * For \( p = \frac{1}{\alpha+\beta} \frac{B}{A} \), produce any \( y^* \in [0, \infty) \)

  * For \( p < \frac{1}{\alpha+\beta} \frac{B}{A} \), produce \( y^* = 0 \)
- $\alpha + \beta > 1$ (IRS):

  * S.o.c. positive

  * Solution of f.o.c. is a minimum!

  * Solution is $y^* \to \infty$.

  * Keep increasing production since higher production is associated with higher returns
− \( \alpha + \beta < 1 \) (DRS):

∗ s.o.c. negative. OK!

∗ Solution of f.o.c. is an interior optimum

∗ This is the only "well-behaved" case under perfect competition

∗ Here can define a supply function
4 Geometry of cost curves

- Nicholson, Ch. 8, pp. 220–228; Ch. 9, pp. 256–259 [OLD: Ch. 12, pp. 307–312 and Ch. 13, pp. 342–346.]

- Marginal costs $MC = \partial c / \partial y \rightarrow$ Cost minimization

\[
p = MC = \frac{\partial c (w, r, y)}{\partial y}
\]

- Average costs $AC = c / y \rightarrow$ Does firm break even?

\[
\pi = py - c (w, r, y) > 0 \text{ iff } \\
\frac{\pi}{y} = p - c (w, r, y) / y > 0 \text{ iff } \\
c (w, r, y) / y = AC < p
\]
• **Supply function** (quantity as function of price).

• Portion of marginal cost $MC$ above average costs. (price equals marginal cost)
• Assume only 1 input (expenditure minimization is trivial)

• **Case 1.** Production function. \( y = L^\alpha \)

  - Cost function? (cost of input is \( w \)):
    \[
    c(w, y) = wL^*(w, y) = wy^{1/\alpha}
    \]

  - Marginal cost?
    \[
    \frac{\partial c(w, y)}{\partial y} = \frac{1}{\alpha}w y^{(1-\alpha)/\alpha}
    \]

  - Average cost \( c(w, y)/y \)?
    \[
    \frac{c(w, y)}{y} = \frac{wy^{1/\alpha}}{y} = wy^{(1-\alpha)/\alpha}
    \]
• **Case 1a.** $\alpha > 1$. Plot production function, total cost, average and marginal. Supply function?

• **Case 1b.** $\alpha = 1$. Plot production function, total cost, average and marginal. Supply function?

• **Case 1c.** $\alpha < 1$. Plot production function, total cost, average and marginal. Supply function?
• **Case 2.** *Non-convex technology.* Plot production function, total cost, average and marginal. Supply function?

• **Case 3.** *Technology with setup cost.* Plot production function, total cost, average and marginal. Supply function?
5 Next Lecture

- Profit Maximization