Outline

1. Implicit Function Theorem

2. Envelope Theorem

3. Convexity and concavity

4. Constrained Maximization
1 Implicit function theorem

- Multivariate implicit function theorem (Dini):
  Consider a set of equations \((f_1(p_1, ..., p_n; x_1, ..., x_s) = 0; ...; f_s(p_1, ..., p_n; x_1, ..., x_s) = 0)\), and a point \((p_0, x_0)\) solution of the equation. Assume:

  1. \(f_1, ..., f_s\) continuous and differentiable in a neighbourhood of \((p_0, x_0)\);

     (a) The following Jacobian matrix \(\frac{\partial f}{\partial x}\) evaluated at \((p_0, x_0)\) has determinant different from 0:

     \[
     \frac{\partial f}{\partial x} = \begin{pmatrix}
     \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_s} \\
     \vdots & \vdots \\
     \frac{\partial f_s}{\partial x_1} & \frac{\partial f_s}{\partial x_s}
     \end{pmatrix}
     \]
Then:

1. There is one and only set of functions $x = g(p)$ defined in a neighbourhood of $p_0$ that satisfy $f(p, g(p)) = 0$ and $g(p_0) = x_0$;

2. The partial derivative of $x_i$ with respect to $p_k$ is

$$
\frac{\partial g_i}{\partial p_k} = - \frac{\det \left( \frac{\partial (f_1, \ldots, f_s)}{\partial (x_1, \ldots, x_{i-1}, p_k, x_{i+1}, \ldots, x_s)} \right)}{\det \left( \frac{\partial f}{\partial x} \right)}
$$
• Example 2 (continued): Max \( h(x_1, x_2) = p_1 * x_1^2 + p_2 * x_2^2 - 2x_1 - 5x_2 \)

• f.o.c. \( x_1 : 2p_1 * x_1 - 2 = 0 = f_1(p,x) \)

• f.o.c. \( x_2 : 2p_2 * x_2 - 5 = 0 = f_2(p,x) \)

• Comparative statics of \( x_1^* \) with respect to \( p_1 \)?

• First compute \( \text{det} \left( \frac{\partial f}{\partial x} \right) \)

\[
\left( \begin{array}{cc}
\frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\
\frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2}
\end{array} \right) = \left( \begin{array}{c}
\ \\
\ 
\end{array} \right)
\]
• Then compute \( \det \left( \frac{\partial (f_1, \ldots, f_s)}{\partial (x_1, \ldots, x_{i-1}, p_k, x_{i+1}, \ldots, x_s)} \right) \)

\[
\begin{pmatrix}
\frac{\partial f_1}{\partial p_1} & \frac{\partial f_1}{\partial x_2} \\
\frac{\partial f_2}{\partial p_1} & \frac{\partial f_2}{\partial x_2}
\end{pmatrix}
= \left( \begin{array} \end{array} \right)
\]

• Finally, \( \frac{\partial x_1}{\partial p_1} = \)

• Why did you compute \( \det \left( \frac{\partial f}{\partial x} \right) \) already?
2 Envelope Theorem

• You now know how $x_1^*$ varies if $p_1$ varies.

• How does the function $h$ vary at the optimum as $p_1$ varies?

• Differentiate $h(x_1^*(p_1, p_2), x_2^*(p_1, p_2), p_1, p_2)$ with respect to $p_1$:

$$
\frac{dh(x_1^*(p_1, p_2), x_2^*(p_1, p_2), p_1, p_2)}{dp_1} = \frac{\partial h(x^*, p)}{\partial x_1} \cdot \frac{\partial x_1^*(x^*, p)}{\partial p_1} + \frac{\partial h(x^*, p)}{\partial x_2} \cdot \frac{\partial x_2^*(x^*, p)}{\partial p_1} + \frac{\partial h(x^*, p)}{\partial p_1}
$$

• Can we say something about the first two terms? They are zero!
• **Envelope Theorem** for unconstrained maximization. Assume that you maximize function $f(x; p)$ with respect to $x$. Consider then the function $f$ at the optimum, that is, $f(x^*(p), p)$. The total differential of this function with respect to $p_i$ equals the partial derivative with respect to $p_i$:

$$\frac{df(x^*(p), p)}{dp_i} = \frac{\partial f(x^*(p), p)}{\partial p_i}.$$ 

• You can disregard the indirect effects. Graphical intuition.
3 Convexity and concavity

- Function $f$ from $C \subset \mathbb{R}^n$ to $\mathbb{R}$ is concave if
  \[
  f(tx + (1 - t)y) \geq tf(x) + (1 - t)f(y)
  \]
  for all $x, y \in C$ and for all $t \in [0, 1]$

- Notice: $C$ must be convex set, i.e., if $x \in C$ and $y \in C$, then $tx + (1 - t)y \in C$, for $t \in [0, 1]$

- Function $f$ from $C \subset \mathbb{R}^n$ to $\mathbb{R}$ is strictly concave if
  \[
  f(tx + (1 - t)y) > tf(x) + (1 - t)f(y)
  \]
  for all $x, y \in C$ and for all $t \in (0, 1)$

- Function $f$ from $\mathbb{R}^n$ to $\mathbb{R}$ is convex if $-f$ is concave.
• Alternative characterization of convexity.

• A function \( f \), twice differentiable, is concave if and only if for all \( x \) the subdeterminants \( |H_i| \) of the Hessian matrix have the property \( |H_1| \leq 0, |H_2| \geq 0, |H_3| \leq 0 \), and so on.

• For the univariate case, this reduces to \( f''' \leq 0 \)

• For the bivariate case, this reduces to \( f''_{x,x} \leq 0 \) and \( f''_{x,x} \cdot f''_{y,y} - (f''_{x,y})^2 \geq 0 \)

• A twice-differentiable function is strictly concave if the same property holds with strict inequalities.
Examples.

1. For which values of $a$, $b$, and $c$ is $f(x) = ax^3 + bx^2 + cx + d$ is the function concave over $R$? Strictly concave? Convex?

2. Is $f(x, y) = -x^2 - y^2$ concave?

For Example 2, compute the Hessian matrix

- $f'_x = \ldots$, $f'_y = \ldots$
- $f''_{x,x} = \ldots$, $f''_{x,y} = \ldots$
- $f''_{y,x} = \ldots$, $f''_{y,y} = \ldots$

Hessian matrix $H$:

$$H = \begin{pmatrix} f''_{x,x} = & f''_{x,y} = \\ f''_{y,x} = & f''_{y,y} = \end{pmatrix}$$

- Compute $|H_1| = f''_{x,x}$ and $|H_2| = f''_{x,x} \ast f''_{y,y} - (f''_{x,y})^2$
• Why are convexity and concavity important?

• Theorem. Consider a twice-differentiable concave (convex) function over $C \subset \mathbb{R}^n$. If the point $x_0$ satisfies the first order conditions, it is a global maximum (minimum).

• For the proof, we need to check that the second-order conditions are satisfied.

• These conditions are satisfied by definition of concavity!

• (We have only proved that it is a local maximum)
4 Constrained maximization

- Nicholson, Ch. 2, pp. 39–46

- So far unconstrained maximization on $\mathbb{R}$ (or open subsets)

- What if there are constraints to be satisfied?

- Example 1: $\max_{x,y} x \cdot y$ subject to $3x + y = 5$

- Substitute it in: $\max_{x,y} x \cdot (5 - 3x)$

- Solution: $x^* =$

- Example 2: $\max_{x,y} xy$ subject to $x \exp(y) + y \exp(x) = 5$

- Solution: ?
• Graphical intuition on general solution.

• Example 3: \( \max_{x, y} f(x, y) = x \cdot y \) s.t. \( h(x, y) = x^2 + y^2 - 1 = 0 \)

• Draw \( 0 = h(x, y) = x^2 + y^2 - 1. \)

• Draw \( x \cdot y = K \) with \( K > 0 \). Vary \( K \)

• Where is optimum?

• Where \( \frac{dy}{dx} \) along curve \( xy = K \) equals \( \frac{dy}{dx} \)
  along curve \( x^2 + y^2 - 1 = 0 \)

• Write down these slopes.
• Idea: Use implicit function theorem.

• Heuristic solution of system

\[
\max_{x,y} f(x, y) \\
\text{s.t. } h(x, y) = 0
\]

• Assume:

  – continuity and differentiability of \( h \)

\[- \quad h'_y \neq 0 \text{ (or } h'_x \neq 0 \)]

• Implicit function Theorem: Express \( y \) as a function of \( x \) (or \( x \) as function of \( y \))!
• Write system as $\max_x f(x, g(x))$

• f.o.c.: $f'_x(x, g(x)) + f'_y(x, g(x)) \cdot \frac{\partial g(x)}{\partial x} = 0$

• What is $\frac{\partial g(x)}{\partial x}$?

• Substitute in and get: $f'_x(x, g(x)) + f'_y(x, g(x)) \cdot (-h'_x/h'_y) = 0$ or

$$\frac{f'_x(x, g(x))}{f'_y(x, g(x))} = \frac{h'_x(x, g(x))}{h'_y(x, g(x))}$$
Lagrange Multiplier Theorem, necessary condition. Consider a problem of the type

$$\max_{x_1,\ldots,x_n} f(x_1, x_2, \ldots, x_n; p)$$

s.t. \[
\begin{align*}
    h_1(x_1, x_2, \ldots, x_n; p) &= 0 \\
    h_2(x_1, x_2, \ldots, x_n; p) &= 0 \\
    &\vdots \\
    h_m(x_1, x_2, \ldots, x_n; p) &= 0
\end{align*}
\]

with \(n > m\). Let \(x^* = x^*(p)\) be a local solution to this problem.

Assume:

- \(f\) and \(h\) differentiable at \(x^*\)

- the following Jacobian matrix at \(x^*\) has maximal rank

\[
J = \begin{pmatrix}
\frac{\partial h_1(x^*)}{\partial x_1} & \cdots & \frac{\partial h_1(x^*)}{\partial x_n} \\
\vdots & \ddots & \vdots \\
\frac{\partial h_m(x^*)}{\partial x_1} & \cdots & \frac{\partial h_m(x^*)}{\partial x_n}
\end{pmatrix}
\]
• Then, there exists a vector \( \lambda = (\lambda_1, ..., \lambda_m) \) such that \((x^*, \lambda)\) maximize the Lagrangean function

\[
L(x, \lambda) = f(x; p) - \sum_{j=0}^{m} \lambda_j h_j(x; p)
\]

• Case \( n = 2, m = 1 \).

• First order conditions are

\[
\frac{\partial f(x; p)}{\partial x_i} - \lambda \frac{\partial h(x; p)}{\partial x_i} = 0
\]

for \( i = 1, 2 \)

• Rewrite as

\[
\frac{f'_x}{f'_x} = \frac{h'_x}{h'_x}
\]
Constrained Maximization, Sufficient condition for the case \( n = 2, m = 1 \).

- If \( x^* \) satisfies the Lagrangean condition, and the determinant of the bordered Hessian

\[
H = \begin{pmatrix}
0 & -\frac{\partial h}{\partial x_1}(x^*) & -\frac{\partial h}{\partial x_2}(x^*) \\
-\frac{\partial h}{\partial x_1}(x^*) & \frac{\partial^2 L}{\partial^2 x_1}(x^*) & \frac{\partial^2 L}{\partial x_2 \partial x_1}(x^*) \\
-\frac{\partial h}{\partial x_2}(x^*) & \frac{\partial^2 L}{\partial x_1 \partial x_2}(x^*) & \frac{\partial^2 L}{\partial x_2 \partial x_2}(x^*)
\end{pmatrix}
\]

is positive, then \( x^* \) is a constrained maximum.

- If it is negative, then \( x^* \) is a constrained minimum.

- Why? This is just the Hessian of the Lagrangean \( L \) with respect to \( \lambda, x_1, \) and \( x_2 \)
Example 4: \( \max_{x,y} x^2 - xy + y^2 \) s.t. \( x^2 + y^2 - p = 0 \)

\[ \max_{x,y,\lambda} x^2 - xy + y^2 - \lambda(x^2 + y^2 - p) \]

- F.o.c. with respect to \( x \):

- F.o.c. with respect to \( y \):

- F.o.c. with respect to \( \lambda \):

- Candidates to solution?

- Maxima and minima?
5 Next Class

• Next class:
  
  – Envelope Theorem II

  – Preferences

  – Utility Maximization (where we get to apply maximization techniques the first time)