Economics 101A
(Lecture 6)

Stefano DellaVigna

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Outline

1. Utility Maximization

2. Utility Maximization – tricky cases

3. Indirect Utility Function

4. Comparative Statics (Introduction)
1 Utility Maximization

• Nicholson, Ch. 4, pp. 94–105 [OLD: 91–103]

• $X = R_+^2$ (2 goods)

• Consumers: choose bundle $x = (x_1, x_2)$ in $X$ which yields highest utility.

• Constraint: income $= M$

• Price of good 1 $= p_1$, price of good 2 $= p_2$

• Bundle $x$ is feasible if $p_1 x_1 + p_2 x_2 \leq M$

• Consumer maximizes

$$\max_{x_1, x_2} u(x_1, x_2)$$

s.t. $p_1 x_1 + p_2 x_2 \leq M$

$x_1 \geq 0, \ x_2 \geq 0$
• Maximization subject to inequality. How do we solve that?

• Trick: $u$ strictly increasing in at least one dimension. ($\geq$ strictly monotonic)

• Budget constraint always satisfied with equality

• Ignore temporarily $x_1 \geq 0, x_2 \geq 0$ and check afterwards that they are satisfied for $x_1^*$ and $x_2^*$. 
Problem becomes

$$\begin{align*}
\max_{x_1, x_2} & \quad u(x_1, x_2) \\
\text{s.t.} & \quad p_1 x_1 + p_2 x_2 - M = 0
\end{align*}$$

$L(x_1, x_2) = u(x_1, x_2) - \lambda(p_1 x_1 + p_2 x_2 = M)$

F.o.c.s:

$$\begin{align*}
& u_{x_i}' - \lambda p_i = 0 \quad \text{for } i = 1, 2 \\
& p_1 x_1 + p_2 x_2 - M = 0
\end{align*}$$
• Moving the two terms across and dividing, we get:

$$MRS = -\frac{u'_{x_1}}{u'_{x_2}} = -\frac{p_1}{p_2}$$

• Graphical interpretation.
• Example with CES utility function.

\[
\max_{x_1, x_2} \left( \alpha x_1^\rho + \beta x_2^\rho \right)^{1/\rho} \\
\text{s.t. } p_1 x_1 + p_2 x_2 - M = 0
\]

• Lagrangean =

• F.o.c.:

• Special case: \( \rho = 0 \) (Cobb-Douglas)
2 Utility maximization – tricky cases

1. Non-convex preferences. Example:

- Second order conditions:

\[ H = \begin{pmatrix}
0 & -p_1 & -p_2 \\
-p_1 & u''_{x_1,x_1} & u''_{x_1,x_2} \\
-p_2 & u''_{x_2,x_1} & u''_{x_2,x_2}
\end{pmatrix} \]

\[
|H| = p_1 \left( -p_1 u''_{x_2,x_2} + p_2 u''_{x_2,x_1} \right) - p_2 \left( -p_1 u''_{x_1,x_2} + p_2 u''_{x_1,x_1} \right) = -p_1^2 u''_{x_2,x_2} + 2p_1 p_2 u''_{x_1,x_2} - p_2^2 u''_{x_1,x_1} \]
2. Solution does not satisfy $x_1^* > 0$ or $x_2^* > 0$. Example:

$$\max x_1 \ast (x_2 + 5)$$

$$s.t. p_1x_1 + p_2x_2 = M$$

- In this case consider corner conditions: what happens for $x_1^* = 0$? And $x_2^* = 0$?
3. Multiplicity of solutions. Example:

- Convex preferences that are not strictly convex
4. Example with CES utility function.

\[ \max_{x_1, x_2} \left( \alpha x_1^\rho + \beta x_2^\rho \right)^{1/\rho} \]
\[ s.t. \, p_1 x_1 + p_2 x_2 - M = 0 \]

- With \( \rho > 1 \) the interior solution is a minimum!

- Draw indifference curves for \( \rho = 1 \) (boundary case) and \( \rho = 2 \)

- Can also check using second order conditions
3 Indirect utility function

- Nicholson, Ch. 4, pp. 106–108 [OLD: 103–105]

- Define the indirect utility $v(p, M) \equiv u(x^*(p, M))$, with $p$ vector of prices and $x^*$ vector of optimal solutions.

- $v(p, M)$ is the utility at the optimum for prices $p$ and income $M$

- Some comparative statics: $\partial v(p, M)/\partial M = ?$

- Hint: Use Envelope Theorem on Lagrangean function
• What is the sign of $\lambda$?

• $\lambda = \frac{u'_{x_i}}{p} > 0$

• $\partial v(p, M)/\partial p_i = ?$

• Properties:
  
  – Indirect utility is always increasing in income $M$

  – Indirect utility is always decreasing in the price $p_i$
4 Comparative Statics (introduction)

- Nicholson, Ch. 5, pp. 121–131 [OLD: 116–128]

- Utility maximization yields $x_i^* = x_i^*(p_1, p_2, M)$

- Quantity consumed as a function of income and price

- What happens to quantity consumed $x_i^*$ as prices or income varies?
• Simple case: Equal increase in prices and income.

• $M' = tM$, $p'_1 = tp_1$, $p'_2 = tp_2$.

• Compare $x^*(tM, tp_1, tp_2)$ and $x^*(M, p_1, p_2)$.

• What happens?

• Write budget line: $tp_1x_1 + tp_2x_2 = tM$

• Demand is homogeneous of degree 0 in $p$ and $M$:
  $$x^*(tM, tp_1, tp_2) = t^0 x^*(M, p_1, p_2) = x^*(M, p_1, p_2).$$
• Consider Cobb-Douglas Case:

\[ x_1^* = \frac{\alpha}{\alpha + \beta} \frac{M}{p_1}, \quad x_2^* = \frac{\beta}{\alpha + \beta} \frac{M}{p_2} \]

• What is \( \frac{\partial x^*}{\partial M} \)?

• What is \( \frac{\partial x^*}{\partial p_x} \)?

• What is \( \frac{\partial x^*}{\partial p_y} \)?

• General results?
5 Next Lecture

• More comparative statics:
  – Income Changes
  – Price Changes

• Expenditure minimization