Economics 101A
(Lecture 6)

Stefano DellaVigna

September 14, 2006
Outline

1. From Preferences to Utility (continued)

2. Common Utility Functions

3. Utility maximization

4. Utility maximization – Tricky Cases
1 From preferences to utility

• Nicholson, Ch. 3

• Economists like to use utility functions $u : X \rightarrow R$

• $u(x)$ is ‘liking’ of good $x$

• $u(a) > u(b)$ means: I prefer $a$ to $b$.

• **Def.** Utility function $u$ represents preferences $\succeq$ if, for all $x$ and $y$ in $X$, $x \succeq y$ if and only if $u(x) \geq u(y)$.

• **Theorem.** If preference relation $\succeq$ is rational and continuous, there exists a continuous utility function $u : X \rightarrow R$ that represents it.
• [Skip proof]

• Example:

\[(x_1, x_2) \succeq (y_1, y_2) \text{ iff } x_1 + x_2 \geq y_1 + y_2\]

• Draw:

• Utility function that represents it: \( u(x) = x_1 + x_2 \)

• But... Utility function representing \( \succeq \) is not unique

• Take \( 3u(x) \) or \( \exp(u(x)) \)

• \( u(a) > u(b) \iff \exp(u(a)) > \exp(u(b)) \)
• If $u(x)$ represents preferences $\succeq$ and $f$ is a strictly increasing function, then $f(u(x))$ represents $\succeq$ as well.

• If preferences are represented from a utility function, are they rational?
  
  – completeness
  
  – transitivity
• Indifference curves: \( u(x_1, x_2) = \bar{u} \)

• They are just implicit functions! \( u(x_1, x_2) - \bar{u} = 0 \)

\[
\frac{dx_2}{dx_1} = -\frac{U'_x}{U'_{x_2}} = MRS
\]

• Indifference curves for:
  
  – monotonic preferences;

  – strictly monotonic preferences;

  – convex preferences
2 Common utility functions

• Nicholson, Ch. 3, pp. 82-86 [OLD: 80–84]

1. Cobb-Douglas preferences: \( u(x_1, x_2) = x_1^\alpha x_2^{1-\alpha} \)

   \[
   \text{MRS} = -\alpha x_1^{\alpha-1} x_2^{1-\alpha} / (1-\alpha) x_1^\alpha x_2^{-\alpha} = \frac{\alpha}{1-\alpha} \frac{x_2}{x_1}
   \]

2. Perfect substitutes: \( u(x_1, x_2) = \alpha x_1 + \beta x_2 \)

   \[
   \text{MRS} = -\alpha / \beta
   \]
3. Perfect complements: \( u(x_1, x_2) = \min(\alpha x_1, \beta x_2) \)
   
   - \( MRS \) discontinuous at \( x_2 = \frac{\alpha}{\beta} x_1 \)

4. Constant Elasticity of Substitution: \( u(x_1, x_2) = (\alpha x_1^\rho + \beta x_2^\rho)^{1/\rho} \)
   
   - \( MRS = -\frac{\alpha}{\beta} \left(\frac{x_1}{x_2}\right)^{\rho-1} \)
   
   - if \( \rho = 1 \), then...
   
   - if \( \rho = 0 \), then...
   
   - if \( \rho \to -\infty \), then...
3 Utility Maximization

• Nicholson, Ch. 4, pp. 94–105 [OLD: 91–103]

• $X = \mathbb{R}^2_+$ (2 goods)

• Consumers: choose bundle $x = (x_1, x_2)$ in $X$ which yields highest utility.

• Constraint: income = $M$

• Price of good 1 = $p_1$, price of good 2 = $p_2$

• Bundle $x$ is feasible if $p_1x_1 + p_2x_2 \leq M$

• Consumer maximizes

$$\max_{x_1, x_2} u(x_1, x_2) \quad s.t. \quad p_1x_1 + p_2x_2 \leq M \quad x_1 \geq 0, \quad x_2 \geq 0$$
• Maximization subject to inequality. How do we solve that?

• Trick: $u$ strictly increasing in at least one dimension. ($\succeq$ strictly monotonic)

• Budget constraint always satisfied with equality

• Ignore temporarily $x_1 \geq 0$, $x_2 \geq 0$ and check afterwards that they are satisfied for $x_1^*$ and $x_2^*$. 
• Problem becomes

$$\max_{x_1, x_2} u(x_1, x_2)$$

$$s.t. \quad p_1 x_1 + p_2 x_2 - M = 0$$

• \(L(x_1, x_2) = u(x_1, x_2) - \lambda (p_1 x_1 + p_2 x_2 = M)\)

• F.o.c.s:

$$u'_{x_i} - \lambda p_i = 0 \quad \text{for} \ i = 1, 2$$

$$p_1 x_1 + p_2 x_2 - M = 0$$
• Moving the two terms across and dividing, we get:

\[ \text{MRS} = -\frac{u'_{x_1}}{u'_{x_2}} = -\frac{p_1}{p_2} \]

• Graphical interpretation.
• Second order conditions:

\[
H = \begin{pmatrix}
0 & -p_1 & -p_2 \\
-p_1 & u''_{x_1,x_1} & u''_{x_1,x_2} \\
-p_2 & u''_{x_2,x_1} & u''_{x_2,x_2}
\end{pmatrix}
\]

\[
|H| = p_1 \left(-p_1 u''_{x_2,x_2} + p_2 u''_{x_2,x_1}\right) \\
- p_2 \left(-p_1 u''_{x_1,x_2} + p_2 u''_{x_1,x_1}\right) \\
= -p_1^2 u''_{x_2,x_2} + 2p_1p_2 u''_{x_1,x_2} - p_2^2 u''_{x_1,x_1}
\]

• Notice: \(u''_{x_2,x_2} < 0\) and \(u''_{x_1,x_1} < 0\) usually satisfied (but check it!).

• Condition \(u''_{x_1,x_2} > 0\) is then sufficient
• Example with CES utility function.

$$\max_{x_1,x_2} \left( \alpha x_1^\rho + \beta x_2^\rho \right)^{1/\rho}$$

$s.t. \ p_1 x_1 + p_2 x_2 - M = 0$

• Lagrangean =

• F.o.c.:

• Special case: $\rho = 0$ (Cobb-Douglas)
4  Utility maximization – tricky cases

1. Non-convex preferences. Example:
• Example with CES utility function.

\[
\max_{x_1, x_2} \left( \alpha x_1^\rho + \beta x_2^\rho \right)^{1/\rho} \\
\text{s.t. } p_1 x_1 + p_2 x_2 - M = 0
\]

• With \( \rho > 1 \) the interior solution is a minimum!

• Draw indifference curves for \( \rho = 1 \) (boundary case) and \( \rho = 2 \)

• Can also check using second order conditions
2. Solution does not satisfy $x_1^* > 0$ or $x_2^* > 0$. Example:

\[
\max x_1 \cdot (x_2 + 5) \\
\text{s.t. } p_1 x_1 + p_2 x_2 = M
\]

- In this case consider corner conditions: what happens for $x_1^* = 0$? And $x_2^* = 0$?
3. Multiplicity of solutions. Example:

- Convex preferences that are not strictly convex
5 Next Class

- Indirect Utility Function

- Comparative Statics:
  - with respect to price
  - with respect to income