Econ 101A – Problem Set 6 Solutions
Due on Monday Dec. 9. No late Problem Sets accepted, sorry!

This Problem set tests the knowledge that you accumulated mainly in lectures 24 to 26. The problem set is focused on dynamic games and general equilibrium. General rules for problem sets: show your work, write down the steps that you use to get a solution (no credit for right solutions without explanation), write legibly. If you cannot solve a problem fully, write down a partial solution. We give partial credit for partial solutions that are correct. Do not forget to write your name on the problem set!

Problem 1. Dynamic Games (48 points) Two companies produce the same good. In the first period, firm1 sells its product as a monopolist on the West Coast. In the second period, firm 1 competes with firm 2 on the East Coast as a Cournot duopolist. There is no discounting between the two periods. Firm 1 produces quantity \( x_W \) at cost \( c/\alpha \) at the West at cost \( cx_W \). On the East Coast, and that’s what makes this problem interesting, firm 1 produces quantity \( x_E \) at cost \( (c - \alpha x_W) x_E \), where \( 0 < \alpha < c < 1/2 \). The parameter \( \alpha \) captures a form of learning by doing. The more firm 1 produces on the West Coast, the lower the marginal costs are going to be on the East Coast. As for firm 2, it produces in the East market with cost \( c x_2 \). The inverse demand functions are \( p_W(x_W) = 1 - x_W \) and \( p_E(x_E, x_2) = 1 - x_E - x_2 \). Each firm maximizes profit.

1. Consider first the case of simultaneous choice. Assume that firm 2 does not observe \( x_W \) before making its production decision. This means that, although formally firm 1 chooses output \( x_W \) first, that you should analyze the game as a simultaneous game between firm 1 and firm 2. Use Nash Equilibrium. Write down the profit function that firm 1 maximizes (careful here) and the profit function that firm 2 maximizes (5 points)

2. Write down the first order conditions of firm 1 with respect to \( x_W \) and \( x_E \), and the first order condition of firm 2 with respect to \( x_2 \). Solve for \( x^*_W, x^*_E, \) and \( x^*_2 \). (4 points)

3. Check the second order conditions for firm 1 and for firm 2. (3 points)

4. What is the comparative statics of \( x^*_W \) and \( x^*_E \) with respect to \( \alpha \)? Does it make sense? How about the comparative statics of \( x^*_2 \) with respect to \( \alpha \)? (4 points)

5. Compute the profits of firm 2 in equilibrium. How do they vary as \( \alpha \) varies? (compute the comparative statics) Why are firm 2’s profits affected by \( \alpha \) even though the parameter \( \alpha \) does not directly affect the costs of firm 2? (5 points)

6. Now consider the case of sequential choice. Assume that firm 2 observes \( x_W \) before making its production decision \( x_2 \). This means that you should analyze the game as a dynamic game between firm 1 and firm 2, and use the concept of subgame-perfect equilibrium. Remember, we start from the last period. Write down the profit functions that firm 1 and firm 2 maximize on the East Coast (4 points)

7. Write down the first order conditions of firm 1 with respect to \( x_E \), and the first order condition of firm 2 with respect to \( x_2 \). Solve for \( x^*_E \) and \( x^*_2 \) as a function of \( x^*_W \). (4 points)

8. Compute the comparative statics of \( x^*_E \) and \( x^*_2 \) with respect to \( x^*_W \). Do these results make sense? (3 points)

9. Compute the profits of firm 1 on the East Coast as a function of \( x^*_W \). (2 points)

10. Using the answer to point 9, write down the maximization problem of firm 1 in the first period, that it, when it decides the production on the West Coast. (3 points)

11. Write down the first order conditions of firm 1 with respect to \( x_W \). Solve for \( x^*_W \) and then, using the solution for \( x^*_W \), find the solution for \( x^*_E \) and \( x^*_2 \). (5 points)
12. Compare the solutions for \(x^*_W\) under simultaneous and under sequential choice. What can you conclude?

Under which conditions the firm does more preemption, that is, produces more on the West Coast in order to reduce the production in equilibrium of firm 2? (6 points)

\[
\text{Solution to Problem 1.}
\]

1. Firm 1 maximizes the sum of the profits on the East and West coast, that is, the maximization program is

\[
\max_{x_W, x_E} x_W (1 - x_W) - cx_W + x_E (1 - x_E - x_W) - (c - \alpha x_W) x_E.
\]

Firm 2 simply maximizes the profits on the East Coast:

\[
\max x_2 (1 - x_E - x_2) - cx_2.
\]

2. The first order conditions of firm 1 with respect to \(x_W\) and \(x_E\) are

\[
1 - 2x_W - c + \alpha x_E = 0 \tag{1}
\]

and

\[
1 - 2x_E - x_2 - (c - \alpha x_W) = 0. \tag{2}
\]

The first order condition of firm 2 with respect to \(x_2\) is

\[
1 - x_E - 2x_2 - c = 0 \tag{3}
\]

From equation (1) we obtain \(x^*_W = (1 - c + \alpha x^*_E)/2\) which we can substitute into (2) to obtain

\[
(2 - \alpha^2/2) x^*_E = 1 - x^*_2 - c + \alpha (1 - c)/2. \tag{4}
\]

From (3) we obtain

\[
x^*_E = 1 - 2x^*_2 - c
\]

which we can substitute into (4) to get

\[
(2 - \alpha^2/2) (1 - 2x^*_2 - c) = 1 - x^*_2 - c + \alpha (1 - c)/2
\]

or

\[
x^*_2 (3 - \alpha^2) = - (-1 + \alpha^2/2 + \alpha/2) (1 - c)
\]

or

\[
x^*_2 = \frac{(2 - \alpha^2 - \alpha) (1 - c)}{6 - 2\alpha^2}. \tag{5}
\]

We can use (3) to obtain

\[
x^*_E = (1 - c) - 2x^*_2 = \frac{[(6 - 2\alpha^2) - 2(2 - \alpha^2 - \alpha)] (1 - c)}{6 - 2\alpha^2} = \frac{(2 + 2\alpha) (1 - c)}{6 - 2\alpha^2}. \tag{6}
\]

Finally, from (1) we get

\[
x^*_W = (1 - c)/2 + \alpha x^*_E/2 = \frac{[(3 - \alpha^2) + \alpha (1 + \alpha)] (1 - c)}{6 - 2\alpha^2} = \frac{(3 + \alpha) (1 - c)}{6 - 2\alpha^2} \tag{7}
\]

3. The Hessian matrix for firm 1 is

\[
H = \begin{bmatrix}
-2 & \alpha \\
\alpha & -2
\end{bmatrix}
\]

where the first minor is -1 and the determinant is \(4 - \alpha^2\) which is positive since \(\alpha < 2\). As for firm 2, the second derivative of the profit function with respect to \(x_2\) is \(-2 < 0\). The second order conditions are satisfied.
4. From expressions (6) and (7) it is clear that both \( x_E^* \) and \( x_W^* \) are increasing in \( \alpha \) : the higher the degree of learning by doing, the higher the production by firm 1 on the East and West market. The learning by doing induces the firm to produce more on the West coast, since this will reduce the costs of production on the East Coast. Once this high production has taken place, the firm has effectively reduced the costs of producing on the East coast and therefore produces more. As for firm 2, we can differentiate expression (5) to get

\[
\frac{\partial x_2^*}{\partial \alpha} = (1 - c) \frac{(-2\alpha - 1)(6 - 2\alpha^2) - (2 - \alpha^2 - \alpha)(-4\alpha)}{(6 - 2\alpha^2)^2} = \frac{1-c}{(6 - 2\alpha^2)^2} \left[ -12\alpha - 6 + 4\alpha^3 + 2\alpha^2 + 8\alpha - 4\alpha^3 - 4\alpha^2 \right] = - \frac{1-c}{(6 - 2\alpha^2)^2} [ -4\alpha - 6 - 2\alpha^2 ] < 0.
\]

Therefore, as the degree of learning-by-doing of firm 1 increases, the production of firm 2 decreases. Essentially, by learning by doing, firm 1 decreases the own production costs and pushes firm 2 to lower and lower levels of production.

5. The profits of firm 2 are

\[
\pi_2 = \frac{(2 - \alpha^2 - \alpha)(1 - c)}{6 - 2\alpha^2} \left( 1 - \frac{(2 + 2\alpha)(1 - c)}{6 - 2\alpha^2} - \frac{(2 - \alpha^2 - \alpha)(1 - c)}{6 - 2\alpha^2} \right) - c \frac{(2 - \alpha^2 - \alpha)(1 - c)}{6 - 2\alpha^2} = \frac{(2 - \alpha^2 - \alpha)(1 - c)}{6 - 2\alpha^2} \left( 1 - \frac{(1 - c)(4 + \alpha - \alpha^2)}{6 - 2\alpha^2} \right) = \left[ \frac{(2 - \alpha^2 - \alpha)(1 - c)}{6 - 2\alpha^2} \right]^2
\]

Since the profits of firm 2 \( \pi_2 \) equal \((x_2^*)^2\), it is clear that \( \partial \pi_2 / \partial \alpha \) has the same sign as \( \partial x_2^* / \partial \alpha \), which is negative. As the learning by doing of firm 1 increases, the profits of firm 2 decrease. This is since the increased production of firm 1 induces firm 2 to produce less and reduces its profits. Interestingly, this occurs despite firm 2 not observing the decisions of firm 1 before it takes its own decisions. It all happens through equilibrium arguments.

6. In the sequential version, the firms produce on the East Coast after observing production \( x_W^* \) on the West Coast. The profit function of firm 1 in period 2 is

\[
x_E \left( 1 - x_E - x_2 \right) - (c - \alpha x_W^*) x_E
\]

and the profit function of firm 2 is

\[
x_2 \left( 1 - x_E - x_2 \right) - c x_2.
\]

7. The first order conditions are

\[
1 - 2x_E^* - x_2^* - c + \alpha x_W^* = 0
\]

and

\[
1 - x_E^* - 2x_2^* - c = 0.
\]

Solving for \( x_2^* \) in the first equation we get \( x_2^* = 1 - 2x_E^* - c + \alpha x_W^* \) which we can substitute in the second expression to get

\[
1 - x_E^* - 2 \left( 1 - 2x_E^* - c + \alpha x_W^* \right) - c = 0
\]

or

\[
x_E^* (x_W^*) = \frac{1-c+2\alpha x_W^*}{3}.
\]

We can then obtain

\[
x_2^* (x_W^*) = \frac{1-c}{2} - \frac{1-c+2\alpha x_W^*}{6} = \frac{(1-c) - \alpha x_W^*}{3}
\]

8. Notice that \( x_E^* \) is increasing in \( x_W^* \) since higher past production decreases current marginal costs for firm 1. As for firm 2, \( x_2^* \) is decreasing in \( x_W^* \) since higher \( \alpha \) implies that firm 1 will produce more on the East coast.
9. The profits for firm 1 are
\[
\pi_1(x_W) = x^*_E (1 - x^*_E - x^*_2 - (c - \alpha x^*_W)) = \frac{1 - c + 2\alpha x^*_W}{3} \left( 1 - c + \alpha x^*_W - \frac{1 - c + 2\alpha x^*_W - (1 - c - \alpha x^*_W)}{3} \right) = \\
= \frac{1 - c + 2\alpha x^*_W}{3} \left( 1 - c + \alpha x^*_W - \frac{2 (1 - c) + \alpha x^*_W}{3} \right) = \frac{(1 - c + 2\alpha x^*_W)^2}{9}
\]

10. The profit function in period 1 for firm 1 is
\[
x_W (1 - x_w) - cx_W + \frac{(1 - c + 2\alpha x_W)^2}{9}
\]

11. The first order condition with respect to \(x_W\) is
\[
1 - 2x^*_w - c + \frac{2}{9} (1 - c + 2\alpha x^*_W) 2\alpha = 0
\]
or
\[
x^*_W \left( 2 - \frac{8}{9} \alpha^2 \right) = (1 - c) \left( 1 + \frac{4}{9} \alpha \right)
\]
or
\[
x^*_w = (1 - c) \frac{9 + 4\alpha}{(18 - 8\alpha^2)}
\]

We can then plug \(x^*_w\) into \(x^*_E(x^*_W)\) to obtain
\[
x^*_E(x^*_W) = \frac{1 - c + 2\alpha (1 - c) (9 + 4\alpha) / (18 - 8\alpha^2)}{3}
\]

Similarly, we get
\[
x^*_2(x^*_W) = \frac{(1 - c - \alpha (1 - c) (9 + 4\alpha) / (18 - 8\alpha^2)}{3}
\]

12. The production on the West coast under simultaneous production is
\[
x^*_W = \frac{(3 + \alpha) (1 - c)}{6 - 2\alpha^2}
\]

whereas in the sequential version we obtain
\[
x^*_W = (1 - c) \frac{(9 + 4\alpha)}{(18 - 8\alpha^2)}
\]

The production in the sequential game is higher than in the simultaneous game if
\[
(1 - c) (9 + 4\alpha) / (18 - 8\alpha^2) \geq \frac{(3 + \alpha) (1 - c)}{6 - 2\alpha^2}
\]
or
\[
(9 + 4\alpha) (6 - 2\alpha^2) \geq (3 + \alpha) (18 - 8\alpha^2)
\]
or
\[
54 - 18\alpha^2 + 24\alpha - 8\alpha^3 \geq 54 - 24\alpha^2 + 18\alpha - 8\alpha^3
\]
or
\[
6\alpha^2 + 6\alpha = 6\alpha (1 + \alpha) \geq 0
\]
or \(\alpha \geq 0\). That is, as long as there is learning by doing (\(\alpha > 0\)), firm 1 produces more on the West coast if the game is sequential than if it is simultaneous. In other words, if firm 2 can observe \(x^*_W\) then, as in a standard Stackelberg duopoly, the leader gets to produce more and earn more profits. I realize that computing the profits here would be very cumbersome (sorry), but I am ready to bet 10:1 that firm 1 overall makes more profits in the sequential than in the simultaneous case. :)

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Problem 2. General Equilibrium (25 points) Consider the case of pure exchange with two consumers. Both consumers have Cobb-Douglas preferences, but with different parameters. Consumer 1 has utility function \( u(x_1^1, x_2^1) = (x_1^1)^\alpha (x_2^1)^{1-\alpha} \). Consumer 2 has utility function \( u(x_1^2, x_2^2) = (x_1^2)^\beta (x_2^2)^{1-\beta} \). The endowment of good \( j \) owned by consumer \( i \) is \( \omega_j^i \). The price of good 1 is \( p_1 \), while the price of good 2 is normalized to 1 without loss of generality.

1. Only for the purpose of this point, assume \( \omega_1^1 = 1, \omega_2^1 = 3, \omega_1^2 = 3, \omega_2^2 = 1 \). (that is, total endowment of each good is 4). Assume further \( \alpha = 1/2, \beta = 1/2 \). Draw the Pareto set and the contract curve for this economy in an Edgeworth box. (you do not need to give the exact solutions, only a graphical representation) What is the set of points that could be the outcome under barter in this economy? (5 points)

2. For each consumer, compute the utility maximization problem. Solve for \( x_j^{i*} \) for \( j = 1, 2 \) and \( i = 1, 2 \) as a function of the price \( p_1 \) and of the endowments. [By now, this part should be sooo familiar to you, you should be able to solve this problem with closed eyes] (5 points)

3. Now comes the general equilibrium part. Require now that the total sum of the demands for good 1 equals the total sum of the endowments, that is, \( x_1^1 + x_1^2 = \omega_1^1 + \omega_1^2 \). Solve for the general equilibrium price \( p_1^* \). (6 points)

4. What is the comparative statics of \( p_1^* \) with respect to the endowment of good 1, that is, with respect to \( \omega_1^i \) for \( i = 1, 2 \)? What about with respect to the endowment of the other good? Does this make sense? What is the comparative statics of \( p_1^* \) with respect to the taste for good 1, that is, with respect to \( \alpha \) and \( \beta \)? Does this make sense? (4 points)

5. Now require the same general equilibrium condition in market 2. Solve for \( p_1^* \) again, and check that this solution is the same as the one you found in the point above. In other words, you found a property that is called Walras' Law. In an economy with \( n \) markets, if \( n-1 \) markets are in equilibrium, the \( n \)th market will be in equilibrium as well. (5 points)

Solution to Problem 2.

1. See Figure.

2. Instead of solving the Lagrangean problem (make sure you know how to do this), i will use a general feature of Cobb-Douglas utility functions. The consumer consumes share \( \alpha \) of income in the first commodity. This implies

\[
x_1^{1*} = \alpha \frac{(p_1^* \omega_1^1 + \omega_2^1)}{p_1^*}
\]

and

\[
x_2^{1*} = (1 - \alpha) \left( p_1^* \omega_1^1 + \omega_2^1 \right).
\]

For consumer 2

\[
x_1^{2*} = \beta \frac{(p_1^* \omega_1^2 + \omega_2^2)}{p_1^*}
\]

and

\[
x_2^{2*} = (1 - \beta) \left( p_1^* \omega_1^2 + \omega_2^2 \right).
\]

3. We have already imposed the first part of the definition of Walrasian Equilibrium, that is, that each consumer chooses the optimal allocation given the price \( p_1^* \). We now impose the second part of the definition, that is, that each market be in equilibrium. We first impose the condition for the first market, that is, \( x_1^{1*} + x_1^{2*} = \omega_1^1 + \omega_2^1 \). This implies

\[
\alpha \frac{(p_1^* \omega_1^1 + \omega_2^1)}{p_1^*} + \beta \frac{(p_1^* \omega_1^2 + \omega_2^2)}{p_1^*} = \omega_1^1 + \omega_2^1
\]
or
\[ \alpha \omega_1 + \beta \omega_2 + \frac{(\alpha \omega_1 + \beta \omega_2)}{p_1^*} = \omega_1^1 + \omega_2^2 \]

or
\[ (\alpha \omega_2^1 + \beta \omega_2^2) = p_1^* \left[ (1 - \alpha) \omega_1^1 + (1 - \beta) \omega_2^1 \right] \]

or
\[ p_1^* = \frac{(\alpha \omega_2^1 + \beta \omega_2^2)}{[(1 - \alpha) \omega_1^1 + (1 - \beta) \omega_2^1]} \]

4. Notice that, as the endowment of good 1 \( \omega_1^1 \) or \( \omega_1^2 \) increases, the price of good 1 decreases. This makes sense. If the quantity available of good 1 increases, while holding constant the quantity of good 2, good 2 becomes relatively scarcer and this induces the relative price of good 1 to decrease. This is like an increase in supply that decreases the price. Notice that \( p_1 \) is not the absolute price of good 1, but rather the relative price of good 1 relative to the price of good 2, which we normalized to 1. Symmetrically, if the endowment of good 2 increases, good 1 becomes scarcer and the price \( p_1^* \) increases. As for the other comparative statics, as the taste for good 1 increases (increase in \( \alpha \) or \( \beta \)) the equilibrium price of good 1 goes up. Since increase in taste for good 1 means a positive shift in demand, we are not surprised that this induces an increase in the equilibrium price of good 1. Notice a difference, though, between the general equilibrium increase in the demand and the partial equilibrium. If both \( \alpha \) and \( \beta \) increase, that is, if both consumers like good 1 more, the price of good 1 will go up, but the quantity consumed of good 1 will go down (or at least not go up) for one of the consumers. In this setting, the total quantity available of each good is constant.

5. We now impose the condition for the second market, that is, \( x_2^1 \ast + x_2^2 \ast = \omega_1^2 + \omega_2^2 \). This implies
\[ (1 - \alpha) \left( p_1^* \omega_1^1 + \omega_2^1 \right) + (1 - \beta) \left( p_1^* \omega_1^2 + \omega_2^2 \right) = \omega_1^1 + \omega_2^2 \]

or
\[ \left[ (1 - \alpha) \omega_1^1 + (1 - \beta) \omega_2^2 \right] p_1^* = (\alpha \omega_2^1 + \beta \omega_2^2) \]

or
\[ p_1^* = \frac{(\alpha \omega_2^1 + \beta \omega_2^2)}{[(1 - \alpha) \omega_1^1 + (1 - \beta) \omega_2^1]} \]

which is the same solution that we found before. If there are \( n \) markets, it’s enough to impose equilibrium conditions in \( n - 1 \) of them, and the \( n \)-th market will automatically be in equilibrium.
Problem 2.1

\[ w_2 \]

\[ w_2^1 \]

\[ w_2^2 \]

\[ \text{CONTRACT CURVE} \]

\[ \text{PARETO SET} \]

\[ \text{CONSUMERS 1} \]

\[ \text{GOOD 1} \]

\[ \text{CONSUMERS 2} \]

\[ \text{GOOD 2} \]