Outline

1. Utility maximization – Tricky Cases

2. Indirect Utility Function

3. Comparative Statics (Introduction)

4. Income Changes
1 Utility maximization – tricky cases

• First, re-solve CES utility function.

$$\max_{x_1, x_2} \left( \alpha x_1^\rho + \beta x_2^\rho \right)^{1/\rho}$$

s.t. \( p_1 x_1 + p_2 x_2 - M = 0 \)

• Solution:

$$x_1^* = \frac{M}{p_1 \left( 1 + \left( \frac{\alpha}{\beta} \right)^{\frac{1}{\rho-1}} \left( \frac{p_2}{p_1} \right)^{\frac{\rho}{\rho-1}} \right)}$$

$$x_2^* = \frac{M}{p_2 \left( 1 + \left( \frac{\beta}{\alpha} \right)^{\frac{1}{\rho-1}} \left( \frac{p_1}{p_2} \right)^{\frac{\rho}{\rho-1}} \right)}$$
• Special case 1: $\rho \to 1^-$ (Perfect Substitutes)

$$\lim_{\rho \to 1^-} \frac{1}{\rho - 1} = \lim_{\rho \to 1^-} \frac{\rho}{\rho - 1} = -\infty$$
(here notice the convergence from the left)

- If $\frac{\alpha p_2}{\beta p_1} > 1$ (or $p_1/p_2 < \alpha/\beta$),

$$\left(\frac{\alpha}{\beta}\right)^{\frac{1}{\rho - 1}} \left(\frac{p_2}{p_1}\right)^{\rho^{\frac{1}{\rho - 1}}} \to 0$$

$$x_1^* \to \frac{M}{p_1}$$

- If $\frac{\alpha p_2}{\beta p_1} < 1$ (or $p_1/p_2 > \alpha/\beta$),

$$\left(\frac{\alpha}{\beta}\right)^{\frac{1}{\rho - 1}} \left(\frac{p_2}{p_1}\right)^{\rho^{\frac{1}{\rho - 1}}} \to \infty$$

$$x_1^* \to 0$$
• Solution for Perfect Substitutes Case is

\[ x_1^* = \begin{cases} 
0 & \text{if } p_1/p_2 > \alpha/\beta \\
M/p_1 & \text{if } p_1/p_2 < \alpha/\beta \\
\text{any } x_1 \in [0, M/p_1] & \text{if } p_1/p_2 = \alpha/\beta
\end{cases} \]

\[ x_2^* = \begin{cases} 
M/p_2 & \text{if } p_1/p_2 > \alpha/\beta \\
0 & \text{if } p_1/p_2 < \alpha/\beta \\
x_2 \text{ such that B.C. holds} & \text{if } p_1/p_2 = \alpha/\beta
\end{cases} \]

• Case \( p_1/p_2 = \alpha/\beta \) has to be analyzed separately

• This is case in which budget line and indifference curves are parallel \( \Rightarrow \) All points on budget line are tangent and hence optimal.
2. Solution does not satisfy $x_1^* > 0$ or $x_2^* > 0$. Example:

$$\max x_1^* (x_2 + 5)$$

$$s.t. \ p_1 x_1 + p_2 x_2 = M$$

• In this case consider corner conditions: what happens for $x_1^* = 0$? And $x_2^* = 0$?

- Example 1: Perfect Substitutes with \( \frac{p_1}{p_2} = \frac{\alpha}{\beta} \)

- Example 2: Non-convex preferences with two optima
2 Indirect utility function

- Nicholson, Ch. 4, pp. 124-127 (106–108, 9th)

- Define the indirect utility \( v(p, M) \equiv u(x^*(p, M)) \), with \( p \) vector of prices and \( x^* \) vector of optimal solutions.

- \( v(p, M) \) is the utility at the optimum for prices \( p \) and income \( M \)

- Some comparative statics: \( \partial v(p, M)/\partial M = ? \)

- Hint: Use Envelope Theorem on Lagrangean function
• What is the sign of \( \lambda \)?

• \( \lambda = u'_x / p > 0 \)

• \( \partial v(p, M) / \partial p_i =? \)

• Properties:
  
  – Indirect utility is always increasing in income \( M \)
  
  – Indirect utility is always decreasing in the price \( p_i \)
3 Comparative Statics (introduction)

• Nicholson, Ch. 5, pp. 141-151 (121–131, 9th)

• Utility maximization yields $x_i^* = x_i^*(p_1, p_2, M)$

• Quantity consumed as a function of income and price

• What happens to quantity consumed $x_i^*$ as prices or income varies?
• Simple case: Equal increase in prices and income.

\[ M' = tM, \quad p'_1 = tp_1, \quad p'_2 = tp_2. \]

• Compare \( x^*(tM, tp_1, tp_2) \) and \( x^*(M, p_1, p_2) \).

• What happens?

• Write budget line: \( tp_1x_1 + tp_2x_2 = tM \)

• Demand is homogeneous of degree 0 in \( p \) and \( M \):

\[ x^*(tM, tp_1, tp_2) = t^0 x^*(M, p_1, p_2) = x^*(M, p_1, p_2). \]
• Consider Cobb-Douglas Case:

\[ x_1^* = \frac{\alpha}{\alpha + \beta} \frac{M}{p_1}, \quad x_2^* = \frac{\beta}{\alpha + \beta} \frac{M}{p_2} \]

• What is \( \partial x_1^*/\partial M \)?

• What is \( \partial x_1^*/\partial p_1 \)?

• What is \( \partial x_1^*/\partial p_2 \)?

• General results?
4 Income changes

• Income increases from $M$ to $M' > M$.

• Budget line $(p_1x_1 + p_2x_2 = M)$ shifts out:

\[ x_2 = \frac{M'}{p_2} - x_1 \frac{p_1}{p_2} \]

• New optimum?
• Engel curve: \( x_i^*(M) \): demand for good \( i \) as function of income \( M \) holding fixed prices \( p_1, p_2 \)

• Does \( x_i^* \) increase with \( M \)?
  
  – Yes. Good \( i \) is normal

  – No. Good \( i \) is inferior
5 Next Class

• More comparative statics:
  – Price Effects
  – Intuition
  – Slutsky Equation

• Then moving on to applications:
  – Labor Supply
  – Intertemporal choice
  – Economics of Altruism