Outline

1. Convexity and concavity II

2. Constrained Maximization

3. Envelope Theorem II

4. Preferences
1 Convexity and concavity

- Alternative characterization of convexity.

- A function $f$, twice differentiable, is concave if and only if for all $x$ the subdeterminants $|H_i|$ of the Hessian matrix have the property $|H_1| \leq 0$, $|H_2| \geq 0$, $|H_3| \leq 0$, and so on.

- For the univariate case, this reduces to $f'' \leq 0$ for all $x$

- For the bivariate case, this reduces to $f''_{x,x} \leq 0$ and $f''_{x,x} * f''_{y,y} - (f''_{x,y})^2 \geq 0$

- A twice-differentiable function is strictly concave if the same property holds with strict inequalities.
Examples.

1. For which values of $a$, $b$, and $c$ is $f(x) = ax^3 + bx^2 + cx + d$ is the function concave over $R$? Strictly concave? Convex?

2. Is $f(x, y) = -x^2 - y^2$ concave?

For Example 2, compute the Hessian matrix

- $f_x' =$, $f_y' =$
- $f_{xx}'' =$, $f_{xy}'' =$
- $f_{yy}'' =$, $f_{yx}'' =$

- Hessian matrix $H$:

$$H = \begin{pmatrix} f_{xx}'' & f_{xy}'' \\ f_{yx}'' & f_{yy}'' \end{pmatrix}$$

- Compute $|H_1| = f_{xx}''$ and $|H_2| = f_{xx}'' * f_{yy}'' - (f_{xy}'')^2$
• Why are convexity and concavity important?

• Theorem. Consider a twice-differentiable concave (convex) function over $C \subset \mathbb{R}^n$. If the point $x_0$ satisfies the first-order conditions, it is a global maximum (minimum).

• For the proof, we need to check that the second-order conditions are satisfied.

• These conditions are satisfied by definition of concavity!

• (We have only proved that it is a local maximum)
2 Constrained Maximization

- Ch. 2, pp. 36-42 (38–44, 9th Ed)

- So far unconstrained maximization on $R$ (or open subsets)

- What if there are constraints to be satisfied?

- Example 1: $\max x, y \ x * y$ subject to $3x + y = 5$

  - Substitute it in: $\max x, y \ x * (5 - 3x)$

  - Solution: $x^* = \quad$

- Example 2: $\max x, y \ xy$ subject to $x \exp(y) + y \exp(x) = 5$

  - Solution: ?
• Graphical intuition on general solution.

• Example 3: \( \max_{x,y} f(x, y) = x \ast y \) s.t. \( h(x, y) = x^2 + y^2 - 1 = 0 \)

• Draw \( 0 = h(x, y) = x^2 + y^2 - 1 \).

• Draw \( x \ast y = K \) with \( K > 0 \). Vary \( K \)

• Where is optimum?

• Where \( \frac{dy}{dx} \) along curve \( xy = K \) equals \( \frac{dy}{dx} \) along curve \( x^2 + y^2 - 1 = 0 \)

• Write down these slopes.
Idea: Use implicit function theorem.

- Heuristic solution of system

\[
\max_{x,y} f(x, y) \quad \text{s.t. } h(x, y) = 0
\]

- Assume:
  - continuity and differentiability of \( h \)
  - \( h'_y \neq 0 \) (or \( h'_x \neq 0 \))

- Implicit function Theorem: Express \( y \) as a function of \( x \) (or \( x \) as function of \( y \))!
• Write system as $\max_x f(x, g(x))$

• f.o.c.: $f'_x(x, g(x)) + f'_y(x, g(x)) \ast \frac{\partial g(x)}{\partial x} = 0$

• What is $\frac{\partial g(x)}{\partial x}$?

• Substitute in and get: $f'_x(x, g(x)) + f'_y(x, g(x)) \ast (-h'_x/h'_y) = 0$ or

$$\frac{f'_x(x, g(x))}{f'_y(x, g(x))} = \frac{h'_x(x, g(x))}{h'_y(x, g(x))}$$
• Lagrange Multiplier Theorem, necessary condition. Consider a problem of the type

\[
\max_{x_1, \ldots, x_n} f(x_1, x_2, \ldots, x_n; p) \\
\text{s.t.} \begin{cases} 
  h_1(x_1, x_2, \ldots, x_n; p) = 0 \\
  h_2(x_1, x_2, \ldots, x_n; p) = 0 \\
  \cdots \\
  h_m(x_1, x_2, \ldots, x_n; p) = 0 
\end{cases}
\]

with \( n > m \). Let \( x^* = x^*(p) \) be a local solution to this problem.

• Assume:

  – \( f \) and \( h \) differentiable at \( x^* \)

  – the following Jacobian matrix at \( x^* \) has maximal rank

\[
J = \begin{pmatrix}
\frac{\partial h_1}{\partial x_1}(x^*) & \cdots & \frac{\partial h_1}{\partial x_n}(x^*) \\
\cdots & \cdots & \cdots \\
\frac{\partial h_m}{\partial x_1}(x^*) & \cdots & \frac{\partial h_m}{\partial x_n}(x^*)
\end{pmatrix}
\]
Then, there exists a vector $\lambda = (\lambda_1, \ldots, \lambda_m)$ such that $(x^*, \lambda)$ maximize the Lagrangean function

$$L(x, \lambda) = f(x; p) - \sum_{j=0}^{m} \lambda_j h_j(x; p)$$

- Case $n = 2, m = 1$.

- First order conditions are

$$\frac{\partial f(x; p)}{\partial x_i} - \lambda \frac{\partial h(x; p)}{\partial x_i} = 0$$

for $i = 1, 2$

- Rewrite as

$$\frac{f'_{x_1}}{f'_{x_2}} = \frac{h'_{x_1}}{h'_{x_2}}$$
• **Constrained Maximization, Sufficient condition for the case** $n = 2, m = 1$.

• If $x^*$ satisfies the Lagrangean condition, and the determinant of the bordered Hessian

$$H = \begin{pmatrix}
0 & -\frac{\partial h}{\partial x_1}(x^*) & -\frac{\partial h}{\partial x_2}(x^*) \\
-\frac{\partial h}{\partial x_1}(x^*) & \frac{\partial^2 L}{\partial x_1^2}(x^*) & \frac{\partial^2 L}{\partial x_2 \partial x_1}(x^*) \\
-\frac{\partial h}{\partial x_2}(x^*) & \frac{\partial^2 L}{\partial x_1 \partial x_2}(x^*) & \frac{\partial^2 L}{\partial x_2^2}(x^*)
\end{pmatrix}$$

is positive, then $x^*$ is a constrained maximum.

• If it is negative, then $x^*$ is a constrained minimum.

• **Why?** This is just the Hessian of the Lagrangean $L$ with respect to $\lambda, x_1,$ and $x_2$
• Example 4: $\max_{x,y} x^2 - xy + y^2 \text{ s.t. } x^2 + y^2 - p = 0$

• $\max_{x,y,\lambda} x^2 - xy + y^2 - \lambda(x^2 + y^2 - p)$

• F.o.c. with respect to $x$:

• F.o.c. with respect to $y$:

• F.o.c. with respect to $\lambda$:

• Candidates to solution?

• Maxima and minima?
3 Envelope Theorem II

- Envelope Theorem II: Ch. 2, pp. 42-43 (44, 9th Ed)

- **Envelope Theorem for Constrained Maximization.** In problem above consider \( F(p) \equiv f(x^*(p); p) \). We are interested in \( dF(p)/dp \). We can neglect indirect effects:

\[
\frac{dF}{dp_i} = \frac{\partial f(x^*(p); p)}{\partial p_i} - \sum_{j=0}^{m} \lambda_j \frac{\partial h_j(x^*(p); p)}{\partial p_i}
\]

- Example 4 (continued).  \( \max_{x,y} x^2 - xy + y^2 \) s.t.  
\( x^2 + y^2 - p = 0 \)

- \( df(x^*(p), y^*(p))/dp \)?

- Envelope Theorem.
4 Preferences

• Part 1 of our journey in microeconomics: Consumer Theory

• Choice of consumption bundle:
  1. Consumption today or tomorrow
  2. work, study, and leisure
  3. choice of government policy

• Starting point: preferences.
  1. 1 egg today $\succ$ 1 chicken tomorrow
  2. 1 hour doing problem set $\succ$ 1 hour in class $\succ$
     ... $\succ$ 1 hour out with friends
  3. War on Iraq $\succ$ Sanctions on Iraq
5 Next Class

- Properties of Preferences
- From Preferences to Utility
- Common Utility Functions