Outline

1. From Preferences to Utility (and viceversa)

2. Common Utility Functions

3. Utility maximization
1 From preferences to utility

- Nicholson, Ch. 3

- Economists like to use utility functions $u : X \rightarrow R$

- $u(x)$ is ‘liking’ of good $x$

- $u(a) > u(b)$ means: I prefer $a$ to $b$.

- **Def.** Utility function $u$ represents preferences $\succeq$ if, for all $x$ and $y$ in $X$, $x \succeq y$ if and only if $u(x) \geq u(y)$.

- **Theorem.** If preference relation $\succeq$ is rational and continuous, there exists a continuous utility function $u : X \rightarrow R$ that represents it.
• [Skip proof]

• Example:

\[(x_1, x_2) \succeq (y_1, y_2) \text{ iff } x_1 + x_2 \geq y_1 + y_2\]

• Draw:

• Utility function that represents it: \(u(x) = x_1 + x_2\)

• But... Utility function representing \(\succeq\) is not unique

• Take \(3u(x)\) or \(\exp(u(x))\)

• \(u(a) > u(b) \iff \exp(u(a)) > \exp(u(b))\)
• If $u(x)$ represents preferences $\succeq$ and $f$ is a strictly increasing function, then $f(u(x))$ represents $\succeq$ as well.

• If preferences are represented from a utility function, are they rational?
  
  – completeness

  – transitivity
• Indifference curves: \( u(x_1, x_2) = \bar{u} \)

• They are just implicit functions! \( u(x_1, x_2) - \bar{u} = 0 \)

\[
\frac{dx_2}{dx_1} = -\frac{U'_{x_1}}{U'_{x_2}} = MRS
\]

• Indifference curves for:
  
  – monotonic preferences;

  – strictly monotonic preferences;

  – convex preferences
2 Common utility functions

- Nicholson, Ch. 3, pp. 100-104 (82-86, 9th)

1. Cobb-Douglas preferences: $u(x_1, x_2) = x_1^\alpha x_2^{1-\alpha}$
   - $MRS = \frac{-\alpha x_1^{\alpha-1} x_2^{1-\alpha} / (1-\alpha) x_1^\alpha x_2^{-\alpha}}{x_1^{1-\alpha}} = \frac{\alpha}{1-\alpha} x_1$

2. Perfect substitutes: $u(x_1, x_2) = \alpha x_1 + \beta x_2$
   - $MRS = -\alpha/\beta$
3. Perfect complements: \( u(x_1, x_2) = \min(\alpha x_1, \beta x_2) \)

- \( MRS \) discontinuous at \( x_2 = \frac{\alpha}{\beta} x_1 \)

4. Constant Elasticity of Substitution: \( u(x_1, x_2) = \left( \alpha x_1^\rho + \beta x_2^\rho \right)^{1/\rho} \)

- \( MRS = -\frac{\alpha}{\beta} \left( \frac{x_1}{x_2} \right)^{\rho-1} \)

- if \( \rho = 1 \), then...

- if \( \rho = 0 \), then...

- if \( \rho \rightarrow -\infty \), then...
3 Utility Maximization

- Nicholson, Ch. 4, pp. 114–124 (94–105, 9th)

- $X = \mathbb{R}_+^2$ (2 goods)

- Consumers: choose bundle $x = (x_1, x_2)$ in $X$ which yields highest utility.

- Constraint: income $= M$

- Price of good 1 $= p_1$, price of good 2 $= p_2$

- Bundle $x$ is feasible if $p_1 x_1 + p_2 x_2 \leq M$

- Consumer maximizes

$$\max_{x_1, x_2} u(x_1, x_2)$$

s.t. $p_1 x_1 + p_2 x_2 \leq M$

$x_1 \geq 0, \ x_2 \geq 0$
• Maximization subject to inequality. How do we solve that?

• Trick: $u$ strictly increasing in at least one dimension. ($\succeq$ strictly monotonic)

• Budget constraint always satisfied with equality

• Ignore temporarily $x_1 \geq 0, x_2 \geq 0$ and check afterwards that they are satisfied for $x_1^*$ and $x_2^*$. 
• Problem becomes

$$\max_{x_1, x_2} u(x_1, x_2)$$
$$s.t. \ p_1 x_1 + p_2 x_2 - M = 0$$

• $L(x_1, x_2) = u(x_1, x_2) - \lambda (p_1 x_1 + p_2 x_2 = M)$

• F.o.c.s:

$$u'_{x_i} - \lambda p_i = 0 \text{ for } i = 1, 2$$
$$p_1 x_1 + p_2 x_2 - M = 0$$
• Moving the two terms across and dividing, we get:

\[ MRS = -\frac{u'_{x_1}}{u'_{x_2}} = -\frac{p_1}{p_2} \]

• Graphical interpretation.
• Second order conditions:

\[ H = \begin{pmatrix} 0 & -p_1 & -p_2 \\ -p_1 & u''_{x_1,x_1} & u''_{x_1,x_2} \\ -p_2 & u''_{x_2,x_1} & u''_{x_2,x_2} \end{pmatrix} \]

\[ |H| = p_1 \left( -p_1 u''_{x_2,x_2} + p_2 u''_{x_2,x_1} \right) \]

\[ -p_2 \left( -p_1 u''_{x_1,x_2} + p_2 u''_{x_1,x_1} \right) \]

\[ = -p_1^2 u''_{x_2,x_2} + 2p_1p_2 u''_{x_1,x_2} - p_2^2 u''_{x_1,x_1} \]

• Notice: \( u''_{x_2,x_2} < 0 \) and \( u''_{x_1,x_1} < 0 \) usually satisfied (but check it!).

• Condition \( u''_{x_1,x_2} > 0 \) is then sufficient
• Example with CES utility function.

\[
\max_{x_1, x_2} \left( \alpha x_1^\rho + \beta x_2^\rho \right)^{1/\rho}
\]

\[
s.t. \ p_1 x_1 + p_2 x_2 - M = 0
\]

• Lagrangean =

• F.o.c.:

• Solution:

\[
x_1^* = \frac{M}{p_1 \left( 1 + \left( \frac{\alpha}{\beta} \right)^{1/\rho} \left( \frac{p_2}{p_1} \right)^{\rho/\rho-1} \right)}
\]

\[
x_2^* = \frac{M}{p_2 \left( 1 + \left( \frac{\beta}{\alpha} \right)^{1/\rho} \left( \frac{p_1}{p_2} \right)^{\rho/\rho-1} \right)}
\]
• Special case 1: $\rho = 0$ (Cobb-Douglas)

\[
x_1^* = \frac{\alpha M}{\alpha + \beta p_1}
\]
\[
x_2^* = \frac{\beta M}{\alpha + \beta p_2}
\]

• Special case 1: $\rho \to 1$ (Perfect Substitutes)

\[
x_1^* = \begin{cases} 
0 & \text{if } p_1/p_2 \geq \alpha/\beta \\
M/p_1 & \text{if } p_1/p_2 < \alpha/\beta 
\end{cases}
\]
\[
x_2^* = \begin{cases} 
M/p_2 & \text{if } p_1/p_2 \geq \alpha/\beta \\
0 & \text{if } p_1/p_2 < \alpha/\beta 
\end{cases}
\]
• Special case 1: \( \rho \rightarrow -\infty \) (Perfect Complements)

\[
    x_1^* = \frac{M}{p_1 + p_2} = x_2^*
\]

• Parameter \( \rho \) indicates substitution pattern between goods:

  - \( \rho > 0 \) \( \rightarrow \) Goods are (net) substitutes
  - \( \rho < 0 \) \( \rightarrow \) Goods are (net) complements
4 Next Class

- Utility Maximization – Tricky Cases

- Indirect Utility Function

- Comparative Statics:
  - with respect to price
  - with respect to income