Outline

1. Utility maximization

2. Utility maximization – Tricky Cases

3. Indirect Utility Function
1 Utility Maximization

• Nicholson, Ch. 4, pp. 114–124 (94–105, 9th)

• $X = R^2_+$ (2 goods)

• Consumers: choose bundle $x = (x_1, x_2)$ in $X$ which yields highest utility.

• Constraint: income $= M$

• Price of good $1 = p_1$, price of good $2 = p_2$

• Bundle $x$ is feasible if $p_1 x_1 + p_2 x_2 \leq M$

• Consumer maximizes

$$\max_{x_1, x_2} u(x_1, x_2)$$

$s.t. \quad p_1 x_1 + p_2 x_2 \leq M$

$x_1 \geq 0, \; x_2 \geq 0$
• Maximization subject to inequality. How do we solve that?

• Trick: \( u \) strictly increasing in at least one dimension. (\( \succeq \) strictly monotonic)

• Budget constraint always satisfied with equality

• Ignore temporarily \( x_1 \geq 0, x_2 \geq 0 \) and check afterwards that they are satisfied for \( x_1^* \) and \( x_2^* \).
• Problem becomes

\[
\max_{x_1, x_2} u(x_1, x_2)
\]

\[s.t. \ p_1x_1 + p_2x_2 - M = 0\]

• \(L(x_1, x_2) = u(x_1, x_2) - \lambda(p_1x_1 + p_2x_2 - M)\)

• F.o.c.s:

\[
\begin{align*}
    u'_{x_i} - \lambda p_i &= 0 \text{ for } i = 1, 2 \\
    p_1x_1 + p_2x_2 - M &= 0
\end{align*}
\]
• Moving the two terms across and dividing, we get:

\[ MRS = -\frac{u'_{x_1}}{u'_{x_2}} = -\frac{p_1}{p_2} \]

• Graphical interpretation.
• Second order conditions:

\[ H = \begin{pmatrix} 0 & -p_1 & -p_2 \\ -p_1 & u''_{x_1,x_1} & u''_{x_1,x_2} \\ -p_2 & u''_{x_2,x_1} & u''_{x_2,x_2} \end{pmatrix} \]

\[ |H| = p_1 \left( -p_1 u''_{x_2,x_2} + p_2 u''_{x_2,x_1} \right) \]
\[ -p_2 \left( -p_1 u''_{x_1,x_2} + p_2 u''_{x_1,x_1} \right) \]
\[ = -p_1^2 u''_{x_2,x_2} + 2p_1p_2 u''_{x_1,x_2} - p_2^2 u''_{x_1,x_1} \]

• Notice: \( u''_{x_2,x_2} < 0 \) and \( u''_{x_1,x_1} < 0 \) usually satisfied (but check it!).

• Condition \( u''_{x_1,x_2} > 0 \) is then sufficient
• Example with CES utility function.

$$\max_{x_1, x_2} \left( \alpha x_1^\rho + \beta x_2^\rho \right)^{1/\rho}$$

s.t. $$p_1 x_1 + p_2 x_2 - M = 0$$

• Lagrangean =

• F.o.c.:

• Solution:

$$x_1^* = \frac{M}{p_1 \left( 1 + \left( \frac{\alpha}{\beta} \right)^{\frac{1}{\rho-1}} \left( \frac{p_2}{p_1} \right)^{\frac{\rho}{\rho-1}} \right)}$$

$$x_2^* = \frac{M}{p_2 \left( 1 + \left( \frac{\beta}{\alpha} \right)^{\frac{1}{\rho-1}} \left( \frac{p_1}{p_2} \right)^{\frac{\rho}{\rho-1}} \right)}$$
• Special case 1: $\rho = 0$ (Cobb-Douglas)

\[
x_1^* = \frac{\alpha M}{\alpha + \beta p_1}
\]
\[
x_2^* = \frac{\beta M}{\alpha + \beta p_2}
\]

• Special case 1: $\rho \rightarrow 1$ (Perfect Substitutes)

\[
x_1^* = \begin{cases} 
0 & \text{if } p_1/p_2 \geq \alpha/\beta \\
M/p_1 & \text{if } p_1/p_2 < \alpha/\beta 
\end{cases}
\]
\[
x_2^* = \begin{cases} 
M/p_2 & \text{if } p_1/p_2 \geq \alpha/\beta \\
0 & \text{if } p_1/p_2 < \alpha/\beta 
\end{cases}
\]
• Special case 1: $\rho \rightarrow -\infty$ (Perfect Complements)

$$x_1^* = \frac{M}{p_1 + p_2} = x_2^*$$

• Parameter $\rho$ indicates substitution pattern between goods:

  - $\rho > 0 \rightarrow$ Goods are (net) substitutes
  - $\rho < 0 \rightarrow$ Goods are (net) complements
2 Utility maximization – tricky cases

1. Non-convex preferences. Example:
2. Example with CES utility function.

\[
\max_{x_1,x_2} \left( \alpha x_1^\rho + \beta x_2^\rho \right)^{1/\rho}
\]
\[
s.t. \quad p_1 x_1 + p_2 x_2 - M = 0
\]

- With \( \rho > 1 \) the interior solution is a minimum!

- Draw indifference curves for \( \rho = 1 \) (boundary case) and \( \rho = 2 \)

- Can also check using second order conditions
2. Solution does not satisfy $x_1^* > 0$ or $x_2^* > 0$. Example:

$$\max x_1 \ast (x_2 + 5)$$
$$s.t. \ p_1x_1 + p_2x_2 = M$$

- In this case consider corner conditions: what happens for $x_1^* = 0$? And $x_2^* = 0$?
3. Multiplicity of solutions. Example:

• Convex preferences that are not strictly convex
3 Indirect utility function

- Nicholson, Ch. 4, pp. 124-127 (106–108, 9th)

- Define the indirect utility $v(p, M) \equiv u(x^*(p, M))$, with $p$ vector of prices and $x^*$ vector of optimal solutions.

- $v(p, M)$ is the utility at the optimum for prices $p$ and income $M$

- Some comparative statics: $\partial v(p, M)/\partial M = ?$

- Hint: Use Envelope Theorem on Lagrangean function
• What is the sign of $\lambda$?

• $\lambda = \frac{u'_x}{p} > 0$

• $\partial v(p, M)/\partial p_i = ?$

• Properties:
  
  – Indirect utility is always increasing in income $M$

  – Indirect utility is always decreasing in the price $p_i$
4 Next Class

- Comparative Statics:
  - with respect to price
  - with respect to income