Outline

1. Two-step Cost Minimization II

2. Cost Minimization: Example

3. Cost Curves and Supply Function
1 Two-step Cost minimization II

- **First Step.** Minimize input costs for given production

- Firm objective function:

\[
\min_{L,K} wL + rK \\
\text{s.t. } f(L, K) \geq y
\]

- Derived demand for inputs:

  - \( L = L^*(w, r, y) \)
  
  - \( K = K^*(w, r, y) \)

- Value function at optimum is **cost function**:

\[
c(w, r, y) = wL^*(r, w, y) + rK^*(r, w, y)\]
• **Second step.** Given cost function, choose optimal quantity of $y$ as well

• Price of output is $p$.

• Firm’s objective:

$$\max p y - c(w, r, y)$$

• First order condition:

$$p - c'_y(w, r, y) = 0$$

• Price equals marginal cost – very important!
• Second order condition:

\[-c''_{y,y}(w, r, y^*) < 0\]

• For maximum, need increasing marginal cost curve.
Cost Minimization: Example

- Continue example above: \( y = f(L, K) = AK^\alpha L^\beta \)

- Cost minimization:

\[
\begin{align*}
\text{min} & \quad wL + rK \\
\text{s.t.} & \quad AK^\alpha L^\beta = y
\end{align*}
\]

- What is the return to scale for this example?

- Increase of all inputs: \( f(tz) \) with \( t \) scalar, \( t > 1 \)

- How much does input increase?
  
  - Decreasing returns to scale: for all \( z \) and \( t > 1 \),

\[
f(tz) < tf(z)
\]
– Constant returns to scale: for all $z$ and $t > 1$,

$$f(tz) = tf(z)$$

– Increasing returns to scale: for all $z$ and $t > 1$,

$$f(tz) > tf(z)$$

• Returns to scale depend on $\alpha + \beta \leq 1$: $f(tK, tL) = A(tK)^\alpha (tL)^\beta = t^{\alpha+\beta} AK^\alpha L^\beta = t^{\alpha+\beta} f(K, L)$
• Solutions:

  – Optimal amount of labor:

    \[
    L^* (r, w, y) = \left( \frac{y}{A} \right)^{\frac{1}{\alpha+\beta}} \left( \frac{w}{\rho} \right)^{-\frac{\alpha}{\alpha+\beta}}
    \]

  – Optimal amount of capital:

    \[
    K^* (r, w, y) = \frac{w}{r} \left( \frac{y}{A} \right)^{\frac{1}{\alpha+\beta}} \left( \frac{w}{\rho} \right)^{-\frac{\alpha}{\alpha+\beta}}
    = \left( \frac{y}{A} \right)^{\frac{1}{\alpha+\beta}} \left( \frac{w}{r} \right)^{\frac{\beta}{\alpha+\beta}}
    \]

• Check various comparative statics:

  – \( \partial L^*/\partial A < 0 \) (technological progress and unemployment)

  – \( \partial L^*/\partial y > 0 \) (more workers needed to produce more output)
- $\partial L^*/\partial w < 0$, $\partial L^*/\partial r > 0$ (substitute away from more expensive inputs)

- Parallel comparative statics for $K^*$
• Cost function

\[ c(w, r, y) = wL^*(r, w, y) + rK^*(r, w, y) = \]

\[ = \left( \frac{y}{A} \right) \frac{1}{\alpha + \beta} \left[ w \left( \frac{w \alpha}{r \beta} \right)^{-\frac{\alpha}{\alpha + \beta}} + \right. \]

\[ \left. + r \left( \frac{w \alpha}{r \beta} \right)^{\frac{\beta}{\alpha + \beta}} \right] \]

• Define \( B := w \left( \frac{w \alpha}{r \beta} \right)^{-\frac{\alpha}{\alpha + \beta}} + r \left( \frac{w \alpha}{r \beta} \right)^{\frac{\beta}{\alpha + \beta}} \)

• Cost-minimizing output choice:

\[ \max p y - B \left( \frac{y}{A} \right)^{\frac{1}{\alpha + \beta}} \]
• First order condition:

\[ p - \frac{1}{\alpha + \beta} \frac{B}{A} \left( \frac{y}{A} \right)^{\frac{1-(\alpha+\beta)}{\alpha+\beta}} = 0 \]

• Second order condition:

\[ -\frac{1}{\alpha + \beta} \frac{1 - (\alpha + \beta)}{\alpha + \beta} \frac{B}{A^2} \left( \frac{y}{A} \right)^{\frac{1-2(\alpha+\beta)}{\alpha+\beta}} \]

• When is the second order condition satisfied?
Solution:

- \( \alpha + \beta = 1 \) (CRS):
  
  * S.o.c. equal to 0
  
  * Solution depends on \( p \)
    
    * For \( p > \frac{B}{A} \), produce \( y^* \to \infty \)
    
    * For \( p = \frac{B}{A} \), produce any \( y^* \in [0, \infty) \)
    
    * For \( p < \frac{B}{A} \), produce \( y^* = 0 \)
- $\alpha + \beta > 1$ (IRS):
  
  * S.o.c. positive
  
  * Solution of f.o.c. is a minimum!

  * Solution is $y^* \rightarrow \infty$.

  * Keep increasing production since higher production is associated with higher returns
\(- \alpha + \beta < 1 \) (DRS):

* s.o.c. negative. OK!

* Solution of f.o.c. is an interior optimum

* This is the only "well-behaved" case under perfect competition

* Here can define a supply function
3 Cost Curves

- Marginal costs $MC = \partial c/\partial y \rightarrow$ Cost minimization
  \[ p = MC = \partial c(w, r, y)/\partial y \]

- Average costs $AC = c/y \rightarrow$ Does firm break even?
  \[ \pi = py - c(w, r, y) > 0 \text{ iff} \]
  \[ \pi/y = p - c(w, r, y)/y > 0 \text{ iff} \]
  \[ c(w, r, y)/y = AC < p \]

- Supply function. Portion of marginal cost $MC$
  above average costs. (price equals marginal cost)
• Assume only 1 input (expenditure minimization is trivial)

• **Case 1.** Production function. \( y = L^\alpha \)

  – Cost function? (cost of input is \( w \)):
    \[
    c(w, y) = wL^*(w, y) = wy^{1/\alpha}
    \]

  – Marginal cost?
    \[
    \frac{\partial c(w, y)}{\partial y} = \frac{1}{\alpha}wy^{(1-\alpha)/\alpha}
    \]

  – Average cost \( c(w, y)/y \)?
    \[
    \frac{c(w, y)}{y} = \frac{wy^{1/\alpha}}{y} = wy^{(1-\alpha)/\alpha}
    \]
• **Case 1a.** \( \alpha > 1 \). Plot production function, total cost, average and marginal. Supply function?

• **Case 1b.** \( \alpha = 1 \). Plot production function, total cost, average and marginal. Supply function?

• **Case 1c.** \( \alpha < 1 \). Plot production function, total cost, average and marginal. Supply function?
• **Case 2.** *Non-convex technology.* Plot production function, total cost, average and marginal. Supply function?

• **Case 3.** *Technology with setup cost.* Plot production function, total cost, average and marginal. Supply function?
3.1 Supply Function

• Supply function: \( y^* = y^* (w, r, p) \)

• What happens to \( y^* \) as \( p \) increases?

• Is the supply function upward sloping?

• Remember f.o.c:

\[
p - c_y'(w, r, y) = 0
\]

• Implicit function:

\[
\frac{\partial y^*}{\partial p} = -\frac{1}{-c_{y,y}''(w, r, y)} > 0
\]

as long as s.o.c. is satisfied.

• Yes! Supply function is upward sloping.
4 Next Lectures

• Profit Maximization

• Aggregation

• Market Equilibrium