Outline

1. Economics of Discrimination II

2. Optimization with 1 variable

3. Multivariate optimization

4. Comparative Statics

5. Implicit function theorem
1 Economics of Discrimination II

• Strong evidence of discrimination against African Americans

• Applied Microeconomics
  – Not (really) covered in this class: See Ec131 (Public), 142 (Applied Metrics) and (partly) 151 (Labor) + 154 (Discrimination)
  – URAP – Get involved in a professor’s research
  – If curious: read Steven Levitt and Stephen Dubner, *Freakonomics*

• At end of class, two more examples:
  – Deductibles and Home Insurance (Sydnor, 2007)
  – Fox News and Voting (DellaVigna and Kaplan, 2007)
2 Optimization with 1 variable

• Nicholson (9th), Ch.2, pp. 20-23 (20-24, 9th Ed)

• Example. Function $y = -x^2$

• What is the maximum?

• Maximum is at 0

• General method?
• Sure! Use derivatives

• Derivative is slope of the function at a point:

\[
\frac{\partial f(x)}{\partial x} = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}
\]

• Necessary condition for maximum \( x^* \) is

\[
\frac{\partial f(x^*)}{\partial x} = 0
\]  \hspace{1cm} (1)

• Try with \( y = -x^2 \).

\[
\frac{\partial f(x)}{\partial x} = \quad = 0 \implies x^* =
\]
• Does this guarantee a maximum? No!

• Consider the function $y = x^3$

  $$\frac{\partial f(x)}{\partial x} = 0 \Rightarrow x^* =$$

• Plot $y = x^3$. 
• Sufficient condition for a (local) maximum:

\[
\frac{\partial f(x^*)}{\partial x} = 0 \text{ and } \left. \frac{\partial^2 f(x)}{\partial^2 x} \right|_{x^*} < 0
\]  \hspace{1cm} (2)

• Proof: At a maximum, \( f(x^* + h) - f(x^*) < 0 \) for all \( h \).

• Taylor Rule: \( f(x^*+h)-f(x^*)=\frac{\partial f(x^*)}{\partial x}h+\frac{1}{2}\frac{\partial^2 f(x^*)}{\partial^2 x}h^2+\text{higher order terms.} \)

• Notice: \( \frac{\partial f(x^*)}{\partial x} = 0. \)

\[
f(x^* + h) - f(x^*) < 0 \text{ for all } h \implies \frac{\partial^2 f(x^*)}{\partial^2 x}h^2 < 0 \implies \frac{\partial^2 f(x^*)}{\partial^2 x} < 0
\]

• Careful: Maximum may not exist: \( y = \exp(x) \)
• Tricky examples:

  - Minimum. \( y = x^2 \)

  - No maximum. \( y = \exp(x) \) for \( x \in (-\infty, +\infty) \)

  - Corner solution. \( y = x \) for \( x \in [0, 1] \)
3 Multivariate optimization

- Nicholson, Ch.2, pp. 23-30 (24-32, 9th Ed)

- Function from $R^n$ to $R$: $y = f(x_1, x_2, ..., x_n)$

- Partial derivative with respect to $x_i$:

$$\frac{\partial f(x_1, ..., x_n)}{\partial x_i} = \lim_{h \to 0} \frac{f(x_1, ..., x_i + h, ..., x_n) - f(x_1, ..., x_i, ..., x_n)}{h}$$

- Slope along dimension $i$

- Total differential:

$$df = \frac{\partial f(x)}{\partial x_1}dx_1 + \frac{\partial f(x)}{\partial x_2}dx_2 + ... + \frac{\partial f(x)}{\partial x_n}dx_n$$
• One important economic example

• Example 1: Partial derivatives of \( y = f(L, K) = L^{0.5}K^{0.5} \)

• \( f'_L = \)  
  (marginal productivity of labor)

• \( f'_K = \)  
  (marginal productivity of capital)

• \( f''_{L,K} = \)
Maximization over an open set (like $R$)

- **Necessary condition for maximum** $x^*$ is
  \[
  \frac{\partial f(x^*)}{\partial x_i} = 0 \quad \forall i
  \]  
  or in vectorial form
  \[
  \nabla f(x) = 0
  \]

- These are commonly referred to as first order conditions (f.o.c.)

- **Sufficient conditions?** Define Hessian matrix $H$:

\[
H = \begin{pmatrix}
  f''_{x_1,x_1} & f''_{x_1,x_2} & \cdots & f''_{x_1,x_n} \\
  \vdots & \ddots & \cdots & \vdots \\
  f''_{x_n,x_1} & f''_{x_n,x_2} & \cdots & f''_{x_n,x_n}
\end{pmatrix}
\]
• Subdeterminant $|H|_i$ of Matrix $H$ is defined as the determinant of submatrix formed by first $i$ rows and first $i$ columns of matrix $H$.

• Examples.

  – $|H|_1$ is determinant of $f''_{x_1,x_1}$, that is, $f''_{x_1,x_1}$

  – $|H|_2$ is determinant of

    $$H = \begin{pmatrix} f''_{x_1,x_1} & f''_{x_1,x_2} \\ f''_{x_2,x_1} & f''_{x_2,x_2} \end{pmatrix}$$

• Sufficient condition for maximum $x^*$.

  1. $x^*$ must satisfy first order conditions;

  2. Subdeterminants of matrix $H$ must have alternating signs, with subdeterminant of $H_1$ negative.
• Case with \( n = 2 \)

• Condition 2 reduces to \( f''_{x_1,x_1} < 0 \) and \( f''_{x_1,x_1} f'''_{x_2,x_2} - (f''_{x_1,x_2})^2 > 0 \).

• Example 2: \( h(x_1, x_2) = p_1 x_1^2 + p_2 x_2^2 - 2x_1 - 5x_2 \)

• First order condition w/ respect to \( x_1 \)?

• First order condition w/ respect to \( x_2 \)?

• \( x_1^*, x_2^* = \)

• For which \( p_1, p_2 \) is it a maximum?

• For which \( p_1, p_2 \) is it a minimum?
4 Comparative statics

• Economics is all about ‘comparative statics’

• What happens to optimal economic choices if we change one parameter?

• Example: Car production. Consumer:
  1. Car purchase and increase in oil price
  2. Car purchase and increase in income

• Producer:
  1. Car production and minimum wage increase
  2. Car production and decrease in tariff on Japanese cars

• Next two sections
5 Implicit function theorem

- Implicit function: Ch. 2, pp. 31-32 (32–33, 9th Ed)

- Consider function \( x_2 = g(x_1, p) \)

- Can rewrite as \( x_2 - g(x_1, p) = 0 \)

- **Implicit function** has form: \( h(x_2, x_1, p) = 0 \)

- Often we need to go from implicit to explicit function

- Example 3: \( 1 - x_1 \ast x_2 - e^{x_2} = 0. \)

- Write \( x_1 \) as function of \( x_2 \):

- Write \( x_2 \) as function of \( x_1 \):
- **Univariate implicit function theorem (Dini):** Consider an equation $f(p, x) = 0$, and a point $(p_0, x_0)$ solution of the equation. Assume:

1. $f$ continuous and differentiable in a neighbourhood of $(p_0, x_0)$;
2. $f_x(p_0, x_0) \neq 0$.

Then:

1. There is one and only function $x = g(p)$ defined in a neighbourhood of $p_0$ that satisfies $f(p, g(p)) = 0$ and $g(p_0) = x_0$;
2. The derivative of $g(p)$ is

$$g'(p) = -\frac{f_p'(p, g(p))}{f_x'(p, g(p))}$$
• Example 3 (continued): $1 - x_1 \cdot x_2 - e^{x_2} = 0$

• Find derivative of $x_2 = g(x_1)$ implicitly defined for $(x_1, x_2) = (1, 0)$

• Assumptions:
  1. Satisfied?
  2. Satisfied?

• Compute derivative
• Multivariate implicit function theorem (Dini):

Consider a set of equations 
\( f_1(p_1, \ldots, p_n; x_1, \ldots, x_s) = 0; \ldots; f_s(p_1, \ldots, p_n; x_1, \ldots, x_s) = 0 \), and a point \((p_0, x_0)\) solution of the equation. Assume:

1. \( f_1, \ldots, f_s \) continuous and differentiable in a neighbourhood of \((p_0, x_0)\);

2. The following Jacobian matrix \( \frac{\partial f}{\partial x} \) evaluated at \((p_0, x_0)\) has determinant different from 0:

\[
\frac{\partial f}{\partial x} = \begin{pmatrix}
\frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_s} \\
\vdots & \vdots & \vdots \\
\frac{\partial f_s}{\partial x_1} & \ldots & \frac{\partial f_s}{\partial x_s}
\end{pmatrix}
\]
• Then:

1. There is one and only set of functions $x = g(p)$ defined in a neighbourhood of $p_0$ that satisfy $f(p, g(p)) = 0$ and $g(p_0) = x_0$;

2. The partial derivative of $x_i$ with respect to $p_k$ is

$$\frac{\partial g_i}{\partial p_k} = -\frac{\det \left( \frac{\partial (f_1, \ldots, f_s)}{\partial (x_1, \ldots, x_{i-1}, p_k, x_{i+1}, \ldots, x_s)} \right)}{\det \left( \frac{\partial f}{\partial x} \right)}$$
• Example 2 (continued): Max $h(x_1, x_2) = p_1 * x_1^2 + p_2 * x_2^2 - 2x_1 - 5x_2$

• f.o.c. $x_1 : 2p_1 * x_1 - 2 = 0 = f_1(p,x)$

• f.o.c. $x_2 : 2p_2 * x_2 - 5 = 0 = f_2(p,x)$

• Comparative statics of $x_1^*$ with respect to $p_1$?

• First compute $\det \left( \frac{\partial f}{\partial x} \right)$

\[
\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix} = \left( \begin{array}{c} \end{array} \right)
\]
• Then compute \( \det \left( \frac{\partial(f_1, \ldots, f_s)}{\partial(x_1, \ldots, x_i-1, p_k, x_{i+1}, \ldots, x_s)} \right) \)

\[
\begin{pmatrix}
\frac{\partial f_1}{\partial p_1} & \frac{\partial f_1}{\partial x_2} \\
\frac{\partial f_2}{\partial p_1} & \frac{\partial f_2}{\partial x_2}
\end{pmatrix} = \left( \begin{array}{c}
\vdots
\end{array} \right)
\]

• Finally, \( \frac{\partial x_1}{\partial p_1} = \)

• Why did you compute \( \det \left( \frac{\partial f}{\partial x} \right) \) already?
6 Next Class

• Next class:
  – Envelope Theorem
  – Convexity and Concavity
  – Constrained Maximization
  – Envelope Theorem II

• Going toward:
  – Preferences
  – Utility Maximization (where we get to apply maximization techniques the first time)