Economics 101A
(Lecture 3)

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Outline

1. Implicit Function Theorem II

2. Envelope Theorem

3. Convexity and concavity

4. Constrained Maximization

5. Envelope Theorem II
1 Implicit function theorem II

• **Univariate implicit function theorem (Dini):** Consider an equation \( f(p, x) = 0 \), and a point \((p_0, x_0)\) solution of the equation. Assume:

  1. \( f \) continuous and differentiable in a neighbourhood of \((p_0, x_0)\);

  2. \( f'_x(p_0, x_0) \neq 0 \).

• Then:

  1. There is one and only function \( x = g(p) \) defined in a neighbourhood of \( p_0 \) that satisfies \( f(p, g(p)) = 0 \) and \( g(p_0) = x_0 \);

  2. The derivative of \( g(p) \) is

     \[
     g'(p) = -\frac{f'_p(p, g(p))}{f'_x(p, g(p))}
     \]
Example 3 (continued): \(1 - x_1 \cdot x_2 - e^{x_2} = 0\)

Find derivative of \(x_2 = g(x_1)\) implicitly defined for \((x_1, x_2) = (1, 0)\)

Assumptions:

1. Satisfied?
2. Satisfied?

Compute derivative
• **Multivariate implicit function theorem (Dini):**

Consider a set of equations \( f_1(p_1, \ldots, p_n; x_1, \ldots, x_s) = 0; \ldots; f_s(p_1, \ldots, p_n; x_1, \ldots, x_s) = 0 \), and a point \((p_0, x_0)\) solution of the equation. Assume:

1. \( f_1, \ldots, f_s \) continuous and differentiable in a neighbourhood of \((p_0, x_0)\);

2. The following Jacobian matrix \( \frac{\partial f}{\partial x} \) evaluated at \((p_0, x_0)\) has determinant different from 0:

\[
\frac{\partial f}{\partial x} = \begin{pmatrix}
\frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_s} \\
\vdots & \ddots & \vdots \\
\frac{\partial f_s}{\partial x_1} & \ldots & \frac{\partial f_s}{\partial x_s}
\end{pmatrix}
\]
Then:

1. There is one and only set of functions $x = g(p)$ defined in a neighbourhood of $p_0$ that satisfy $f(p, g(p)) = 0$ and $g(p_0) = x_0$;

2. The partial derivative of $x_i$ with respect to $p_k$ is

$$\frac{\partial g_i}{\partial p_k} = -\frac{\det \left( \frac{\partial(f_1, \ldots, f_s)}{\partial(x_1, \ldots, x_{i-1}, p_k, x_{i+1}, \ldots, x_s)} \right)}{\det \left( \frac{\partial f}{\partial x} \right)}$$
• Example 2 (continued): Max $h(x_1, x_2) = p_1 * x_1^2 + p_2 * x_2^2 - 2x_1 - 5x_2$

• f.o.c. $x_1 : 2p_1 * x_1 - 2 = 0 = f_1(p,x)$

• f.o.c. $x_2 : 2p_2 * x_2 - 5 = 0 = f_2(p,x)$

• Comparative statics of $x_1^*$ with respect to $p_1$?

• First compute $\det \left( \frac{\partial f}{\partial x} \right)$

$$\left( \begin{array}{cc}
\frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\
\frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2}
\end{array} \right) = \left( \begin{array}{c}
\vdots
\end{array} \right)$$
Then compute \( \det \left( \frac{\partial (f_1, \ldots, f_s)}{\partial (x_1, \ldots, x_{i-1}, p_k, x_{i+1}, \ldots, x_s)} \right) \)

\[
\begin{pmatrix}
\frac{\partial f_1}{\partial p_1} & \frac{\partial f_1}{\partial x_2} \\
\frac{\partial f_2}{\partial p_1} & \frac{\partial f_2}{\partial x_2}
\end{pmatrix}
= \begin{pmatrix}
\vdots
\end{pmatrix}
\]

Finally, \( \frac{\partial x_1}{\partial p_1} = \)

Why did you compute \( \det \left( \frac{\partial f}{\partial x} \right) \) already?
2 Envelope Theorem

- Ch. 2, pp. 32-36 (33–37, 9th Ed)

- You now know how $x_1^*$ varies if $p_1$ varies.

- How does $h(x^*(p))$ vary as $p_1$ varies?

- Differentiate $h(x_1^*(p_1, p_2), x_2^*(p_1, p_2), p_1, p_2)$ with respect to $p_1$:

$$
\frac{dh(x_1^*(p_1, p_2), x_2^*(p_1, p_2), p_1, p_2)}{dp_1} = \frac{\partial h(x^*, p)}{\partial x_1} \cdot \frac{\partial x_1^*(x^*, p)}{\partial p_1} + \frac{\partial h(x^*, p)}{\partial x_2} \cdot \frac{\partial x_2^*(x^*, p)}{\partial p_1} + \frac{\partial h(x^*, p)}{\partial p_1}
$$

- The first two terms are zero.
• **Envelope Theorem** for unconstrained maximization. Assume that you maximize function $f(x; p)$ with respect to $x$. Consider then the function $f$ at the optimum, that is, $f(x^*(p), p)$. The total differential of this function with respect to $p_i$ equals the partial derivative with respect to $p_i$:

$$\frac{df(x^*(p), p)}{dp_i} = \frac{\partial f(x^*(p), p)}{\partial p_i}.$$

• You can disregard the indirect effects. Graphical intuition.
3 Convexity and concavity

• Function $f$ from $C \subset R^n$ to $R$ is concave if
  \[ f(tx + (1 - t)y) \geq tf(x) + (1 - t)f(y) \]
  for all $x, y \in C$ and for all $t \in [0, 1]$

• Notice: $C$ must be convex set, i.e., if $x \in C$ and
  $y \in C$, then $tx + (1 - t)y \in C$, for $t \in [0, 1]$

• Function $f$ from $C \subset R^n$ to $R$ is strictly concave if
  \[ f(tx + (1 - t)y) > tf(x) + (1 - t)f(y) \]
  for all $x, y \in C$ and for all $t \in (0, 1)$

• Function $f$ from $R^n$ to $R$ is convex if $-f$ is concave.
• Alternative characterization of convexity.

• A function \( f \), twice differentiable, is concave if and only if for all \( x \) the subdeterminants \( |H_i| \) of the Hessian matrix have the property \( |H_1| \leq 0, |H_2| \geq 0, |H_3| \leq 0 \), and so on.

• For the univariate case, this reduces to \( f''' \leq 0 \)

• For the bivariate case, this reduces to \( f''_{x,x} \leq 0 \) and \( f''_{x,x} \times f''_{y,y} - (f'''_{x,y})^2 \geq 0 \)

• A twice-differentiable function is strictly concave if the same property holds with strict inequalities.
• Examples.

1. For which values of \(a, b,\) and \(c\) is \(f(x) = ax^3 + bx^2 + cx + d\) is the function concave over \(R\)? Strictly concave? Convex?

2. Is \(f(x, y) = -x^2 - y^2\) concave?

• For Example 2, compute the Hessian matrix

\[
\begin{align*}
\frac{\partial^2 f}{\partial x^2} &= f_{xx} = \ , f_{xy} = \\
\frac{\partial^2 f}{\partial y^2} &= f_{yy} = \ , f_{yx} = \\
\frac{\partial^2 f}{\partial x \partial y} &= f_{xy} = \ , f_{yx} = \\
\text{Hessian matrix } H : \\
H &= \begin{pmatrix}
\frac{\partial^2 f}{\partial x^2} = f_{xx} & f_{xy} \\
\frac{\partial^2 f}{\partial y^2} = f_{yy} & f_{yy}
\end{pmatrix}
\end{align*}
\]

• Compute \(|H_1| = f_{xx}^\prime\) and \(|H_2| = f_{xx}^\prime \cdot f_{yy}^\prime - (f_{xx}^\prime)^2\)
• Why are convexity and concavity important?

• Theorem. Consider a twice-differentiable concave (convex) function over $C \subset R^n$. If the point $x_0$ satisfies the first order conditions, it is a global maximum (minimum).

• For the proof, we need to check that the second-order conditions are satisfied.

• These conditions are satisfied by definition of concavity!

• (We have only proved that it is a local maximum)
4 Constrained maximization

- Ch. 2, pp. 36-42 (38–44, 9th Ed)

- So far unconstrained maximization on \( R \) (or open subsets)

- What if there are constraints to be satisfied?

- Example 1: \( \max_{x,y} x \cdot y \) subject to \( 3x + y = 5 \)

- Substitute it in: \( \max_{x,y} x \cdot (5 - 3x) \)

- Solution: \( x^* = \)

- Example 2: \( \max_{x,y} xy \) subject to \( x \exp(y) + y \exp(x) = 5 \)

- Solution: ?
• Graphical intuition on general solution.

• Example 3: \( \max_{x,y} f(x, y) = xy \) s.t. \( h(x, y) = x^2 + y^2 - 1 = 0 \)

• Draw \( 0 = h(x, y) = x^2 + y^2 - 1 \).

• Draw \( xy = K \) with \( K > 0 \). Vary \( K \)

• Where is optimum?

• Where \( \frac{dy}{dx} \) along curve \( xy = K \) equals \( \frac{dy}{dx} \) along curve \( x^2 + y^2 - 1 = 0 \)

• Write down these slopes.
• Idea: Use implicit function theorem.

• Heuristic solution of system

\[
\max_{x,y} f(x, y) \\
\text{s.t. } h(x, y) = 0
\]

• Assume:
  
  – continuity and differentiability of \( h \)
  
  – \( h'_y \neq 0 \) (or \( h'_x \neq 0 \))

• Implicit function Theorem: Express \( y \) as a function of \( x \) (or \( x \) as function of \( y \))!
• Write system as $\max_x f(x, g(x))$

• f.o.c.: $f'_x(x, g(x)) + f'_y(x, g(x)) \cdot \frac{\partial g(x)}{\partial x} = 0$

• What is $\frac{\partial g(x)}{\partial x}$?

• Substitute in and get: $f'_x(x, g(x)) + f'_y(x, g(x)) \cdot \left(-\frac{h'_x}{h'_y}\right) = 0$ or

\[
\frac{f'_x(x, g(x))}{f'_y(x, g(x))} = \frac{h'_x(x, g(x))}{h'_y(x, g(x))}
\]
• **Lagrange Multiplier Theorem, necessary condition.** Consider a problem of the type

\[
\max_{x_1, \ldots, x_n} f(x_1, x_2, \ldots, x_n; p)
\]

s.t.

\[
\begin{align*}
  h_1(x_1, x_2, \ldots, x_n; p) &= 0 \\
  h_2(x_1, x_2, \ldots, x_n; p) &= 0 \\
  &\ldots \\
  h_m(x_1, x_2, \ldots, x_n; p) &= 0
\end{align*}
\]

with \( n > m \). Let \( x^* = x^*(p) \) be a local solution to this problem.

• Assume:

  – \( f \) and \( h \) differentiable at \( x^* \)

  – the following Jacobian matrix at \( x^* \) has maximal rank

\[
J = \begin{pmatrix}
  \frac{\partial h_1}{\partial x_1}(x^*) & \ldots & \frac{\partial h_1}{\partial x_n}(x^*) \\
  \ldots & \ldots & \ldots \\
  \frac{\partial h_m}{\partial x_1}(x^*) & \ldots & \frac{\partial h_m}{\partial x_n}(x^*)
\end{pmatrix}
\]
• Then, there exists a vector \( \lambda = (\lambda_1, ..., \lambda_m) \) such that \((x^*, \lambda)\) maximize the Lagrangean function

\[
L(x, \lambda) = f(x; p) - \sum_{j=0}^{m} \lambda_j h_j(x; p)
\]

• Case \( n = 2, m = 1 \).

• First order conditions are

\[
\frac{\partial f(x; p)}{\partial x_i} - \lambda \frac{\partial h(x; p)}{\partial x_i} = 0
\]

for \( i = 1, 2 \)

• Rewrite as

\[
\frac{f'_{x_1}}{f'_{x_2}} = \frac{h'_{x_1}}{h'_{x_2}}
\]
Constrained Maximization, Sufficient condition for the case $n = 2, m = 1$.

- If $x^*$ satisfies the Lagrangean condition, and the determinant of the bordered Hessian

$$H = \begin{pmatrix}
0 & -\frac{\partial h}{\partial x_1}(x^*) & -\frac{\partial h}{\partial x_2}(x^*) \\
-\frac{\partial h}{\partial x_1}(x^*) & \frac{\partial^2 L}{\partial x_1\partial x_1}(x^*) & \frac{\partial^2 L}{\partial x_2\partial x_1}(x^*) \\
-\frac{\partial h}{\partial x_2}(x^*) & \frac{\partial^2 L}{\partial x_1\partial x_2}(x^*) & \frac{\partial^2 L}{\partial x_2\partial x_2}(x^*)
\end{pmatrix}$$

is positive, then $x^*$ is a constrained maximum.

- If it is negative, then $x^*$ is a constrained minimum.

- Why? This is just the Hessian of the Lagrangean $L$ with respect to $\lambda, x_1, \text{and } x_2$.
Example 4: \( \max_{x,y} x^2 - xy + y^2 \) s.t. \( x^2 + y^2 - p = 0 \)

\[ \max_{x,y,\lambda} x^2 - xy + y^2 - \lambda(x^2 + y^2 - p) \]

- F.o.c. with respect to \( x \):
- F.o.c. with respect to \( y \):
- F.o.c. with respect to \( \lambda \):

Candidates to solution?

Maxima and minima?
5 Envelope Theorem II

- Envelope Theorem II: Ch. 2, pp. 42-43 (44, 9th Ed)

- **Envelope Theorem for Constrained Maximization.** In problem above consider $F(p) \equiv f(x^*(p); p)$. We are interested in $dF(p)/dp$. We can neglect indirect effects:

$$\frac{dF}{dp_i} = \frac{\partial f(x^*(p); p)}{\partial p_i} - \sum_{j=0}^{m} \lambda_j \frac{\partial h_j(x^*(p); p)}{\partial p_i}$$

- Example 4 (continued). $\max_{x,y} x^2 - xy + y^2$ s.t. $x^2 + y^2 - p = 0$

- $df(x^*(p), y^*(p))/dp$?

- Envelope Theorem.
6 Next Class

- Next class:
  - Preferences
  - Utility Maximization (where we get to apply maximization techniques the first time)