1 Consumption and Risky Assets

Consumer’s lifetime utility:

\[ U = u(c_1) + E[u(c_2)] \]

Income: \( Y_1 = \bar{Y} \) certain and \( Y_2 \sim F(\bar{Y}, \sigma^2_Y) \) is random variable. Initial wealth \( A_0 \) is zero.

a. With no financial assets and a storing technology available, the problem is given by:

\[ \begin{align*}
U &= u(c_1) + E[u(c_2)] \\
c_1 + s &= \bar{Y} \\
c_2 &= Y_2 + s
\end{align*} \]

FOC wrt \( s \) (by choosing how much to save the consumer pins down \( c_1, c_2 \)):

\[ u'(c_1) = E[u'(c_2)] \]

Now, we add a risky asset to this economy, with return \( r \sim (\bar{r}, \sigma^2_r) \), with \( E(rY_2) = 0 \).

b. The consumer will buy a strictly positive amount iff \( \bar{r} > 0 \), i.e., if the expected return on this investment exceeds the risk free rate (in our case zero), it will be optimal for the consumer to invest at least an \( \varepsilon > 0 \) amount of its income in the risky asset. This is true because for a small \( \varepsilon \), consumption is period 2 is "almost" uncorrelated with the return on this asset (since \( C_2 = Y_2 + \varepsilon (1 + r) \)), so the covariance is zero and it has positive expected return \( \Rightarrow \) the consumer should invest some positive amount. Also, notice that for very small amounts of risk, the utility function can be approximated by a linear function (think of a taylor expansion around \( Y_2 \)), i.e. agents are 'almost' risk neutral when it comes to small amounts of risk. This are both reasons why the agent always takes some risk (as long as the risk is small, the linear approximation works).

c. This statement is not true because the consumer is risk averse \( u'' < 0 \), and thus won’t be willing to leverage up to infinity to go long on the risky asset. There is an optimal level (interior solution) that will determine the optimal amount to be invested in the risky asset. The difference with out answer to part b. is that for not small amounts of investment in the risky asset, consumption and the return on the asset become highly correlated, and thus the asset becomes less and less attractive since remember the agent ultimately cares about correlation between returns and marginal utility of consumption!
d. Consumer now has to choose how much to invest in the risk free and how much to invest in risky asset. Let \( \alpha \) be the fraction of income invested in risky asset and \( \beta \) the fraction invested in risk free. The consumer needs to choose \( \alpha \) and \( \beta \) to max expected utility:

\[
\max_{\{\alpha, \beta\}} u(c_1) + E[u(c_2)]
\]

\[
c_1 = \bar{Y}(1 - \alpha - \beta)
\]

\[
c_2 = Y_2 + [(1 + r)\alpha + \beta] \bar{Y}
\]

FOC

\[
(\alpha) \quad - u'(c_1) \bar{Y} + E[u'(c_2)(1 + r)] \bar{Y} = 0
\]

\[
(\beta) \quad - u'(c_1) \bar{Y} + E[u'(c_2)] \bar{Y} = 0
\]

Can be re-written as follows:

\[
u'(c_1) = E[u'(c_2)(1 + r)] \quad (1)
\]

\[
u'(c_1) = E[u'(c_2)] \quad (2)
\]

Where (1) is the Euler equation for the risky asset and (2) the Euler equation for the risk-free asset (equal to the ne derived in part a.)

2 The Risk Free Rate Puzzle

We solved this in section a long time ago! The quickest way is to use the result:

\[
\frac{\dot{C}(t)}{C(t)} = \frac{r(t) - \rho}{\frac{C(t)U''(C(t))}{U'(C(t))}}
\]

where \( CRRA = -\frac{C(t)U''(C(t))}{U'(C(t))} \) is the coefficient of relative risk aversion. By pugging in the calibration for \( CRRA, \rho, \) and consumption growth, we obtain

\[
1.5 = \frac{r - 5}{4} \implies r = 11
\]

Really, really high!
3 Romer, Problem 8.10 (in the end)

4 Asset Pricing I

Our starting point is:

\[ P_t = E_t \left[ \sum_{k=1}^{\infty} \left( \frac{1}{1 + \rho} \right)^k \frac{U''(C_{t+k})}{U'(C_t)} Z_{t+k} \right] \]

which implies:

\[ P_{t+1} = E_{t+1} \left[ \sum_{k=2}^{\infty} \left( \frac{1}{1 + \rho} \right)^{k-1} \frac{U''(C_{t+k})}{U'(C_{t+1})} Z_{t+k} \right] \]

Going back to our initial equation:

\[ P_t = E_t \left[ \sum_{k=1}^{\infty} \left( \frac{1}{1 + \rho} \right)^k \frac{U''(C_{t+k})}{U'(C_t)} Z_{t+k} \right] \]

\[ P_t = E_t \left[ \left( \frac{1}{1 + \rho} \right)^{k} \frac{U''(C_{t+1})}{U'(C_t)} Z_{t+1} + \sum_{k=2}^{\infty} \left( \frac{1}{1 + \rho} \right)^{k-1} \frac{U''(C_{t+k})}{U'(C_{t+1})} Z_{t+k} \right] \]

\[ P_t = E_t \left[ \left( \frac{1}{1 + \rho} \right)^{k} \frac{U''(C_{t+1})}{U'(C_t)} Z_{t+1} + \frac{U''(C_{t+1})}{U'(C_t)} P_{t+1} \right] = E_t \left[ \left( \frac{1}{1 + \rho} \right)^{k} \frac{U''(C_{t+1})}{U'(C_t)} (Z_{t+1} + P_{t+1}) \right] \]

EXTRA PROBLEMS

5 Romer, Problem 9.14 (in the end)

6 Asset Pricing II

If the expected return on Asset A exceeds the expected return on Asset B:

A. Financial markets cannot be in equilibrium.

B. Financial markets can be in equilibrium, but only if there are noise traders.

C. A rational investor would choose to hold Asset B only if the covariance of Asset B’s return with the marginal utility of consumption is greater than the covariance of Asset A’s return with the marginal utility of consumption.

D. A rational investor would choose to hold Asset B only if the covariance of Asset B’s return with the marginal utility of consumption is less than the covariance of Asset A’s return with the marginal utility of consumption.
To see this, check out the Euler equation for risky assets. You should have two Euler equations, one for asset A and one for asset B. Compare them to get the answer.

7 Asymmetric Information and Investment

Consider the model of investment under asymmetric information in Section 8.9 of Romer. Suppose that initially the entrepreneur is undertaking the project, and that \((1 + r)(1 - W)\) is strictly less than \(R^{MAX}\). Describe how each of the following affects \(D\):

a. A small increase in \(W\): An increase in \(W\) implies an increase in the downpayment done by the entrepreneur, and a decrease in the loan made by the lender. This decreases the probability of being in a state that has to be monitored, and thus should decrease \(D\).

b. A small increase in \(r\): An increase in \(r\) implies an increase in the outside option lenders have, and thus \(D\) would have to be higher to satisfy the participation constraint of the lenders (to continue to leave the lender indifferent between investing in the risky project of saving at the risk free rate).

c. A small increase in \(c\): An increase in \(c\) has two effects. First, it decreases the incentives to monitor, since it is now more expensive to do so, and so this puts pressure on \(D\) to decrease. Second, by increasing the monitoring costs, it decreases the expected payoff of the lenders, and thus \(D\) would have to increase to satisfy the participation constraint. I don’t think we can tell what happens to \(D\) (or at least it is not obvious to me without doing the math).

d. Instead of being distributed uniformly on \([0, 2\gamma]\), the output of the project is distributed uniformly on \([\gamma - b, \gamma + b]\), and there is a small increase in \(b\). In this case, the increase in \(b\) is an increase in the dispersion of the returns. Since for the lender this only means an increase in the dispersion of the bad payments and not the good ones, \(D\) has to increase to compensate him.

e. Instead of being distributed uniformly on \([0, 2\gamma]\), the output of the project is distributed uniformly on \([b, 2\gamma + b]\), and there is a small increase in \(b\). An increase in \(b\) here should be interpreted as an upward shift in the support of returns, which implies an upward shift in the support of payments to the bank for the non-monitoring zone. To satisfy the participation constraint with equality (do not leave any surplus to the lender), \(D\) would have to decrease.
Problem 8.10

(a) Suppose the individual reduces her consumption by a small (formally infinitesimal) amount $dC$ in period $t$. The utility cost of doing this equals the marginal utility of consumption in period $t$, $1/C_t$, times $dC$. Thus we have

$$\text{utility cost} = \frac{dC}{C_t}.$$

This reduction in consumption allows the individual to purchase $dC/P_t$ trees in period $t$. In period $t+1$, the individual receives the extra output from her additional holdings of trees. She gets to consume an extra $[dC/P_t]Y_{t+1}$. The individual then sells her additional holdings of trees for $[dC/P_t]P_{t+1}$ and consumes the proceeds. Thus her total extra consumption in period $t+1$ is given by $[dC/P_t]Y_{t+1} + [dC/P_t]P_{t+1}$. The marginal utility of consumption in period $t+1$ is $1/C_{t+1}$. Thus the expected discounted utility benefit from this action is

$$\text{(2) expected utility benefit} = E_t \left[ \frac{1}{1 + \rho} \left( \frac{dC}{P_t} Y_{t+1} + \frac{dC}{P_t} P_{t+1} \right) \right].$$

If the individual is optimizing, a marginal change of this type must leave expected utility unchanged. This means that the utility cost must equal the expected utility benefit, or

$$\text{(3) } \frac{dC}{C_t} = E_t \left[ \frac{1}{1 + \rho} \left( \frac{1}{C_{t+1}} \left( Y_{t+1} + P_{t+1} \right) \right) \right].$$

Canceling the $dC$'s (which is somewhat informal) gives us

$$\text{(4) } \frac{1}{C_t} = E_t \left[ \frac{1}{1 + \rho} \left( \frac{1}{C_{t+1}} \frac{1}{P_t} \left( Y_{t+1} + P_{t+1} \right) \right) \right].$$

We can now solve equation (4) for $P_t$ in terms of $Y_t$ and expectations involving $Y_{t+1}$, $P_{t+1}$ and $C_{t+1}$. Note that we can replace $C_t$ with $Y_t$ and that $P_t$ is not uncertain at time $t$. Using these facts, equation (4) can be rewritten as

$$\text{(5) } \frac{1}{Y_t} = \frac{1}{P_t} E_t \left[ \frac{1}{1 + \rho} \left( \frac{1}{C_{t+1}} \left( Y_{t+1} + P_{t+1} \right) \right) \right].$$

Solving equation (5) for the price of a tree in period $t$ gives us

$$\text{(6) } P_t = \frac{Y_t}{1 + \rho} E_t \left[ \frac{Y_{t+1} + P_{t+1}}{C_{t+1}} \right].$$

(b) Since $C_{t+s} = Y_{t+s}$ for all $s \geq 0$, equation (6) can be written as

$$\text{(7) } P_t = \frac{Y_t}{1 + \rho} E_t \left[ \frac{Y_{t+1} + P_{t+1}}{Y_{t+1}} \right].$$

Equation (7) holds for all periods and so we can write the price of a tree in period $t+1$ as

$$\text{(8) } P_{t+1} = \frac{Y_{t+1}}{1 + \rho} E_{t+1} \left[ \frac{P_{t+2}}{Y_{t+2}} \right].$$

Substituting equation (8) into equation (7) yields

$$\text{(9) } P_t = \frac{Y_t}{1 + \rho} + \frac{Y_t}{1 + \rho} E_t \left[ \frac{1}{1 + \rho} + \frac{1}{1 + \rho} E_{t+1} \left( \frac{P_{t+2}}{Y_{t+2}} \right) \right].$$

Now use the law of iterated projections that states that for any variable $x$, $E_t E_{t+1} x_{t+2} = E_t x_{t+2}$, to obtain
After repeated substitutions, we will have
\begin{equation}
(11) \quad P_t = Y_t \left[ \frac{1}{1 + \rho} + \frac{1}{(1 + \rho)^2} + K \right].
\end{equation}

Imposing the no-bubbles condition that \( \lim_{s \to \infty} E_t \left[ \frac{(P_{t+s}/Y_{t+s})}{(1 + \rho)^s} \right] = 0 \), the price of a tree in period \( t \) can be written as
\begin{equation}
(12) \quad P_t = Y_t \left[ \frac{1}{1 + \rho} + \frac{1}{(1 + \rho)^2} + K \right].
\end{equation}

Since \( 1/(1 + \rho) < 1 \), the sum converges and we can write
\begin{equation}
(13) \quad P_t = Y_t \left[ \frac{1/(1 + \rho)}{1 - 1/(1 + \rho)} \right] = Y_t \left[ \frac{1/(1 + \rho)}{\rho/(1 + \rho)} \right].
\end{equation}

Thus, finally, the price of a tree in period \( t \) is
\begin{equation}
(14) \quad P_t = Y_t / \rho.
\end{equation}

(c) There are two effects of an increase in the expected value of dividends at some future date. First, at a given marginal utility of consumption, the higher expected dividends increase the attractiveness of owning trees. This tends to raise the current price of a tree. Since consumption equals dividends in this model, however, higher expected dividends in that future period mean higher consumption and lower marginal utility of consumption in that future period. This tends to reduce the attractiveness of owning trees – the tree is going to pay off more in a time when marginal utility is expected to be low – and thus tends to lower the current price of a tree. In the case of logarithmic utility, these two forces exactly offset each other, leaving the current price of a tree unchanged in the face of a rise in expected future dividends.

(d) The path of consumption is equivalent to the path of output. Thus if output follows a random walk, so does consumption. But if output does not follow a random walk, then consumption does not either.

**Problem 9.14**

(a) Consider the value of a unit of debt. It pays off one unit of output at time \( t + \tau \), for all \( \tau \geq 0 \). The consumer values this payoff according to the marginal utility of consumption at each time \( t + \tau \). Thus the value of having one unit of output at time \( t + \tau \) rather than at \( t \) is equal to the discounted marginal utility of consumption at time \( t + \tau \) relative to the marginal utility of consumption at time \( t \), which is given by \( e^{-\rho \tau} \frac{u'(C(t + \tau))}{u'(C(t))} \). Thus the value of a unit of debt at time \( t \) is simply the appropriately discounted "sum" of all the future payoffs, or
\begin{equation}
(1) \quad P(t) = \int_{\tau=0}^{\infty} e^{-\rho \tau} E_t \left[ \frac{u'(C(t + \tau))}{u'(C(t))} \right] d\tau.
\end{equation}

Equity holders are the residual claimant and thus at time \( t + \tau, \tau \geq 0 \), they receive the additional profit generated by the marginal unit of capital, \( \pi(K(t + \tau)) \), minus the total amount paid to bond holders, which is \( b \) (the total number of outstanding bonds). Again, individuals value this payoff at time \( t + \tau \) according to the discounted marginal utility of consumption at time \( t + \tau \) relative to the marginal utility of consumption at time \( t \). Thus the value of the equity in the marginal unit of capital is
\begin{equation}
(2) \quad V(t) = \int_{\tau=0}^{\infty} e^{-\rho \tau} E_t \left[ \frac{u'(C(t + \tau))}{u'(C(t))} \left( \pi(K(t + \tau)) - b \right) \right] d\tau.
\end{equation}
(b) Adding equation (2) to b times equation (1) gives us the following market value of the claim on the marginal unit of capital:

\[
P(t)b + V(t) = \int_{\tau=0}^{\infty} e^{-\rho\tau} b E_t \left[ \frac{u'(C(t+\tau))}{u'(C(t))} \right] d\tau + \int_{\tau=0}^{\infty} e^{-\rho\tau} E_t \left[ \frac{u'(C(t+\tau))}{u'(C(t))} \{ \pi(K(t+\tau)) - b \} \right] d\tau.
\]

Combining the integrals yields

\[
P(t)b + V(t) = \int_{\tau=0}^{\infty} e^{-\rho\tau} E_t \left[ \frac{u'(C(t+\tau))}{u'(C(t))} \{ \pi(K(t+\tau)) - b \} \right] d\tau,
\]

and thus

\[
P(t)b + V(t) = \int_{\tau=0}^{\infty} e^{-\rho\tau} E_t \left[ \frac{u'(C(t+\tau))}{u'(C(t))} \pi(K(t+\tau)) \right] d\tau.
\]

The division of financing between bonds and equity as captured by \( b \), the number of outstanding bonds, does not affect the size of \( \pi(K(t+\tau)) \). Since the division of \( \pi(K(t+\tau)) \) between bonds and equity does not affect the market value of the claims on the marginal unit of capital. The present discounted value of that unit of capital is determined by its expected effect on the path of profits. Since the division of \( \pi(K(t+\tau)) \) between bonds and equity does not affect the size of \( \pi(K(t+\tau)) \), it does not affect the market value of the claim on the unit of capital.

(c) The market value of each of the \( n \) assets is given by

\[
V_i(t) = \int_{\tau=0}^{\infty} e^{-\rho\tau} E_t \left[ \frac{u'(C(t+\tau))}{u'(C(t))} d_i(t+\tau) \right] d\tau.
\]

There will be \( n \) equations of the form of (6). Adding these \( n \) equations together gives us the following total value of the \( n \) financial instruments:

\[
V_1(t)+K+V_n(t) = \int_{\tau=0}^{\infty} e^{-\rho\tau} E_t \left[ \frac{u'(C(t+\tau))}{u'(C(t))} \{ d_1(t+\tau)+K+d_n(t+\tau) \} \right] d\tau.
\]

Since \( d_1(t+\tau) + \ldots + d_n(t+\tau) = \pi(K(t+\tau)) \), we can rewrite equation (7) as

\[
V_1(t)+K+V_n(t) = \int_{\tau=0}^{\infty} e^{-\rho\tau} E_t \left[ \frac{u'(C(t+\tau))}{u'(C(t))} \pi(K(t+\tau)) \right] d\tau.
\]

The total market value of the \( n \) financial instruments is determined by the expected effect on the path of profits of the marginal unit of capital. It does not depend upon the individual payoffs to the assets.

(d) The value of a unit of debt continues to be given by equation (1). The value of a unit of equity is now

\[
V(t) = \int_{\tau=0}^{\infty} e^{-\rho\tau} E_t \left[ \frac{u'(C(t+\tau))}{u'(C(t))} \{ (1-0)\pi(K(t+\tau)) - b \} \right] d\tau.
\]

Adding equation (9) to \( b \) times equation (1) gives the following market value of the claims on the marginal unit of capital:

\[
P(t)b + V(t) = \int_{\tau=0}^{\infty} e^{-\rho\tau} b E_t \left[ \frac{u'(C(t+\tau))}{u'(C(t))} \right] d\tau + \int_{\tau=0}^{\infty} e^{-\rho\tau} E_t \left[ \frac{u'(C(t+\tau))}{u'(C(t))} \{ (1-0)\pi(K(t+\tau)) - b \} \right] d\tau.
\]

Combining the integrals yields

\[
P(t)b + V(t) = \int_{\tau=0}^{\infty} e^{-\rho\tau} E_t \left[ \frac{u'(C(t+\tau))}{u'(C(t))} \{ (1-0)\pi(K(t+\tau)) + \theta b \} \right] d\tau.
\]

Now the division of the financing between bonds and equity does matter. The number of bonds issued, \( b \), does affect the market value of the claim on the marginal unit of capital. The division of the additional profits between bonds and capital does affect the size of those profits. Specifically, a switch toward debt financing increases profits since interest payments are tax deductible.