Economics 131
Section Notes
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Constrained Optimization, Shadow Prices, Inefficient Markets, and Government Projects

1 Constrained Optimization

1.1 Unconstrained Optimization

Consider the case with two variable x and y, where $x, y \in R$, i.e. x, y can take on any real values. We wish to maximize the **objective function** $f(x, y)$ and there are no constraints. Thus we solve

$$\max_{x,y} f(x, y)$$

Assuming $f(x, y)$ has a maximum (for example f does not go to ∞)[1] and that it is differentiable everywhere, then any maximum (x^*, y^*) must solve the following **first order necessary conditions** (FOC)

$$\frac{\partial f}{\partial x}(x^*, y^*) = 0 \qquad \text{(Unconstrained FOC)}$$
$$\frac{\partial f}{\partial y}(x^*, y^*) = 0$$

This is a system of two equations in two unknowns, and thus should usually be solvable for both x^* and y^*. I won't look at second order conditions here.

1.2 Equality Constraints

Say there is a **constraint** on x and y which we can write as $g(x, y) = c$. There are two ways of solving the ensuing maximization problem. The first via **substitution**, the second via a **Lagrangean**

1.2.1 Substitution

One way of dealing with this is to try and use this equation to solve for one variable in terms of the other, i.e. we find another function $y = h(x)$, such that $g(x, h(x)) = c$. This can then be substituted into the objective function to get a maximization in a single variable

$$\max_{x} f(x, h(x))$$

which yields the FOC

$$\frac{\partial f}{\partial x}(x^*, h(x^*)) + \frac{\partial f}{\partial y}(x^*, h(x^*)) \frac{dh(x^*)}{dx} = 0 \qquad \text{(Constrained FOC)}$$

We could use this one equation to solve for x^*, and then find $y^* = h(x^*)$.

Differentiating the constraint with respect to x can let us solve for the slope of $h(x)$

$$\frac{\partial g(x, h(x))}{\partial x} + \frac{\partial g(x, h(x))}{\partial y} \frac{dh(x)}{dx} = 0 \Rightarrow \frac{dh(x)}{dx} = -\frac{\frac{\partial g(x,h(x))}{\partial x}}{\frac{\partial g(x,h(x))}{\partial y}}$$

[1] **Weierstrauss's Theorem** say shat $f(x, y)$ will be guaranteed to have a maximum if f is continuous and x and y are defined over a closed and bounded domain. See an analysis book for more.

Plugging this into the FOC and rearranging means that at the optimum

$$\frac{\frac{\partial f(x^*,h(x^*))}{\partial x}}{\frac{\partial f(x^*,h(x^*))}{\partial y}} = \frac{\frac{\partial g(x^*,h(x^*))}{\partial x}}{\frac{\partial g(x^*,h(x^*))}{\partial y}} \qquad \text{(Tangency)}$$

which is a general tangency condition, which we see many versions of throughout economics. It says that the constraint g and the level curve of f at $f^* = f(x^*, y^*)$ should be tangent at the maximum. A level curve of f at z is just all the points x, y such that $f(x, y) = z$ (like with indifference curves)

1.2.2 Lagrangean

An equivalent way of solving this problem which at first may seem more difficult, but which in fact can be very useful (and sometimes easier) is to maximize a **Lagrangean**

$$\mathfrak{L}(x, y, \lambda) = f(x, y) + \lambda [c - g(x, y)]$$

where λ is a **Lagrange multiplier**, which we maximize just as we would an unconstrained problem, taking into account the extra variable λ. The FOC are then

$$\frac{\mathfrak{L}(x^*, y^*, \lambda^*)}{\partial x} = \frac{\partial f}{\partial x}(x^*, y^*) - \lambda^* \frac{\partial g}{\partial x}(x^*, y^*) = 0 \qquad \text{(i)}$$

$$\frac{\mathfrak{L}(x^*, y^*, \lambda^*)}{\partial y} = \frac{\partial f}{\partial y}(x^*, y^*) - \lambda^* \frac{\partial g}{\partial y}(x^*, y^*) = 0 \qquad \text{(ii)}$$

$$\frac{\mathfrak{L}(x^*, y^*, \lambda^*)}{\partial \lambda} = c - g(x^*, y^*) = 0 \qquad \text{(iii)}$$

This is a system of three equations with three unknowns, and therefore should be relatively straightforward to solve. Notice that third FOC is just the constraint restated. Solve (i) and (ii) for λ and you will get

$$\lambda^* = \frac{\frac{\partial f}{\partial x}(x^*, y^*)}{\frac{\partial g}{\partial x}(x^*, y^*)} = \frac{\frac{\partial f}{\partial y}(x^*, y^*)}{\frac{\partial g}{\partial y}(x^*, y^*)} \qquad \text{(Lambda)}$$

The second equation eliminates λ for the purposes of problem solving, which combined with the constraint (iii) gives a 2 equation system in two unknowns (x^*, y^*). λ^* can then be solved for by plugging back into (Lambda). Notice that the second part of (Lambda) can be rearranged to produce the same result as (Tangency), making the equivalence of the two approaches obvious.

1.2.3 Interpreting the Lagrange Multiplier

So why all the fuss about the Lagrange multiplier? One reason why is that the Lagrange multiplier has an interesting interpretation. If we were to increase c by one unit how much higher would f go? The answer is λ^*.

More formally, take c to be a parameter (like prices or income) that is fixed over the optimization. Then really all of our solutions x^*, y^*, λ^* are dependent on c. We can then write $x^*(c), y^*(c), \lambda^*(c)$, as the parameter c changes, these solutions will change. Now plug $x^*(c), y^*(c)$ into f and we will get a new function $F(c) = f(x^*(c), y^*(c))$ which gives us the maximum value of f we can get for a given c. Now take the derivative of F

$$\begin{aligned}
\frac{dF(c)}{dc} &= \frac{\partial f(x^*(c), y^*(c))}{\partial x} \frac{dx^*(c)}{dc} + \frac{\partial f(x^*(c), y^*(c))}{\partial y} \frac{dy^*(c)}{dc} \\
&= \lambda^*(c) \frac{\partial g(x^*(c), y^*(c))}{\partial x} \frac{dx^*(c)}{dc} + \lambda^*(c) \frac{\partial g(x^*(c), y^*(c))}{\partial y} \frac{dy^*(c)}{dc} \\
&= \lambda^*(c) \left[\frac{\partial g(x^*(c), y^*(c))}{\partial x} \frac{dx^*(c)}{dc} + \frac{\partial g(x^*(c), y^*(c))}{\partial y} \frac{dy^*(c)}{dc} \right]
\end{aligned}$$

where the second equality comes from using FOC (i) and (ii). The constraint implies that $g(x^*(c), y^*(c)) = c$. Differentiating this gives us that

$$\frac{\partial g(x^*(c), y^*(c))}{\partial x}\frac{dx^*(c)}{dc} + \frac{\partial g(x^*(c), y^*(c))}{\partial y}\frac{dy^*(c)}{dc} = 1$$

so the entire term in brackets above equals one leaving us with

$$\frac{dF(c)}{dc} = \lambda^*(c)$$

which is what we wanted to show.

1.3 Inequality Constraints

In economics it is much more common to start with inequality constraints of the form $g(x, y) \leq c$. The constraint is said to be **binding** if at the optimum $g(x^*, y^*) = c$, and it is said to be **slack** if at the optimum $g(x^*, y^*) = c$, clearly it must be one or the other. In this case it is not clear whether or not we can use the substitution method, since that would be invalid if the constraint is slack. The Lagrangean can be used with the same FOC's except now we replace equation (iii) with the following three **"Kuhn-Tucker"** conditions

$$\begin{aligned} c - g(x^*, y^*) &\geq 0 \\ \lambda^* &\geq 0 \\ \lambda^*[c - g(x^*, y^*)] &= 0 \end{aligned} \tag{KT conditions}$$

The first condition is just a restatement of the constraint. The second condition says that λ^* is always non-negative. This depends crucially of the fact on how we wrote the Lagrangean with λ times $c - g(x, y)$ where $c - g(x, y)$ is *greater than zero*. Typically, this depends on the economic meaning of the constraint. The third condition says that either λ^* or $c - g(x^*, y^*)$ must be zero. If $\lambda^* = 0$, then the constraint can be slack the FOC turn into the (Unconstrained FOC) considered in 1.1 and we can pretty much forget about the Kuhn-Tucker conditions. If $\lambda^* > 0$ then the constraint must be binding then the problem turns into the standard Lagrangean considered above. Thus the Kuhn-Tucker condition provide a neat mathematical way of turning the problem into either an unconstrained problem or a constrained one. Unfortunately you typically have to check both cases, unless you know for sure it's one or the other.

Note that the interpretation of λ^*, still stands. Now if $\lambda^* = 0$, this means that the constraint is slack $g(x, y) < c$. Changing c by a "tiny bit" won't make the constraint binding and so the constraint will still be slack, and therefore $F(c)$ won't change at all.

Example 1 *A* ***nonnegativity constraint*** *is just a constraint of the form $x \geq 0$. Consider the one variable case where we wish to maximize $f(x)$ subject to $x \geq 0$. Then the Lagrangean will be $\mathfrak{L}(x, \lambda) = f(x) + \lambda x$. The first order Kuhn-Tucker conditions are then*

$$f'(x^*) + \lambda = 0, \;\; x^* \geq 0, \; \lambda^* \geq 0, \; x^*\lambda^* = 0$$

If the nonnegativity constraint does not bind then $f'(x^*) = 0$. If it does bind, then $f'(x^*) = -\lambda^* < 0$ and $x^* = 0$. Thus we can rephrase the FOC's without mentioning λ^* as

$$f'(x^*) \leq 0 \text{ with } "=" \text{ if } x^* > 0$$

which you may have seen in earlier notes

1.4 More than One Constraint

When there is more than one constraints say $g(x, y) \leq c$ and $h(x, y) \leq d$ we can just write a longer Lagrangean with two multipliers λ and μ

$$\mathfrak{L}(x, y, \lambda, \mu) = f(x, y) + \lambda[c - g(x, y)] + \mu[d - h(x, y)]$$

which yields the following FOC's

$$\frac{\mathfrak{L}(x^*, y^*, \lambda^*, \mu^*)}{\partial x} = \frac{\partial f}{\partial x}(x^*, y^*) - \lambda^* \frac{\partial g}{\partial x}(x^*, y^*) - \mu^* \frac{\partial g}{\partial x}(x^*, y^*) = 0 \quad \text{(i)}$$

$$\frac{\mathfrak{L}(x^*, y^*, \lambda^*, \mu^*)}{\partial y} = \frac{\partial f}{\partial y}(x^*, y^*) - \lambda^* \frac{\partial g}{\partial y}(x^*, y^*) - \mu^* \frac{\partial g}{\partial x}(x^*, y^*) = 0 \quad \text{(ii)}$$

$$c - g(x^*, y^*) \geq 0, \quad \lambda^* \geq 0, \quad \lambda^* [c - g(x^*, y^*)] \quad \text{(iii)}$$

$$d - h(x^*, y^*) \geq 0, \quad \mu^* \geq 0, \quad \mu^* [c - g(x^*, y^*)] \quad \text{(iv)}$$

While this may look intimidating, really it just a system of equation of 4 variables in 4 unknowns. Moreover if both constraints bind and are independent, then (x^*, y^*) will simply be whatever satisfies both constraints.

More constraints simply imply adding more Lagrange multipliers and more FOC's like (iii) and (iv). More control variables like x and y simply imply more conditions like (i) and (ii). There are many other interesting issues related to optimization like this that are very interesting and worth learning more about, but we won't go over them here.[2]

Example 2 ***"Bang-Bang" Solution*** *Consider the very straightforward problem of a firm maximizing $f(x) = ax$ a simple linear function where $a \gtreqless 0$ is a constant of indeterminate sign. Without a constraint there is no solution to this problem (infinity isn't technically considered a "solution"). However consider the constraints $\times \geq 0$ and $x \leq 100$. The Lagrangean will then be*

$$\mathfrak{L}(x, \lambda, \mu) = ax + \lambda x + \mu(100 - x)$$

with FOC

$$\begin{aligned} a + \lambda^* - \mu^* = 0 & \\ & x^* \geq 0,\ \lambda^* \geq 0,\ x^* \lambda^* = 0 \\ & x^* \leq 100,\ \mu^* \geq 0,\ x^* \mu^* = 0 \end{aligned}$$

Realize that both constraints cannot be binding at the same time so either $\lambda^* = 0$ or $\mu^* = 0$. Also the FOC implies $a = \mu^* - \lambda^*$ There are then 3 cases worth considering here (1) $a > 0$ then $a =$it must be that $a = \mu^* > 0$ and $\lambda^* = 0$ as $a = -\lambda^* > 0$ contradicts $\lambda^* \geq 0$. The μ constraint must be binding and so $x^* = 100$ (2) $a < 0$ in which case $a = \lambda^*$ and $x^* = \mu^* = 0$ and (3) $a = 0$ in which case $\mu^* = \lambda^* = 0$ and x^* can take any value between 0 and 100 as they all yield $ax^* = 0x^* = 0$. This kind of a solution is known as a "bang-bang" solution - either you "bang" x at the top or at the bottom. Think of a driver who either floors the gas or slams the break.

2 Shadow Prices

We saw that Lagrange multipliers can be interpreted as the change in the objective function by relaxing the constraint by one unit, assuming that unit is very small. In economics that change can be seen as a value or "**shadow price**" on that constraint, namely on c. With shadow prices it is possible to put a price on *any* constraint.

Consider a firm that has a stock of X to sell on the market at price p. The firm maximizes px, where $p > 0$ subject to the constraint that he cannot sell less than zero $x \leq 0$ and more than $x \geq X$. This is identical to the above example with $p = a$ and $x = 100$, yielding the "bang-bang" solution (1), with $x^* = X$ and $\mu^* = p$. Thus the value of an extra unit of stock X is $\mu^* = p$, the value of a unit of x on the market.

2.1 Remember Units!

Unfortunately we tend to ignore units in economics, but they are always there. For example labor L is in units of time, say hours, and consumption x is in units of consumption. w is the amount of dollars ($) per

[2] A typical multivariate calculus book should have a chapter on this subject for equality constraints. Simon and Blume (1994) is a pretty good reference with economic applications.

hour, while p is the amount \$ per unit of consumption. $wL - px$ is then in \$ as the other units cancel out. Profits and costs are also in terms of \$. Utility can be seen to be evaluated in units of "utils" although typically this shouldn't be taken too deeply. Marginal utility is in utils per unit of consumption while marginal cost is in \$ per unit of production. Marginal rates of substitution are in one unit of consumption relative to a different unit of consumption.

An objective function can be in one unit, while the control variables are in another, and the constraint may be in perhaps another. For example with utility maximization, the objective function is in utils, the control variables are in units of consumption, and the budget constraint is expressed in \$. A shadow price will be expressed in the unit of the objective function per unit of the constraint. In the Lagrangean it "turns" the units of the constraint into the units of the objective function so that the Lagrangean is in just one unit. Be warned that expressing the constraint in a different way may change the shadow price or its interpretation.

2.2 Old Problems Reconsidered

We now consider the old problems from the "*Micro Review*" Section Notes. All of the solutions, supplies and demands are exactly the same as before except now there are extra variables, namely the shadow prices. To save effort we ignore non-negativity constraints and assume all solutions are positive.

2.2.1 Profit Maximization

Firms maximize profits $\pi(x, L) = px - wL$ subject to the constraint $x \leq f(L)$. The Lagrangean is

$$\mathfrak{L}(x, L, \mu) = px - wL + \mu[f(L) - x]$$

We know the constraint will bind since the firm will maximize profits by selling all that it produces and so the FOC are

$$\frac{\partial \mathfrak{L}}{\partial x} = p - \hat{\mu} = 0$$
$$\frac{\partial \mathfrak{L}}{\partial L} = -w + \hat{\mu} f'\left(L^D\right) = 0$$
$$\frac{\partial \mathfrak{L}}{\partial \mu} = f\left(L^D\right) - x^S = 0$$

The first equation says that $\hat{\mu}(w, p) = p$, shadow price of production in \$ is equal to the price, while the second says that $\hat{\mu} = \frac{w}{f'(L^D)} = MC\left(x^S\right)$. both the price and the marginal cost of production at the optimum. I write $\hat{\mu}$ to distinguish it from μ^* in general equilibrium.

2.2.2 Utility Maximizaton

Individuals maximize $U(l, x)$ subject to the budget constraint $px \leq wL + M$ and the time constraint $l + L \leq T$. The Lagrangean is

$$\mathfrak{L}(l, x, L, \alpha, \omega) = U(l, x) + \alpha[M + wL - px] + \omega[T - l - L]$$

Knowing that the individual will not waste time or money we know that both constraints will bind and so the FOC are

$$\frac{\partial \mathfrak{L}}{\partial l} = \frac{\partial U}{\partial l} - \hat{\omega} = 0$$
$$\frac{\partial \mathfrak{L}}{\partial x} = \frac{\partial U}{\partial x} - \hat{\alpha} p = 0$$
$$\frac{\partial \mathfrak{L}}{\partial L} = \hat{\alpha} w - \hat{\omega} = 0$$
$$\frac{\partial \mathfrak{L}}{\partial \lambda} = M + wL^S - px^D = 0$$
$$\frac{\partial \mathfrak{L}}{\partial \tau} = T - l^D - L^S = 0$$

The first equation tells us that at the optimum that the shadow price of time T in utils per hour $\hat{\omega}(w,p,M) = \frac{\partial U}{\partial l}\left[l^D(w,p,M), x^D(w,p,M)\right]$ is the marginal utility of leisure, evaluated at the the proper demands. The second equation say that the shadow price of earnings wL or money M in utils per dollars $\hat{\alpha}(w,p,M) = \frac{1}{p}\frac{\partial U}{\partial x}\left[l^D(w,p,M), x^D(w,p,M)\right]$ the marginal utility of consumption divided by the price of consumption. The third equation tells us that the shadow price of time is w times the shadow price of money $\hat{\omega} = w\hat{\alpha}$ condition. This can also be rewritten as $\frac{\hat{\omega}}{\hat{\alpha}} = w$ which now expresses the the shadow price of time in \$, which equals the market wage. In general one can divide by $\hat{\alpha}$ to translate prices in utility to prices in \$.

2.2.3 Planner's Problem

This problem is very similar to the last one with the production constraint $x = f(L)$ in place of the budget constraint.

$$\mathfrak{L}(l,x,L,\phi,\omega) = U(l,x) + \phi[f(L) - x] + \omega[T - l - L]$$

Knowing that the individual will not waste time or money we know that both constraints will bind and so the FOC are

$$\begin{aligned}
\frac{\partial \mathfrak{L}}{\partial l} &= \frac{\partial U}{\partial l} - \omega^* = 0 \\
\frac{\partial \mathfrak{L}}{\partial x} &= \frac{\partial U}{\partial x} - \phi^* = 0 \\
\frac{\partial \mathfrak{L}}{\partial L} &= \phi^* f'(L^*) - \omega^* = 0 \\
\frac{\partial \mathfrak{L}}{\partial \lambda} &= M + wL^* - px^* = 0 \\
\frac{\partial \mathfrak{L}}{\partial \omega} &= T - l^* - L^* = 0
\end{aligned}$$

The value of time $\omega^* = \frac{\partial U}{\partial l}[l^*, x^*]$ is now time constraint is the same. The second equation says that at the optimum, the shadow price of production $\phi^* = \frac{\partial U}{\partial x}(l^*, x^*)$ is just the marginal utility of consumption. The third equation $\omega^* = \phi^* f'(L^*)$says that the shadow price of time is equal to the marginal rate of transformation (recall $MRT_{lx} = f'(L)$) times the shadow price of consumption.

2.2.4 Equivalence of the Planner and the Market

The key condition to remember about the equivalence between the planner's problem and the decentralized market outcome is that $MRS_{l^*x^*} = MRT_{l^*x^*}$. In the planner's problem production efficiency implies

$$MRS_{l^*x^*} = \frac{\partial U/\partial l}{\partial U/\partial x} = \frac{\omega^*}{\phi^*} = f'(L^*) = MRT_{l^*x^*}$$

In the decentralized market outcome we also get production efficiency

$$MRS_{l^*x^*} = \frac{\partial U/\partial l}{\partial U/\partial x} = \frac{w^*}{p^*} = f'(L^*) = MRT_{l^*x^*}$$

Therefore the ratio of shadow prices in the optimal planner's economy is equal to the ratio of market prices in the market economy, $\frac{\omega^*}{\phi^*} = \frac{w^*}{p^*}$. The shadow prices are in terms of utils and the market prices are in terms of \$, while the ratio of the prices, shadow or market, are in terms of units of consumption per unit of leisure. Now write $\alpha^* = \hat{\alpha}(w^*, p^*, \pi^*), \mu^* = \hat{\mu}(w^*, p^*)$ and $,\omega^* = \omega(w^*, p^*, \pi^*)$ as the equilibrium values of the shadow prices. We saw before that $\omega^* = \frac{w^*}{\alpha^*}$, now we can also write $\phi^* = \frac{p^*}{\alpha^*}$ as the efficiency condition tells us $\frac{p^*}{\alpha^*} = \frac{w^*}{\alpha^*}\frac{\phi^*}{\omega^*} = \omega^*\frac{\phi^*}{\omega^*} = \phi^*$.

2.3 The Inescapability of Scarcity

The big point of all of this is not only to show you all sorts of efficiency properties, but to point out some fundamentals about economics. A resource (time, money, consumption, air, etc.) is scarce if the amount that people would like to use exceeds the amount available - i.e. the resource constraint associated with it is binding. So long as resources can be used either directly (straight into utility) or indirectly (through production) to improve people's lives there will be a positive price on it, market or shadow, if it is scarce. Even if something is not sold for money it still has a price.

Of course if a good is not scarce then its shadow price is zero. For example imagine a world where consumption is not scarce, and people consume unto the point until they are satiated so $\frac{\partial U}{\partial x} = 0$ and so $\phi^* = 0$. An interesting point to be taken from the third FOC in the planner's problem is that of $\omega^* = \phi^* MRT_{lx}$, so long as the $MRT_{lx} > 0$ then at the optimum if time is scarce then consumption must be scarce, too, and vice-versa. The only way we cannot have scarcity is if neither time nor consumption is scarce - an unlikely situation, and that's why if you grow up to be economists, you should always have a job! Said otherwise, this principle means that at the optimum nonscarce (free) goods cannot be costlessly transformed into scarce goods.

3 Inefficient Markets

There are many reasons why markets may not work efficiently. A particular inefficiency we consider here is when $x < x^*$ the optimal amount. Four main ways this could happen is through (i) market power of firms, such as through monopoly, (ii) distortionary taxation, (iii) regulation which restricts production or (iv) failure of markets to clear because of sticky prices and macroeconomic fluctuations.

3.0.1 Firm's Problem

To find the shadow price of x in this situation we can use a constraint of the form $x \leq \bar{x} < x^*$, to which we attach the multiplier ξ.The Lagrangean in this case is then

$$\mathfrak{L}(x, L, \mu, \xi) = px - wL + \mu [f(L) - x] + \xi [\bar{x} - x]$$

We assume in advance that both constraints bind[3] and so x and L are determined before we even look for the optimum: $x^S = \bar{x}$, and $L^D = f^{-1}(\bar{x}) = \bar{L}$, which by the concavity of production implies $\bar{L} < L^*$, which leads to unemployment. Profits are $\bar{\pi} = p\bar{x} - w\bar{L}$. Warning: although you know the quantities you should not put them in the Lagrangean ahead of time: treat x and L as variables and plug them in as solutions to the FOC. For the purposes of calculation we only need the first two FOC to tell us about the shadow prices

$$\begin{aligned} p - \bar{\mu} - \bar{\xi} &= 0 \\ -w + \bar{\mu} f'\left(\bar{L}\right) &= 0 \end{aligned}$$

The second equation tells us that the shadow price of production still equals marginal cost $\bar{\mu} = \frac{w}{f'(\bar{L})} = MC(\bar{x})$. However the price is now greater than marginal cost as $p = \bar{\mu} + \bar{\xi} = MC(\bar{x}) + \bar{\xi}$, and $\bar{\xi} > 0$. The shadow price of increasing production $\bar{\xi} = p - \bar{\mu} < p$, the market price minus the marginal cost of production, is the marginal surplus of producing another unit.

At this point the marginal rate of transformation is given by

$$MRT_{\bar{l}\bar{x}} = f'\left(\bar{L}\right) = \frac{w}{\bar{\mu}} = \frac{w}{p - \bar{\xi}} > \frac{w}{p}$$

which states that the has a higher rate of transformation of leisure into consumption than the market.

3.0.2 Consumer's Problem

For the consumer problem we now consider the fact the employment is limited[4] to $\bar{L} < L^*$ and so leisure is taken to be $\bar{l} = T - \bar{L}$. This will imply that consumption will be too low at $\bar{x} = \left(w\bar{L} + M\right)/p$, which in the constrained general equilibrium will equal $\bar{x}$. The Lagrangean of the consumer's problem is now

[3] This means we look at a limited class of prices (w, p) which imply $x^D(w, p) < x^*$

[4] If you remember Walras' Law implied that if the labor market cleared so did the goods market. The logically equivalent contrapositive is that if the goods market does not clear neither does the labor market.

$$\mathfrak{L}(l,x,L,\alpha,\omega,\lambda) = U(l,x) + \alpha[M + wL - px] + \omega[T - l - L] + \lambda\left[\bar{L} - L\right]$$

We take just first three FOC (you should be able to guess the other 3)

$$\frac{\partial \mathfrak{L}}{\partial l} = \frac{\partial U}{\partial l} - \bar{\omega} = 0$$
$$\frac{\partial \mathfrak{L}}{\partial x} = \frac{\partial U}{\partial x} - \bar{\alpha}p = 0$$
$$\frac{\partial \mathfrak{L}}{\partial L} = \bar{\alpha}w - \bar{\omega} - \bar{\lambda} = 0$$

The shadow prices are now $\bar{\omega} = \frac{\partial U}{\partial l}\left(\bar{l},\bar{x}\right)$, $\bar{\alpha} = \frac{\partial U}{\partial x}\left(\bar{l},\bar{x}\right)$ and $\bar{\lambda} = \bar{\alpha}w - \bar{\omega}$. The big difference comes in the third equation which implies that the shadow price of time is now less then the shadow value of the consumption the consumer could get from working $\bar{\omega} = \bar{\alpha}w - \bar{\lambda} < \bar{\alpha}w$ as $\bar{\lambda} < 0$. Solving for w we find $w = \frac{\bar{\omega}+\bar{\lambda}}{\bar{\alpha}} > \frac{\bar{\omega}}{\bar{\alpha}}$ which states that the market wage is greater than the shadow price of (leisure) time in \$ $\frac{\bar{\omega}}{\bar{\alpha}}$. The consumer would like to sell more leisure and buy more consumption, but cannot.

The marginal rate of substitution in this case is

$$MRS_{\bar{l}\bar{x}} = \frac{\partial U/\partial l}{\partial U/\partial x} = \frac{\bar{\omega}}{\bar{\alpha}p} = \frac{\bar{\alpha}w - \bar{\lambda}}{\bar{\alpha}p} = \frac{w}{p} - \frac{\bar{\lambda}}{\bar{\alpha}p} < \frac{w}{p}$$

so the consumer is more willing to translate leisure into market goods than the market.

3.0.3 Constrained General Equilibrium

Since this model presumes that markets no longer clear, the prices are indeterminate. Recall that in general equilibrium only the ratio of prices matters so we can set $\bar{p} = 1$. The wage $\bar{w}$ is then just the amount that makes supply equal demand, where $M = \bar{\pi}$, i.e. $\bar{x} = w\bar{L} + \bar{\pi} = w\bar{L} + \left(\bar{x} - w\bar{L}\right) = \bar{x}$. The wage only determines what we call "profits" (or "nonlabor" income) and what we call "labor income".

Pareto inefficiency is shown by the fact that production efficiency fails

$$MRS_{\bar{l}\bar{x}} < \frac{w}{p} < MRT_{\bar{l}\bar{x}}$$

In this case, society would be better off transforming more leisure into consumption, although it cannot.[5]

4 Government Projects

Any project, public or otherwise, will always have some price associated with its benefits and costs. The greatest challenge in cost-benefit analysis is determining the correct price which reflects the social benefits and social costs, i.e. the "social price". As we saw above, if the markets work efficiently, then relative market price should reflect relative shadow prices in terms of utility. However, if they are inefficient then this relationship breaks down. We consider only cases where the government project is too small to affect relative prices.

4.1 Government Purchases of Goods

Assume the government makes the firm produce a small purchase of goods $G \geq \bar{G} \approx 0$, so that total output which is $x+G$. We can find the price of a tiny change in G by just putting in a constraint for $G \geq \bar{G} \approx 0$ into the Lagrangean- a neat trick since the multiplier, p_G, will give us the value of changing that constraint, i.e., the loss of a profits the firm will lose by producing one unit of G, which is its shadow price, or alternatively

[5] One extreme is where firms will charge the highest price they can, which is equivalent to where the unconstrained demand $x^D(w,1,\pi(\bar{x})) = \bar{x}$ and corresponds to the lowest wage $\bar{w}$. What is funny about this case is that there is no excess demand for consumption (prices are high enough) or excess supply of labor (wages are low enough), but the market allocation is still Pareto inefficient. This is a borderline case as $MRS_{\bar{l}\bar{x}} = w/p$, as $\bar{\nu} = 0$, but $MRT_{\bar{l}\bar{x}} > w/p$ as $\bar{\lambda} > 0$

how much the government need to pay the firm to compensate it for producing a unit of G at the margin. [6] What will determine the shadow price p_G of G is (i) whether or not the constraint on production $x \leq \bar{x}$ is binding and (ii) whether or not G belongs in the constraint.

In the case where G belongs in the constraint $x + G \leq \bar{x}$ and this constraint is binding. We have a Lagrangean

$$\mathfrak{L}(x, L, G, \mu, \xi, p_G) = px - wL + \mu [f(L) - x - G] + \xi [\bar{x} - x - G] + p_G [G - \bar{G}]$$

Because at the optimum $G = \bar{G} \approx 0$, G won't have an effect on any of the other quantities considered previously. The only difference is now we have an extra FOC with respect to G to tell us about p_G :

$$-\bar{\mu} - \bar{\xi} + p_G = 0$$

In this case $p_G = \bar{\mu} + \bar{\xi} = \bar{p}$, so that the shadow price of a government good equals its market price which is over cost. The case where where G does not belong in the constraint or the constraint is not binding can just ignore the production constraint and thus ξ. The FOC is then

$$-\mu + p_G = 0$$

which means that the shadow price of the government goods is just its marginal cost, $p_G = \mu = MC(x)$. Now if the goods market is inefficient $p_G = MC(\bar{x}) < \bar{p}$, while if it is efficient then $p_G = MC(x^*) = p^*$.

4.2 Government Purchases of Labor

Now assume that the government forces a worker to work a small amount of labor $L_G \geq \bar{L}_G \approx 0$ for its own use. We can find the shadow price of government labor by attaching a multiplier to it as , say ω_G. Alternatively ω_G is how much utility the government would have to give the person in exchange for that person voluntarily providing a unit of labor. Again we have to consider the cases when government labor adds to employment and when it doesn't. We start by giving the Lagrangean for when it doesn't.

$$\mathfrak{L}(l, x, L, \alpha, \omega, \lambda, \omega_G) = U(l, x) + \alpha [M + wL - px] + \omega [T - l - L - L_G] + \lambda [\bar{L} - L - L_G] + \omega_G [L_G - \bar{L}_G]$$

The FOC with respect to L we get

$$-\bar{\omega} - \bar{\lambda} + \bar{\omega}_G = 0$$

which implies that the shadow price of government labor is the same as it is for regular labor $\bar{\omega}_G = \bar{\omega} + \bar{\lambda} = \bar{\alpha}\bar{w}$. Dividing by α to get the money price of government labor $w_G = \frac{\bar{\omega}_G}{\bar{\alpha}} = \bar{w}$. Now if government labor can add to regular employment (i.e. it can reduce unemployment) then we can just ignore λ, since L_G doesn't have to obey this constraint. The FOC is then simple

$$-\omega + \omega_G = 0$$

and so the the utility cost of government labor is the same as the utility cost of leisure which may be less than that of regular labor $\omega_G = \omega = \alpha w - \lambda$. Dividing by α we get $w_G = w - \frac{\lambda}{\alpha}$ which is strictly less than w if $\lambda > 0$, i.e., if there is involuntary unemployment.

4.3 Concluding Remarks

In order to keep things simple this analysis brushed aside any distributional issues, and this could have a potential effect on shadow prices of goods. In fact even with perfectly efficient markets where market, $ prices are the same for everyone, the utility price may be quite different. Another thing to keep in mind is that prices refer to marginal quantities. You may value one hour of work at the going wage, but for the 24th hour in the day you would need to be paid much, much more. Finally, there is much that is left out of the analysis. Those who use wages as a measure of the value of life assume implicitly that an hour of work is an hour lost just as an hour of premature death. If just being *alive* during work has its own shadow price, then you would want to add the value of being alive for an hour to your wage to get the value of an hour of your life - a difficult number to calculate, indeed.

[6] If you want to see the price of large projects just set $G \geq \bar{G}$ for some $\bar{G} > 0$. This will tend to change the relative prices of goods however.