CHAPTER 10: SINGLE-INDEX AND MULTIFACTOR MODELS

1. a. To optimize this portfolio one would need:

\[
\begin{align*}
 n &= 60 \text{ estimates of means} \\
 n &= 60 \text{ estimates of variances} \\
 \frac{n^2 - n}{2} &= 1770 \text{ estimates of covariances} \\
 \frac{n^2 + 3n}{2} &= 1890 \text{ estimates}
\end{align*}
\]

b. In a single index model:

\[
 r_i - r_f = \alpha_i + \beta_i (r_M - r_f) + e_i
\]

or equivalently, using excess returns

\[
 R_i = \alpha_i + \beta_i R_M + e_i
\]

the variance of the rate of return on each stock can be decomposed into the components:

1. \( \beta_i^2 \sigma_M^2 \) - The variance due to the common market factor
2. \( \sigma^2(e_i) \) - The variance due to firm specific unanticipated events

In this model \( \text{Cov}(r_i, r_j) = \beta_i \beta_j \sigma_M^2 \). The number of parameter estimates would be:

- \( n = 60 \) estimates of the mean \( E(r_i) \),
- \( n = 60 \) estimates of the sensitivity coefficient \( \beta_i \),
- \( n = 60 \) estimates of the firm-specific variance \( \sigma^2(e_i) \), and
- 1 estimate of the market mean \( E(r_M) \)
- 1 estimate for the market variance \( \sigma_M^2 \)

\[
\frac{n^2 + 3n}{2} = 182 \text{ estimates}
\]

Thus, the single index model reduces the total number of required parameter estimates from 1,890 to 182, and in general from \( (n^2 + 3n)/2 \) to \( 3n + 2 \).

2. a. The standard deviation of each individual stock is given by:

\[
\sigma_i = [\beta_i^2 \sigma_M^2 + \sigma^2(e_i)]^{1/2}
\]
Since $\beta_A = .8$, $\beta_B = 1.2$, $\sigma(e_A) = 30\%$, $\sigma(e_B) = 40\%$, and $\sigma_M = 22\%$ we get:

$$\sigma_A = (0.8^2 \times 22^2 + 30^2)^{1/2} = 34.78\%$$

$$\sigma_B = (1.2^2 \times 22^2 + 40^2)^{1/2} = 47.93\%$$

b. The expected rate of return on a portfolio is the weighted average of the expected returns of the individual securities:

$$E(r_p) = w_A E(r_A) + w_B E(r_B) + w_f r_f$$

where $w_A$, $w_B$, and $w_f$ are the portfolio weights of stock A, stock B, and T-bills, respectively.

Substituting in the formula we get:

$$E(r_p) = 0.30 \times 13 + 0.45 \times 18 + 0.25 \times 8 = 14\%$$

The beta of a portfolio is similarly a weighted average of the betas of the individual securities:

$$\beta_p = w_A \beta_A + w_B \beta_B + w_f \beta_f$$

The beta of T-bills ($\beta_f$) is zero. The beta of the portfolio is therefore:

$$\beta_p = 0.30 \times 0.8 + 0.45 \times 1.2 + 0 = 0.78$$

The variance of this portfolio is:

$$\sigma_p^2 = \beta_p^2 \sigma_M^2 + \sigma^2(e_p)$$

where $\beta_p^2 \sigma_M^2$ is the systematic component and $\sigma^2(e_p)$ is the nonsystematic component. Since the residuals, $e_i$ are uncorrelated, the non-systematic variance is:

$$\sigma^2(e_p) = w_A^2 \sigma^2(e_A) + w_B^2 \sigma^2(e_B) + w_f^2 \sigma^2(e_f)$$

$$= 0.30^2 \times 30^2 + 0.45^2 \times 40^2 + 0.25^2 \times 0 = 405$$

where $\sigma^2(e_A)$ and $\sigma^2(e_B)$ are the firm-specific (nonsystematic) variances of stocks A and B, and $\sigma^2(e_f)$, the nonsystematic variance of T-bills, is zero. The residual standard deviation of the portfolio is thus:
\[ \sigma(c_p) = (405)^{1/2} = 20.12\% \]

The total variance of the portfolio is then:

\[ \sigma_P^2 = .78^2 \times 22^2 + 405 = 699.47 \]

and the standard deviation is 26.45%.

3.  a. The two figures depict the stocks' security characteristic lines (SCL). Stock A has a higher firm-specific risk because the deviations of the observations from the SCL are larger for A than for B. Deviations are measured by the vertical distance of each observation from the SCL.

b. Beta is the slope of the SCL, which is the measure of systematic risk. Stock B's SCL is steeper, hence stock B's systematic risk is greater.

c. The R\(^2\) (or squared correlation coefficient) of the SCL is the ratio of the explained variance of the stock's return to total variance, and the total variance is the sum of the explained variance plus the unexplained variance (the stock's residual variance).

\[ R^2 = \frac{\beta_i^2 \sigma_M^2}{\beta_i^2 \sigma_M^2 + \sigma^2(e_i)} \]

Since stock B's explained variance is higher (its explained variance is \( \beta_B^2 \sigma_M^2 \), which is greater since its beta is higher), and its residual variance \( \sigma^2(e_B) \) is smaller, its R\(^2\) is higher than stock A's.

d. Alpha is the intercept of the SCL with the expected return axis. Stock A has a small positive alpha whereas stock B has a negative alpha; hence stock A's alpha is larger.

e. The correlation coefficient is simply the square root of R\(^2\), so stock B’s correlation with the market is higher.

4.  a. Firm-specific risk is measured by the residual standard deviation. Thus, stock A has more firm-specific risk: 10.3% > 9.1%.

b. Market risk is measured by beta, the slope coefficient of the regression. A has a larger beta coefficient: 1.2 > .8.

c. R\(^2\) measures the fraction of total variance of return explained by the market return. A's R\(^2\) is larger than B's: .576 > .436.
d. The average rate of return in *excess* of that predicted by the CAPM is measured by alpha, the intercept of the SCL. $\alpha_A = 1\%$ is larger than $\alpha_B = -2\%$.

e. Rewriting the SCL equation in terms of total return $(r)$ rather than excess return $(R)$:

\[
\begin{align*}
    r_A - r_f &= \alpha + \beta(r_M - r_f) \\
    r_A &= \alpha + r_f(1 - \beta) + \beta r_M
\end{align*}
\]

The intercept is now equal to:

\[
\alpha + r_f(1 - \beta) = 1 + r_f(1 - 1.2)
\]

Since $r_f = 6\%$, the intercept would be: $1 - 1.2 = -2\%$.

5. The standard deviation of each stock can be derived from the following equation for $R^2$:

\[
R_i^2 = \frac{\beta_i^2 \sigma_M^2}{\sigma_i^2} = \frac{\text{Explained variance}}{\text{Total variance}}
\]

Therefore,

\[
\sigma_A^2 = \frac{\beta_A^2 \sigma_M^2}{R_A^2} = \frac{.7^2 \times 20^2}{.20} = 980
\]

$\sigma_A = 31.30\%$

For stock B

\[
\sigma_B^2 = \frac{1.2^2 \times 20^2}{.12} = 4800
\]

$\sigma_B = 69.28\%$

6. The systematic risk for A is

\[
\beta_A^2 \sigma_M^2 = .70^2 \times 20^2 = 196
\]

and the firm-specific risk of A (the residual variance) is the difference between A's total risk and its systematic risk,

\[
980 - 196 = 784
\]
B's systematic risk is:

\[ \beta_B^2 \sigma_M^2 = 1.2^2 \times 20^2 = 576 \]

and B's firm-specific risk (residual variance) is:

\[ 4800 - 576 = 4224 \]

7. The covariance between the returns of A and B is (since the residuals are assumed to be uncorrelated):

\[ \text{Cov}(r_A, r_B) = \beta_A \beta_B \sigma_M^2 = .70 \times 1.2 \times 400 = 336 \]

The correlation coefficient between the returns of A and B is:

\[ \rho_{AB} = \frac{\text{Cov}(r_A, r_B)}{\sigma_A \sigma_B} = \frac{336}{31.30 \times 69.28} = .155 \]

8. Note that the correlation is the square root of \( R^2 \):

\[ \rho = \sqrt{R^2} \]

\[ \text{Cov}(r_A, r_M) = \rho \sigma_A \sigma_M = .20^{1/2} \times 31.30 \times 20 = 280 \]

\[ \text{Cov}(r_B, r_M) = \rho \sigma_B \sigma_M = .12^{1/2} \times 69.28 \times 20 = 480 \]

9. The non-zero alphas from the regressions are inconsistent with the CAPM. The question is whether the alpha estimates reflect sampling errors or real mispricing. To test the hypothesis of whether the intercepts (3% for A, and –2% for B) are significantly different from zero, we would need to compute t-values for each intercept.

10. For portfolio P we can compute:

\[ \sigma_P = [.6^2 \times 980 + .4^2 \times 4800 + 2 \times .4 \times .6 \times 336]^{1/2} \]

\[ = [1282.08]^{1/2} = 35.81\% \]

\[ \beta_P = .6 \times .70 + .4 \times 1.2 = .90 \]
\[ \sigma^2(e_p) = \sigma_p^2 - \beta_p^2 \sigma_M^2 = 1282.08 - .90^2 \times 400 = 958.08 \]

\[ \text{Cov}(r_p, r_M) = \beta_p \sigma_M^2 = .90 \times 400 = 360 \]

This same result can also be attained using the covariances of the individual stocks with the market:

\[ \text{Cov}(r_p, r_M) = \text{Cov}( .6r_A + .4r_B, r_M) = .6 \text{Cov}(r_A, r_M) + .4 \text{Cov}(r_B, r_M) \]

\[ = .6 \times 280 + .4 \times 480 = 360 \]

11. Note that the variance of T-bills and their covariance with any asset are zero. Therefore, for portfolio Q

\[ \sigma_Q = w_p \sigma_p^2 + w_M \sigma_M^2 + 2 \times w_p \times w_M \times \text{Cov}(r_p, r_M) \]

\[ \sigma_Q = [\.5^2 \times 1282.08 + .3^2 \times 400 + 2 \times .5 \times .3 \times 360]^{1/2} \]

\[ = [464.52]^{1/2} = 21.55\% \]

\[ \beta_Q = .5 \times .90 + .3 \times 1 + 0 = .75 \]

\[ \sigma^2(e_Q) = \sigma_Q^2 - \beta_Q^2 \sigma_M^2 = 464.52 - .75^2 \times 400 = 239.52 \]

\[ \text{Cov}(r_Q, r_M) = \beta_Q \sigma_M^2 = .75 \times 400 = 300 \]

12. In a two-stock capital market, the capitalization of A being twice that of B implies that

\[ w_A = 2/3 \text{ and } w_B = 1/3. \]

\[ \sigma_A = 30\%, \quad \sigma_B = 50\%, \quad \rho_{AB} = .7 \]

a. The variance of the market index portfolio is:

\[ \sigma_M^2 = w_A^2 \sigma_A^2 + w_B^2 \sigma_B^2 + 2w_Aw_B\rho \sigma_A \sigma_B \]

\[ = (2/3)^230^2 + (1/3)^250^2 + 2(2/3)(1/3).7 \times 30 \times 50 = 1144.44 \]

\[ \sigma_M = 33.83\% \]

b. The beta of stock A is:

\[ \beta_A = \frac{\text{Cov}(r_A, r_M)}{\sigma_M^2} \]
where
\[ \text{Cov}(r_A, r_M) = \text{Cov}(r_A, (2/3 r_A + 1/3 r_B)) = 2/3 \times \sigma_A^2 + 1/3 \times \text{Cov}(r_A, r_B) \]
\[ = (2/3) \times 30^2 + (1/3) \times 0.7 \times 30 \times 50 = 950 \]
so that
\[ \beta_A = \frac{950}{1144.44} = 0.83 \]
For stock B,
\[ \text{Cov}(r_B, r_M) = \text{Cov}(r_B, (2/3 r_A + 1/3 r_B)) = 2/3 \times \text{Cov}(r_A, r_B) + 1/3 \times \sigma_B^2 \]
\[ = 2/3 \times 0.7 \times 30 \times 50 + 1/3 \times 50^2 = 1533.33 \]
so that,
\[ \beta_B = \frac{1533.33}{1144.44} = 1.34 \]
c. The residual variance of each stock is:
\[ \sigma^2(e_A) = \sigma_A^2 - \beta_A^2 \sigma_M^2 = 30^2 - (0.83^2 \times 1144.44) = 111.60 \]
and
\[ \sigma^2(e_B) = \sigma_B^2 - \beta_B^2 \sigma_M^2 = 50^2 - (1.34^2 \times 1144.44) = 445.04 \]
d. If the index model holds, then the following holds too:
\[ (r_A - r_f) = \beta_A (r_M - r_f) \]
\[ 11\% = 0.83(r_M - r_f) \]
Thus the market risk premium must be
\[ r_M - r_f = \frac{11\%}{0.83} = 13.25\% \]
Since A's beta is smaller than 1.0, its risk premium is smaller than the market's risk premium.
13. a. Merrill Lynch adjusts beta by taking the sample estimate of beta and averaging it with 1.0, using the weights of 2/3 and 1/3, as follows:

\[
\text{adjusted beta} = \frac{2}{3} \times 1.24 + \frac{1}{3} \times 1 = 1.16
\]

b. If you use your current estimate of beta to be \( \beta_{t-1} = 1.24 \), then

\[
\beta_t = .3 + .7 \times (1.24) = 1.168
\]

which is the prediction of beta for next year.


ABC: \( \beta \) for ABC was .60, considerably below the average stock’s \( \beta \) of 1.0, indicating that when the S&P 500 rose or fell by 1 percentage point, ABC’s return on average rose or fell only 0.60 percentage point. As such it indicates that ABC’s systematic risk or market risk was low relative to the typical value for stocks. ABC's alpha (the intercept of the regression) was -3.2%, indicating that when the market return was 0%, the average return on ABC was -3.2%. ABC's unsystematic or residual risk, as measured by \( \sigma(e) \), was 13.02%. Its \( R^2 \) was .35, indicating closeness of fit to the linear regression above the value for a typical stock.

XYZ: \( \beta \) for XYZ was somewhat higher at .97, indicating XYZ’s return pattern was very similar to the market index’s \( \beta \) of 1.0 and the stock therefore had average systematic risk over the period examined. Alpha for XYZ was positive and quite large, indicating an almost 7.3% return, on average, for XYZ independent of market return. Residual risk was 21.45%, half again as much as ABC’s, indicating a wider scatter of observations around the regression line for XYZ. Correspondingly, the fit of the regression model was considerably less, consistent with an \( R^2 \) of only .17.

The effects of including one stock or the other in a diversified portfolio may be quite different, if it can be assumed that both stocks' betas will remain stable over time, since there is such a large difference in their systematic risk level. The betas obtained from the two brokerage houses may help the analyst draw inferences for the future. ABC’s \( \beta \) estimates are similar regardless of the sample period of the underlying data. They range from .60 to .71, all well below the market average \( \beta \) of 1.0. XYZ's \( \beta \) varies significantly among the three sources of calculations, ranging as high as 1.45 for the weekly price change observations over the most recent two years. One could infer that XYZ's beta for the future might be well above 1.0, meaning it may have somewhat more systematic risk than was implied by the monthly regression for the 1992 - 2001 period.
The upshot is that these stocks appear to have significantly different systematic risk characteristics. If these stocks are added to a diversified portfolio, XYZ will add more to total volatility.

15. a. The $\alpha$ of stock A is:

$$\alpha_A = r_A - [r_f + \beta_A(r_M - r_f)]$$

$$= 11 - [6 + .8(12 - 6)] = .2\%$$

For stock B:

$$\alpha_B = 14 - [6 + 1.5(12 - 6)] = -1\%$$

Stock A would be a good addition. A short position in B may be desirable.

b. The reward to variability ratio of the stocks is:

$$S_A = \frac{11 - 6}{10} = .5$$

$$S_B = \frac{14 - 6}{11} = .73$$

Stock B is superior when only one can be held.

c. The $R^2$ of the regression is $.7^2 = .49$, leaving 51\% of total variance unexplained by the market, and therefore, interpreted as firm-specific.

17. b. $9 = 3 + \beta (11 - 3)$ implies that $\beta = .75$.