CHAPTER 11: ARBITRAGE PRICING THEORY

1. The revised estimate of the expected rate of return on the stock would be the old estimate plus the sum of the products of the unexpected change in each factor times the respective sensitivity coefficient, i.e.,

\[
\text{revised estimate} = 12\% + [(1 \times 2\%) + (.5 \times 3\%)] = 15.5\%
\]

2. Equation 11.6 applies here:

\[
E(r_p) = r_f + \beta_{p1}[E(r_1) - r_f] + \beta_{p2}[E(r_2) - r_f]
\]

We need to find the risk premium, \(RP\), for each of the two factors; \(RP_1 = [E(r_1) - r_f]\) and \(RP_2 = [E(r_2) - r_f]\). To do so, the following system of two equations with two unknowns must be solved:

\[
\begin{align*}
31 & = 6 + 1.5 \times RP_1 + 2.0 \times RP_2 \\
27 & = 6 + 2.2 \times RP_1 + (-.2) \times RP_2
\end{align*}
\]

The solution to this set of equations is

\[
RP_1 = 10\% \quad \text{and} \quad RP_2 = 5\%
\]

Thus, the expected return-beta relationship is:

\[
E(r_p) = 6\% + \beta_{p1} \times 10\% + \beta_{p2} \times 5\%
\]

3. The expected return of portfolio F equals the risk-free rate since its beta equals 0. Portfolio A's ratio of risk premium to beta is: \((12 - 6)/1.2 = 5\), whereas, portfolio E's ratio is lower at \((8 - 6)/.6 = 3.33\). This implies that an arbitrage opportunity exists. For instance, you can create a portfolio G with beta equal to .6 (the same as E's) by mixing portfolios A and F in equal weights. The expected return and beta of G is then:

\[
\begin{align*}
E(r_G) & = .5 \times 12\% + .5 \times 6\% = 9\% \\
\beta_G & = .5 \times 1.2 + .5 \times 0 = 0.6
\end{align*}
\]

Comparing G to E, G has the same beta and higher return. Therefore, an arbitrage opportunity exists by buying portfolio G and selling an equal amount of portfolio E.
If you do so, your profit will be

\[ r_G - r_E = (9\% + .6 \times F) - (8\% + .6 \times F) = 1\% \]

of the funds (long or short) in each portfolio.

4. a. As a first step, convert the scenario rates of return to dollar payoffs per share, as shown in the following table:

<table>
<thead>
<tr>
<th>Scenarios</th>
<th>Price</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>A $10</td>
<td>10(1 - .15) = $8.50</td>
<td>10(1 + .20) = $12.00</td>
<td>10(1 + .30) = $13.00</td>
<td></td>
</tr>
<tr>
<td>B $15</td>
<td>15(1 + .25) = $18.75</td>
<td>15(1 + .10) = $16.50</td>
<td>15(1 - .10) = $13.50</td>
<td></td>
</tr>
<tr>
<td>C $50</td>
<td>50(1 + .12) = $56.00</td>
<td>50(1 + .15) = $57.50</td>
<td>50(1 + .12) = $56.00</td>
<td></td>
</tr>
</tbody>
</table>

Identifying an arbitrage opportunity always involves constructing a zero investment portfolio. This portfolio must show non-negative payoffs in all scenarios.

For example, the proceeds from selling short two shares of A and two shares of B will be sufficient to buy one share of C.

\[ (-2)10 + (-2)15 + 50 = 0 \]

The payoff table for this zero investment portfolio in each scenario is:

<table>
<thead>
<tr>
<th>Price</th>
<th># of shares</th>
<th>Investment</th>
<th>Scenarios</th>
</tr>
</thead>
<tbody>
<tr>
<td>A $10</td>
<td>-2</td>
<td>-20</td>
<td>-17</td>
</tr>
<tr>
<td>B $15</td>
<td>-2</td>
<td>-30</td>
<td>-37.5</td>
</tr>
<tr>
<td>C $50</td>
<td>+1</td>
<td>50</td>
<td>56</td>
</tr>
</tbody>
</table>

This portfolio qualifies as an arbitrage portfolio because it is both a zero investment portfolio and has positive returns in all scenarios.

b. Should prices of A and B go down due to excess short selling and price of C go up because of buying pressures, then the rate of return on \((A + B)\) will go up and the rate of return on C will fall.
We now find a price change that will eliminate the arbitrage opportunity shown above. First note that as the price of C changes, so will the number of shares of stock that can be purchased for the investment portfolio. Second, the weakest scenario for a portfolio long on C and short on (A + B) appears to be scenario 2. Focusing on scenario 2, we call X the number of shares of A and B that will drive profits in that scenario down to zero. Next we find the price of C that will enable us to buy only that number of shares. Denote by X the number of shares of A and B sold short and let the proceeds be sufficient to long one share of C. First we set the payoff in scenario 2 to zero.

\[ 12X + 16.5X + 57.5 = 0 \quad ; \quad X = -2.0175 \]

meaning that we short 2.0175 shares of A and 2.0175 shares of B for each share of C held long. (Note that in the previous arbitrage portfolio \( X = -2 \)).

Next, with this number of shares of A and B short, we ask: what is the price of one share of C that would make the portfolio investment zero? In other words, how high a price for C will enable us to buy only one share of C with the proceeds of the short sale?

\[ 10X + 15X + P_C = 0 \]

where \( P_C \) is the new price of C. Substituting \( X = -2.0175 \) we find that:

\[ 10 \times (-2.0175) + 15 \times (-2.0175) + P_C = 0 \]

\[ P_C = 50.4375 \]

This means that the minimum price change that is needed to eliminate the arbitrage opportunity is 43.75 cents. To check our result, let's look at the following payoff table for a price change of 50 cents, that is, \( P_C = \$50.50 \).

<table>
<thead>
<tr>
<th></th>
<th>Price</th>
<th># of shares</th>
<th>Investment</th>
<th>Scenarios</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>$10</td>
<td>-2.02</td>
<td>-20.20</td>
<td>-17.17</td>
</tr>
<tr>
<td>B</td>
<td>$15</td>
<td>-2.02</td>
<td>-30.30</td>
<td>-37.875</td>
</tr>
<tr>
<td>C</td>
<td>$50.50</td>
<td>+1</td>
<td>50.50</td>
<td>56.00</td>
</tr>
</tbody>
</table>

Note that the zero investment portfolio must be recalculated \( (X = -2.02) \) and indeed the payoffs are no longer positive across the board, becoming negative for scenario 2. Thus the arbitrage opportunity has been eliminated. This exercise proves that the price increase of C will eliminate the arbitrage opportunity we found in 4a, the one that uses an equal number of shares of A and B to go short. However, it does not eliminate all arbitrage opportunities. For example, with \( P_C = \$50.50 \) an arbitrage portfolio can be formed with \( X_A = -1.95 \), and \( X_B = -2.07 \) for the number of shares of A and B sold short respectively.
5. Substituting the portfolio return and the betas in the expected return-beta relationship, we obtain two equations in the unknowns, the risk-free rate and the factor risk premium, $\text{RP}$.

\[ 12 = r_f + 1.2 \times \text{RP} \]
\[ 9 = r_f + 0.8 \times \text{RP} \]

Solving these equations, we obtain

$r_f = 3\%$ and $\text{RP} = 7.5\%$

6. a. Shorting equally the 10 negative-alpha stocks and investing the proceeds equally in the 10 positive-alpha stocks eliminates the market exposure and creates a zero-investment portfolio. Denoting the systematic market factor as $R_M$, the expected dollar return is (noting that the expectation of non-systematic risk, $e$, is zero):

\[
\$1,000,000 \times [.03 + 1.0 \times R_M] - \$1,000,000 \times [-.03 + 1.0 \times R_M]
\]

\[= \$1,000,000 \times .06 = $60,000\]

The sensitivity of the payoff of this portfolio to the market factor is zero because the exposures of the positive alpha and negative alpha stocks cancel out. (Notice that the terms involving $R_M$ sum to zero.) Thus, the systematic component of total risk also is zero. The variance of the analyst's profit is not zero, however, since this portfolio is not well diversified.

For $n = 20$ stocks (i.e., long 10 stocks and short 10 stocks) the investor will have a $\$100,000$ position (either long or short) in each stock. Net market exposure is zero, but firm-specific risk has not been fully diversified. The variance of dollar returns from the positions in the 20 firms is

\[20 \times [(100,000 \times .30)^2] = 18,000,000,000\]

and the standard deviation of dollar returns is $\$134,164$.

b. If $n = 50$ stocks (25 long and 25 short), $\$40,000$ is placed in each position, and the variance of dollar returns is

\[50 \times [(40,000 \times .30)^2] = 7,200,000,000\]

The standard deviation of dollar returns is $\$84,853$. 

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Similarly, if \( n = 100 \) stocks (50 long and 50 short), $20,000 is placed in each position, and the variance of dollar returns is

\[
100 \times [(20,000 \times .30)^2] = 3,600,000,000
\]

The standard deviation of dollar returns is $60,000.

Notice that when the number of stocks increased by a factor of 5, from 20 to 100, standard deviation fell by a factor of \( \sqrt{5} = 2.236 \), from $134,164 to $60,000.

7  a. \( \sigma^2 = \beta^2 \sigma_M^2 + \sigma^2(e) \)

The standard deviations are:

<table>
<thead>
<tr>
<th></th>
<th>( \sigma(e) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>25</td>
</tr>
<tr>
<td>B</td>
<td>10</td>
</tr>
<tr>
<td>C</td>
<td>20</td>
</tr>
</tbody>
</table>

and \( \sigma_M = 20 \). Thus,

\[
\sigma_A^2 = .8^2 \times 20^2 + 25^2 = 881
\]
\[
\sigma_B^2 = 1.0^2 \times 20^2 + 10^2 = 500
\]
\[
\sigma_C^2 = 1.2^2 \times 20^2 + 20^2 = 976
\]

b. If there are an infinite number of assets with identical characteristics, a well-diversified portfolio of each type will have only systematic risk since the non-systematic risk will approach zero with large \( n \). The mean will equal that of the individual (identical) stocks.

c. There is no arbitrage opportunity because the well-diversified portfolios all plot on the security market line (SML). Because they are fairly priced, there is no arbitrage.

8. a. A long position in a portfolio, \( P \), comprised of stocks A and B will offer an expected return-beta tradeoff lying on a straight line between points A and B. Therefore, we can choose weights so that \( \beta_P = \beta_C \) but with expected return higher than that of portfolio C. Hence, combining \( P \) with a short position in \( C \) will create an arbitrage portfolio with zero investment, zero beta, and positive rate of return.

b. The argument in (a) leads to the proposition that the coefficient of \( \beta^2 \) must be zero to preclude arbitrage opportunities.
9. Any pattern of returns can be "explained" if we are free to choose an indefinitely large number of explanatory factors. If a theory of asset pricing is to have value, it must explain returns using a reasonably limited number of explanatory variables (systematic factors).

10. The APT factors must correlate with major sources of uncertainty, i.e., sources of uncertainty that are of concern to many investors. Researchers should investigate factors that correlate with uncertainty in consumption and investment opportunities. GDP, the inflation rate, and interest rates are among the factors that can be expected to determine risk premiums. In particular, industrial production (IP) is a good indicator of changes in the business cycle. Thus, IP is a candidate for a factor that is highly correlated with uncertainties that have to do with investment and consumption opportunities in the economy.

11. a. \[ E(r) = 6 + 1.2 \times 6 + .5 \times 8 + .3 \times 3 = 18.1\% \]

b. Surprises in the macroeconomic factors will result in surprises in the return of the stock: Unexpected return from macro factors = \[1.2 \times (4 - 5) + .5 \times (6 - 3) + .3 \times (0 - 2) = -0.3\%\]

12. The APT required (i.e., equilibrium) rate of return on the stock based on \(r_f\) and the factor betas is:

\[ \text{required } E(r) = 6 + 1 \times 6 + .5 \times 2 + .75 \times 4 = 16\% \]

According to the equation for the return on the stock, the actually expected return on the stock is 15% (because the expected surprises on all factors by definition are zero). Because the actually expected return based on risk is less than the equilibrium return, we conclude that the stock is overpriced.

13. b.

14. c.

15. d.

16. d.

17. (c) Investors will take on as large a position as possible only if the mispricing opportunity is an arbitrage. Otherwise, considerations of risk and diversification will limit the position they attempt to take in the mispriced security.

18. d.

19. d.