1. (15) Define or state each of the following.
   (a) Brouwer’s Fixed Point Theorem
   (b) contraction mapping
   (c) metric

2. (30) Prove that for every $n \in \mathbb{N}$,
   $$\sum_{k=1}^{n} k^3 = \frac{1}{4} (n(n + 1))^2$$

3. (30) Let $X$ and $Y$ be vector spaces over the same field $F$, and $T \in L(X,Y)$, that is, $T : X \rightarrow Y$ is a linear transformation.
   (a) Show that $\ker T$ is a vector subspace of $X$ and that $\text{Im} T$ is a vector subspace of $Y$.
   (b) Suppose $\dim X = \dim Y$ and $\ker T = \{0\}$. Show that if $V = \{v_1, \ldots, v_n\}$ is a basis for $X$, then $\{T(v_1), \ldots, T(v_n)\}$ is a basis for $Y$. 
4. (30) Consider \( \mathbb{R} \) with the usual metric.

(a) Let \( C = \{ \frac{n}{n^2+1} : n = 0, 1, 2, 3, \ldots \} \). Show directly from the definition that \( C \) is compact.

(Note: An otherwise correct answer that does not use the open cover definition will receive 10 points.)

(b) Let \( C_1 = C \setminus \{0\} = \{ \frac{n}{n^2+1} : n = 1, 2, 3, \ldots \} \). Is \( C_1 \) compact? Justify your answer.

(Note: Answers with no justification will receive no points.)

5. (30) Let \( f : A \to \mathbb{R}^m \) be continuous, where \( A \subseteq \mathbb{R}^n \) is open and convex. Show that if \( f \) is differentiable on \( A \) and \( \| df_x \| \) is bounded on \( A \), then \( f \) is uniformly continuous on \( A \).

6. (30) Suppose \( \Psi_1, \Psi_2 : X \to 2^Y \) are closed-valued, upper hemicontinuous correspondences, where \( X \subseteq \mathbb{R}^n \), \( Y \subseteq \mathbb{R}^m \) for some \( n, m \). Suppose that \( \Psi_1(x) \cap \Psi_2(x) \neq \emptyset \) for each \( x \in X \). Show that \( \Psi_1 \cap \Psi_2 \) is upper hemicontinuous, where \( \Psi_1 \cap \Psi_2 : X \to 2^Y \) is defined by

\[
(\Psi_1 \cap \Psi_2)(x) = \Psi_1(x) \cap \Psi_2(x) \quad \forall x \in X
\]

(Note: For full credit, the answer will have to directly use the definition of upper hemicontinuity. An otherwise correct answer that uses alternative characterizations of upper hemicontinuity will receive 50% credit, provided any necessary additional assumptions are clearly stated.)