1. Let $X$ be a finite set. A random choice function is a function $P : 2^X \setminus \emptyset \to [0,1]^X$ such that $P(A)(x) \geq 0$, $\sum_{a \in A} P(A)(a) = 1$, and $\sum_{b \notin A} P(A)(b) = 0$. For notational ease, let $P_A = P(A)$. That is, a random choice function takes a menu of options and outputs a probability distribution over the menu, where $P_A(x)$ denotes the probability that $x$ is chosen from menu $A$.

A random choice function admits a Luce representation if there exists a set of weights $\{w(x) : x \in X\}$ such that

$$P_A(x) = \begin{cases} \frac{w(x)}{\sum_{a \in A} w(a)} & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

Let $\mathcal{P}$ denote the set of all bijections from $X$ to $\{1,2,\ldots,|X|\}$. Note that $\mathcal{P}$ is equivalent to the set of all strict rankings of objects in $X$, where $f(x) = 1$ is interpreted as $x$ being the most preferred object under ranking $f$. A random choice function admits a Falmagne representation if there exists a probability distribution $\mu \in [0,1]^\mathcal{P}$ (that is, $\mu(f) \geq 0$ and $\sum_{f \in \mathcal{P}} \mu(f) = 1$) such that

$$P_A(x) = \begin{cases} \sum \{\mu(f) : f \text{ such that } f(x) \leq f(a) \text{ for all } a \in A\} & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

A random choice function satisfies Random Independence of Irrelevant Alternatives if

$$\frac{P_A(x)}{P_A(y)} = \frac{P_B(x)}{P_B(y)}$$

whenever $P_A(x), P_A(y), P_B(x), P_B(y) > 0$.

(a) Interpret the Luce and the Falmagne representations.

(b) Interpret the Random IIA condition.

(c) Prove or provide a counterexample to the following statement: Let $P$ admit a Luce representation. Define $C(A) = \{x \in A : P_A(x) > 0\}$. Then $C$ is rationalizable.

(d) Prove or provide a counterexample to the following statement: If $P$ admits a Luce representation, then it satisfies Random IIA.

(e) Suppose $P_A(a) > 0$ whenever $a \in A$. Prove that if $P$ satisfies Random IIA, then $P$ admits a Luce representation. (Hint: Consider a candidate for $w$.)

(f) Prove or provide a counterexample to the following statement: If $P$ admits a Falmagne representation, then $P$ satisfies Random IIA.

2. Consider the following classical condition:

**Definition 0.1.** A choice rule $\mathcal{C}$ satisfies path independence if, for all $A, B \in 2^X \setminus \emptyset$,

$$\mathcal{C}(A \cup B) = \mathcal{C}(\mathcal{C}(A) \cup \mathcal{C}(B))$$

(a) Prove the following claim: If $\mathcal{C}$ is nonempty and rationalizable, then $\mathcal{C}$ satisfies path independence.

(b) Provide an example of a nonempty choice rule that satisfies path independence, but is not rationalizable.