Questions

• Question 1
Consider a group of individuals $A$, $B$ and $C$ and the relation at least as tall as in $A$ is at least as tall as $B$. Does this relation satisfy the completeness and transitivity properties? Take the same group of individuals as above and consider the relation strictly taller than. Is it complete? Is this relation transitive?

• Question 2
Determine if completeness and transitivity are satisfied for the following preferences defined on $x = (x_1, x_2)$ and $y = (y_1, y_2)$.

- $x \succeq y$ iff (if and only if) $x_1 \geq y_1$ and $x_2 \geq y_2$ (solved as an example).
- $x \succeq y$ iff $\min\{x_1, x_2\} \geq \min\{y_1, y_2\}$, and
- $x \succeq y$ iff $x_1 > y_1$ or $x_1 = y_1$ and $x_2 > y_2$.

• Question 3
Determine if completeness and transitivity are satisfied for the following preferences defined on $x = (x_1, x_2)$ and $y = (y_1, y_2)$

$$x \succeq y \text{ iff } \max\{x_1, x_2\} \geq \max\{y_1, y_2\}$$

Illustrate a typical indifference curve graphically (Hint: pick a bundle $x = (x_1, x_2)$ and think what are the set of bundles that the consumer indifferent between them and $x = (x_1, x_2)$). Accordingly, determine and explain graphically whether this preference relation satisfies convexity.
Answers

• Question 1

The relation \textit{at least as tall as} is complete and transitive.

– To verify completeness, pick any two individuals $A$ and $B$. Clearly, either individual $A$ is at least as tall as individual $B$ or individual $B$ is at least as tall as individual $A$ or both.

– For transitivity, pick three individuals $A$, $B$ and $C$ and suppose that individual $A$ is at least as tall as individual $B$ and individual $B$ is at least as tall as individual $C$. Obviously, individual $A$ must be at least as tall as individual $C$. Thus, the relation at least as tall as satisfies the transitivity property.

The relation \textit{strictly taller than} does not satisfy completeness but is transitive.

– In order to see that completeness fails, pick two individuals $A$ and $B$ with the same height. Clearly, it is not true that individual $A$ is strictly taller than individual $B$ and not true that individual $B$ is strictly taller than individual $A$. Thus, the relation strictly taller than is not complete since two individuals of the same height cannot be compared.

– For transitivity, pick three individuals $A$, $B$ and $C$ and suppose that individual $A$ is strictly taller than individual $B$ and individual $B$ is strictly taller than individual $C$. Obviously, individual $A$ must be also strictly taller than individual $C$. Thus, the relation strictly taller than satisfies the transitivity property.

• Question 2

$x \succ y$ iff (if and only if) $x_1 \geq y_1$ and $x_2 \geq y_2$.

– Not complete: consider the following counter example: $x = (0, 1)$ and $y = (1, 0)$. Clearly, neither $x_i \geq y_i$ for all $i$ nor $y_i \geq x_i$ for all $i$. So, neither $x \succ y$ nor $y \succ x$. Hence, the bundles $x = (0, 1)$ and $y = (1, 0)$ cannot be compared.

– Transitive: pick $x = (x_1, x_2)$, $y = (y_1, y_2)$ and $z = (z_1, z_2)$ and suppose that $x \succ y$ and $y \succ z$ towards showing that $x \succ z$. By assumption, since $x \succ y$ then $x_i \geq y_i$ for all $i$ and since $y \succ z$ $y_i \geq z_i$ for all $i$. That is,

\[
x_1 \geq y_1 \text{ and } x_2 \geq y_2
\]

and

\[
y_1 \geq z_1 \text{ and } y_2 \geq z_2
\]
Hence,

\[ x_1 \geq z_1 \text{ and } x_2 \geq z_2 \]

Therefore, \( x \succ y \) and \( y \succ z \) imply that \( x \succ z \).

- Strongly monotonic: first, let’s recall the definitions of monotonicity: we say that the preference relation \( \succ \) is monotonic if for any two bundles \( x \) any \( y \) such that \( x \gg y \), \( x \succ y \) (by \( x \gg y \) we mean that each component of \( x \) is strictly larger than the corresponding component of \( y \)). And, we say that it is strongly monotonic if for any two bundles \( x \) any \( y \) such that \( x \geq y \) and \( x \neq y \) (by \( x \geq y \) we mean that \( x \) has at least as much of all components and strictly more of at least of one component). You should be able to show that if preference relation \( \succ \) is strongly monotonic, then it is monotonic. According to these definitions, the above preference relation is strongly monotonic: pick \( x = (x_1, x_2) \) and \( y = (y_1, y_2) \) such that

\[ x_1 > y_1 \text{ and } x_2 \geq y_2 \]

then \( x \succ y \) but not \( y \succ x \). Hence, \( x \succ y \). Since it is strongly monotonic it is also weakly monotonic.

\[ x \succ y \iff \min\{x_1, x_2\} \geq \min\{y_1, y_2\}. \]

- Complete: pick any \( x = (x_1, x_2) \) and \( y = (y_1, y_2) \). Clearly, either

\[ \min\{x_1, x_2\} \geq \min\{y_1, y_2\} \]

holds or,

\[ \min\{x_1, x_2\} \leq \min\{y_1, y_2\} \]

holds or both. Hence, either \( x \succ y \) or \( y \succeq x \) or both.

- Transitive: pick any \( x = (x_1, x_2), y = (y_1, y_2) \) and \( z = (z_1, z_2) \) and suppose that \( x \succ y \) and \( y \succeq z \). To show that transitivity we need that \( x \succeq y \). Since \( x \succeq y \)

\[ \min\{x_1, x_2\} \geq \min\{y_1, y_2\} \]

and since \( y \succeq z \)

\[ \min\{y_1, y_2\} \geq \min\{z_1, z_2\} \]

So, we conclude that

\[ \min\{x_1, x_2\} \geq \min\{z_1, z_2\} \]

which implies that \( x \succeq z \). Therefore, \( x \succeq y \) and \( y \succeq z \) imply that \( x \succeq z \).

\[ x \succeq y \iff x_1 > y_1 \text{ or } x_1 = y_1 \text{ and } x_2 > y_2. \]
- Not complete: for a counter example pick two bundles $x$ and $y$ such that $x = y$. For example, $x = (1, 1)$ and $y = (1, 1)$. Clearly, since $x_1 = y_1$ and $x_2 = y_2$ neither $x \succ y$ nor $y \succ x$. Hence, the two bundles $x = (1, 1)$ and $y = (1, 1)$ cannot be compared by this preference relation.

- Transitive: Pick $x = (x_1, x_2)$, $y = (y_1, y_2)$ and $z = (z_1, z_2)$ and suppose that $x \succ y$ and $y \succ z$ towards showing that $x \succ z$. By assumption, since $x \succ y$ then
  
either x_1 > y_1 \text{ or if } x_1 = y_1 \text{ then } x_2 > y_2 $$
  and since $y \succ z$ then
  
either y_1 > z_1 \text{ or if } y_1 = z_1 \text{ then } y_2 > z_2 $$
  Hence, it must hold that
  
either x_1 > z_1 \text{ or if } x_1 = z_1 \text{ then } x_2 > z_2 $$
  which implies that $x \succ z$.

- Question 3

Completeness and transitivity

- Complete: pick any $x = (x_1, x_2)$ and $y = (y_1, y_2)$. Clearly, either
  $$\max\{x_1, x_2\} \geq \max\{y_1, y_2\}$$
  holds or,
  $$\max\{x_1, x_2\} \leq \max\{y_1, y_2\}$$
  holds or both. Hence, either $x \succ y$ or $y \succ x$ or both.

- Transitive: pick any $x = (x_1, x_2)$, $y = (y_1, y_2)$ and $z = (z_1, z_2)$ and suppose that $x \succ y$ and $y \succ z$. To show that transitivity we need that $x \succ y$. Since $x \succ y$
  $$\max\{x_1, x_2\} \geq \max\{y_1, y_2\}$$
  and since $y \succ z$
  $$\max\{y_1, y_2\} \geq \max\{z_1, z_2\}$$
  So, we conclude that
  $$\max\{x_1, x_2\} \geq \max\{z_1, z_2\}$$
  which implies that $x \succ z$. Therefore, $x \succ y$ and $y \succ z$ imply that $x \succ z$.

A typical indifference curve is illustrated graphically in the figure attached from which it is obvious that this preference relation does not satisfy convexity.