This problem set is due in class on Monday, April 13th

1. Let \( \{X_n\} \) be a sequence of random variables. Assume that
   \[
   \sqrt{n}X_n \xrightarrow{d} N(0, 1)
   \]
   (a) Find the asymptotic distribution of \( e^{\sqrt{n}X_n} - \text{plim} X_n \) and \( \sqrt{n}(X_n^2 - \text{plim} X_n^2) \).
   (b) Find the asymptotic distribution of \( e^{\sqrt{n}X_n} \) and \( \sqrt{n}X_n^2 \).

2. Suppose that \( X_1, \ldots, X_n \) are independent and that it is known that \( (X_i)^\lambda - 10 \) has a standard normal distribution, \( i = 1, \ldots, n \). This is called the Box-Cox transformation.
   • Derive the second-round estimator \( \hat{\lambda}_2 \) of the Newton-Raphson iteration, starting from an initial guess that \( \hat{\lambda}_1 = 1 \).
   • For the following data, compute \( \hat{\lambda}_2 \):
     96, 125, 146, 76, 114, 69, 130, 119, 85, 106
   • Write a computer program to iterate to convergence or to 100 times.

3. Consider a discrete random variable \( N \) having probability mass function
   \[
   p_N(n; \theta^0) = \text{Prob}(N = n; \theta^0) = \frac{-(\theta^0)^n}{n \log(1 - \theta^0)} \quad n = 1, 2, \ldots, 0 < \theta^0 < 1
   \]
   which is often referred to as the logarithmic series distribution for reasons that will become clear later in the problem.
   (a) Prove that
      \[
      \sum_{n=1}^{\infty} p_N(n; \theta^0) = 1.
      \]
      (Hint: consider the infinite order taylor series expansion of \( \log(1 + x) \) and substitute in \( x = -\theta^0 \).)
   (b) Find the expected value of \( N \), \( E(N) \). (Hint: \( \sum_{n=1}^{\infty} \rho^n \frac{e}{1 - \rho} \).)
   (c) Find the variance of \( N \), \( V(N) \). (Hint: remember that the derivative of a sum is the sum of the derivatives of each of the sum’s parts.)
   (d) Define the maximum-likelihood estimator \( \hat{\theta}_{\text{mle}} \) of \( \theta^0 \).
   (e) After considerable effort, a researcher has obtained a random sample of one thousand measurements on \( N \). These data are summarized in Table 1.

\[
\begin{array}{c|c|c|c|c|c|c|c|c|c|c}
N & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
\hline
\text{Frequency} & 700 & 205 & 50 & 26 & 10 & 6 & 1 & 1 & 1
\end{array}
\]
(f) Write a matlab program that implements Newton’s method to calculate the maximum-likelihood estimate of \( \theta^0 \) using the above data.

(g) Write another matlab program to implement the bisection method to calculate the maximum likelihood estimate of \( \theta^0 \) using the above data. An introduction to the bisection method for solving a nonlinear equation of one variable can be found at:

http://www.library.cornell.edu/nr/bookcpdf/c9-1.pdf