1. *Capital asset pricing model* ("Classic" CAPM). Assume a two-period model in which each investor $i$ maximizes the expected value of a quadratic function of next period’s random wealth, $W^i$:

$$U^i = \mathbb{E}\left\{ \alpha W^i - \frac{\gamma}{2} (W^i)^2 \right\}.$$ 

Next period’s wealth $W^i$ depends on current wealth, $W_0^i$, the realized (net) returns on the $N$ risky assets in which wealth can be held, $\{r_j\}_{j=1}^N$, the nonrandom return $r_F$ on a riskless asset, and the investment shares $\{x^i_j\}_{j=1}^N$ of initial wealth that investor $i$ selects for the available risky assets:

$$W^i = W_0^i \left[ \sum_{j=1}^N x^i_j (1 + r_j) + \left( 1 - \sum_{j=1}^N x^i_j \right) (1 + r_F) \right].$$

(a) Show how to write the last equation as

$$W^i = W_0^i \left[ (1 + r_F) + \sum_{j=1}^N x^i_j (r_j - r_F) \right].$$

(b) Derive investor $i$’s first-order optimum condition with respect to $x^i_j$.

(c) Sum these optimum conditions over all $M$ investors $i$ to derive an equilibrium condition involving aggregate second-period wealth, $W \equiv \sum_{i=1}^M W^i$.

(d) Define the coefficient

$$\rho \equiv \frac{\gamma \mathbb{E}\{W\} / M}{\alpha - \gamma \mathbb{E}\{W\} / M}.$$ 

Show that we can interpret $\rho$ as a measure of “average” relative risk aversion.
(e) Define the (gross) “return on the market” as

\[ 1 + r_M = \frac{W}{W_0}, \]

where \( W_0 \equiv \sum_{i=1}^{M} W_0^i \). Show that the equilibrium condition from part c, above, can be put into the form:

\[ \mathbb{E}\{r_j - r_F\} = \frac{\rho \text{Cov}\{r_j, r_M\}}{\mathbb{E}\{1 + r_M\}}. \]

(f) What is the intuitive interpretation of the last condition?

(g) Show how to write the condition from part e, above, as

\[ \mathbb{E}\{r_j - r_F\} = \beta_j \mathbb{E}\{r_M - r_F\}, \]

where

\[ \beta_j \equiv \frac{\text{Cov}\{r_j, r_M\}}{\text{Var}(r_M)}. \]

The CAPM framework predicts that a risky asset’s “beta,” as defined here, determines the degree to which it can be expected to outperform the market as a whole. (This form of the model can be tested from market returns data alone, without assumptions on the degree of risk aversion.)