Notes on PTM Model with Interest Rule

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We start with the interest-rate rules
\[
\begin{align*}
\log(1 + i_t) &= \tau + \beta p_t - \alpha_w u^w_t - \alpha_o u^o_t, \\
\log(1 + i^*_t) &= \tau + \beta p^*_t - \alpha_w^* u^w_t + \alpha_o^* u^o_t,
\end{align*}
\]
where the technology processes follow
\[
\begin{align*}
a_{t+1} &= \lambda a_t + u_{t+1}, \\
a^*_t &= \lambda a^*_t + u^*_t,
\end{align*}
\]
and the innovations share the same variance \(\sigma^2_u\).

Pricing and ex ante consumption (using complete markets, no nontradables case)
\[
\frac{P(h)}{P(f)} = \frac{E\left(\frac{C}{A}\right)}{E\left(\frac{C^*}{A^*}\right)}, \quad \frac{P(f)^*}{P(h)^*} = \frac{E\left(\frac{C^*}{A^*}\right)}{E\left(\frac{C}{A}\right)}.
\]

From the last two equations we can derive
\[
E_t c_{t+1} = \frac{1}{\rho} \left[ \log \left( \frac{\theta - 1}{\theta \kappa} \right) + E_t a^w_{t+1} - \rho \left( 1 - \frac{\rho}{2} \right) \sigma^2_z - \frac{1}{2} \sigma^2_u + \sigma^2_c \right],
\]
in which the variances will be constant, with endogenous values to be derived below.

Solutions for price level and ex post consumption

Start from the Euler equation
\[
\frac{C^\rho_{t-1}}{P_t} = \beta (1 + i_t) E_t \left\{ \frac{C^\rho_{t+1}}{P_{t+1}} \right\}
\]
As \(P_{t+1}\) is predetermined on the basis of date \(t\) information, we may write this (after substitution of the interest-rate policy rule) as
\[
\rho \left( E_t c_{t+1} - c_t \right) = \log \beta + \bar{\tau} - [p_{t+1} - (1 + \gamma) p_t] + \rho^2 \sigma^2_z - \alpha_w u^w_t - \alpha_o u^o_t
\]
Taking expectations based on date \(t - 1\) information yields
\[
\rho \left( E_{t-1} c_{t+1} - E_{t-1} c_t \right) = \log \beta + \bar{\tau} - [E_{t-1} p_{t+1} - (1 + \gamma) p_t] + \rho^2 \sigma^2_z
\]
which can be solved for $p_t$ to yield the first-order difference equation

$$p_t = \frac{1}{1 + \gamma} \mathcal{E}_{t-1} p_{t+1} + \frac{1}{1 + \gamma} \left\{ \rho(\mathcal{E}_{t-1} c_{t+1} - \mathcal{E}_{t-1} c_t) - \left( \log \beta + \tau + \frac{\rho^2}{2} \sigma_c^2 \right) \right\}$$

Notice that

$$\mathcal{E}_{t-1} c_{t+1} - \mathcal{E}_{t-1} c_t = \frac{1}{\rho} (\lambda^2 - \lambda) a_t^{w}$$

so

$$p_t = \frac{1}{1 + \gamma} \mathcal{E}_{t-1} p_{t+1} + \frac{1}{1 + \gamma} \left\{ \lambda(\lambda - 1) a_t^{w} - \left( \log \beta + \tau + \frac{\rho^2}{2} \sigma_c^2 \right) \right\}$$

Solving this equation forward (and ruling out $p$ bubbles) gives

$$p_t = \frac{\lambda(\lambda - 1)}{1 + \gamma - \lambda} a_t^{w} - \frac{1}{\gamma} \left( \log \beta + \tau + \frac{\rho^2}{2} \sigma_c^2 \right)$$

Consider next ex post consumption. We again use the Euler equation to write

$$c_t = \mathcal{E}_{t+1} c_t - \frac{1}{\rho} \left[ \log \beta + \log(1 + i_t) - (p_{t+1} - p_t) + \frac{\rho^2}{2} \sigma_c^2 \right]$$

Here $i_t$ is predetermined unless it reacts to current shocks; assume temporarily it does not. Then we can assess the response of $c_t$ to an innovation in $a_t^{w}$ from

$$c_t = \frac{\lambda}{\rho} a_t^{w} + \text{(constants)}$$

so that

$$\frac{dc_t}{du_t^{w}} = \frac{1}{\rho} \left[ \log \beta + \log(1 + i_t) - \left( \frac{\lambda(\lambda - 1)}{1 + \gamma - \lambda} a_t^{w} - p_t \right) + \frac{\rho^2}{2} \sigma_c^2 \right]$$

which is zero if $\lambda = 0$.

Now let $i_t$ respond to current shocks and write consumption in terms of innovations:

$$c_t - \mathcal{E}_{t-1} c_t = \frac{\gamma \lambda}{\rho(1 + \gamma - \lambda)} u_t^{w} + \frac{1}{\rho} \left( \alpha_w u_t^{w} + \alpha_o u_t^{p} \right) = \frac{\gamma \lambda + \alpha_w (1 + \gamma - \lambda)}{\rho(1 + \gamma - \lambda)} u_t^{w} + \frac{\alpha_o u_t^{p}}{\rho}$$

The corresponding innovation in Foreign consumption equals:

$$c_t - \mathcal{E}_{t-1} c_t = \frac{\gamma \lambda + \alpha_w (1 + \gamma - \lambda)}{\rho(1 + \gamma - \lambda)} u_t^{w} - \frac{\alpha_o u_t^{p}}{\rho}$$

We may now compute the conditional variances and covariances

$$\sigma_c^2 = \left[ \frac{\gamma \lambda + \alpha_w (1 + \gamma - \lambda)}{\rho(1 + \gamma - \lambda)} \right]^2 \sigma_u^{w} + \left( \frac{\alpha_o}{\rho} \right)^2 \sigma_u^{p}$$

$$\sigma_p^2 = \left[ \frac{\gamma \lambda + \alpha_w (1 + \gamma - \lambda)}{\rho(1 + \gamma - \lambda)} \right]^2 \sigma_u^{w} + \left( \frac{\alpha_o}{\rho} \right)^2 \sigma_u^{p}$$

$$\sigma_{cu} = \frac{\gamma \lambda + \alpha_w (1 + \gamma - \lambda)}{\rho(1 + \gamma - \lambda)} \sigma_u^{w}$$

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Expected utility

We can write one-period Home expected utility as

\[
E_{t-1} \left\{ \frac{c_t^{1-\rho}}{1-\rho} - \frac{1}{2} \left( \frac{\theta - 1}{\theta} \right) \left( C_t^{1-\rho} + C_t^{1-\rho} \right) \right\}
\]

\[
= E_{t-1} \left\{ \frac{\theta - \frac{1}{2}(\theta - 1)(1-\rho)}{(1-\rho)\theta} C_t^{1-\rho} - \frac{1}{2} \left( \frac{\theta - 1}{\theta} \right) C_t^{1-\rho} \right\}
\]

\[
= \theta - \frac{1}{2}(\theta - 1)(1-\rho) \exp \left\{ (1-\rho)E_{t-1}c_t + \frac{(1-\rho)^2}{2} \sigma_e^2 \right\} - \frac{1}{2} \left( \frac{\theta - 1}{\theta} \right) \exp \left\{ (1-\rho)E_{t-1}c_t^* + \frac{(1-\rho)^2}{2} \sigma_e^2 \right\}
\]

The first part of this (Home welfare) boils down to

\[
\frac{-\sigma_e^2}{2} + \frac{\sigma_{eu^*}}{\rho}
\]