Macroeconomic Dynamics, **1**, 1997, 588–614. Printed in the United States of America.

DYNAMIC SEIGNIORAGE THEORY

An Exploration

MAURICE OBSTFELD
University of California, Berkeley
National Bureau of Economic Research
and
Centre for Economic Policy Research

This paper develops a dynamic model of seigniorage in which economies' equilibrium paths reflect the ongoing strategic interaction between an optimizing government and a rational public. The model extends existing positive models of monetary policy and inflation by explicitly incorporating the intertemporal linkages among budget deficits, debt, and inflation. A central finding is that the public's rational responses to government policies may well create incentives for the government to reduce inflation and the public debt over time. A sufficiently myopic government may, however, provoke a rising equilibrium path of inflation and public debt.

Keywords: Seigniorage, Dynamic Games, Time Consistency, Markov Perfect Equilibrium

1. INTRODUCTION

This paper develops a dynamic model of seigniorage whose equilibrium paths are generated by the ongoing strategic interaction of an optimizing government with a rational public. The model extends existing positive models of inflation by explicitly incorporating the intertemporal linkages among budget deficits, debt, and inflation. A central finding is that the public's rational responses to policies may lead the government to reduce inflation and the public debt over time, even in the absence of self-supporting reputational expectation mechanisms.

Recent research aiming to explain observed inflation patterns has proceeded along two main lines. The first focuses on the temptation to effect resource transfers

I am grateful for conversations with several individuals, particularly Guillermo Calvo, Torsten Persson, and Lars Svensson. Thanks for helpful comments and suggestions also go to Matthew Canzoneri, Drew Fudenberg, Alberto Giovannini, Vittorio Grilli, Dale Henderson, Eric Maskin, Jeffrey Miron, Dani Rodrik, Eric van Damme, innumerable anonymous referees, and participants in seminars at Harvard, Boston University, Georgetown, the World Bank, Rochester, Princeton, McGill, Montréal, Queen's, Chapel Hill, Virginia, MIT, Tilburg, and the NBER. Essential research support was provided by the NBER's Olin Visiting Scholar Program and by the National Science Foundation. The initial work on this paper was done during a visit to the Institute of International Economic Studies, University of Stockholm, in May 1988. Earlier versions were circulated as NBER working paper 2869 (February 1989) and CEPR discussion paper 519 (March 1991). Address correspondence to: Maurice Obstfeld, Department of Economics, 549 Evans Hall, #3880, University of California, Berkeley, Berkeley, CA 94720-3880, USA; e-mail: obstfeld@econ. Berkeley. EDU.

 1365-1005/97 $9.00 + .10

from the private sector to the government through surprise inflation. The second stresses a dynamic aspect of the public-finance problem: the optimal distribution over time of inflation distortions. A brief review of the predictions and limitations of these two approaches—which may be called the *discretionary-policy* approach and the *inflation-smoothing* approach—puts the goals of the present exploration into perspective.

Calvo (1978), Barro (1983), and others have observed that the temptation to tax cash balances through surprise inflation may lead to higher inflation and lower seigniorage revenue than would result if the government were deprived of its discretionary powers and bound instead to a prior choice of the price level's path. The incentive to violate such prior commitments in later periods is an example of the general problem of time inconsistency: Optimal government plans that affect current household choices may no longer seem optimal after households have made those choices.[1]

Whereas the discretionary-policy approach suggests that inflation will be higher than is socially optimal, the inflation-smoothing approach suggests that inflation, whatever its level today, will be persistent. The inflation-smoothing approach builds on Ramsey's principle of optimal taxation, which directs governments to adopt contingency plans for tax rates that equate the expected marginal losses from tax distortions in all future periods [Barro (1979)]. Mankiw (1987) and Grilli (1988, 1989) argue that because inflation is a distorting tax that inflicts economic costs on society, optimizing governments will base inflation plans on the Ramsey principle. These authors make the empirical prediction that the stochastic process for inflation will be a martingale, with expected future inflation equal to current inflation. A corollary of their results is that total government spending commitments—debt plus the present value of nondiscretionary outlays—also will follow a martingale.[2]

Examples of high and seemingly chronic inflation certainly abound, but there are many notable episodes as well of successful inflation reduction, often coupled with government fiscal consolidation. Fischer (1986, p. 14) observes that

> it is clear that inflationary bias is only a sometime thing. At the ends of the Napoleonic and Civil Wars, and World War I, Britain and the United States deflated to get back to fixed gold parities. These episodes too deserve attention in the dynamic inconsistency literature.

Needless to say, there are numerous much more recent examples.[3]

Available models of both discretionary policy and inflation smoothing suffer from theoretical limitations that leave them unable to throw light on such important episodes of government behavior. Most discretionary-policy models are intertemporal only in a superficial sense, because they lack any intrinsic sources of dynamic evolution. In particular, the models make no allowance for the dynamics of public debt or for the role that government budgets might play in the inflationary expectations of the public. Inflation-smoothing models, in contrast, place the determination of the public debt at center stage, but it is well known that the optimal plans that produce Ramsey tax rules are dynamically inconsistent except in very special cases. The behavior predicted by these models generally will not be observed when the government can set policy anew each period at its discretion.[4]

The model developed in this paper synthesizes elements of the discretionary-policy and inflation-smoothing approaches in a genuine dynamic setting that assumes rational private-sector expectations. Consonant with the first approach, the model predicts that *at each point in time* at which inflation is positive, it will be higher than it would be if the government could commit itself in advance to future tax policies.[5] But, consonant with the second approach, the theory also predicts that for plausible parameters, government tax-smoothing behavior can generate an inflation rate with a tendency to fall *over time* toward the socially preferred long-run rate (zero in my model).[6] The basic reason is that government budgetary conditions affect inflationary expectations, thus giving the authorities additional incentives to retire debt and thereby reduce future seigniorage needs. Equilibria with persistently high inflation cannot, however, be ruled out in general.

In technical terms, the investigation is an application of dynamic game theory to interactions between public and private sectors. The endogenous variable responsible for economic dynamics is the stock of government spending commitments, including the public debt. In the equilibria that I construct, the government's monetary policy actions are always optimal, given household behavior and the economy's aggregate physical state; at the same time, private forecasts are always rational, given the government's strategy. Players' strategies are restricted, however, to be *memoryless*. Although this restriction rules out many potential equilibria, it serves to highlight recursive, Markov perfect equilibria that can be characterized in terms of a minimal set of currently relevant economic state variables. Even under a Markov restriction, equilibrium may not be unique. One somewhat novel aspect of the equilibria that I define is that government strategies prescribe choices of the money supply rather than of inflation itself, contrary to most of the literature.

Section 2 of the paper sets up a model monetary economy and describes the objectives of households and the government. Section 3 develops the definition of equilibrium. In Section 4, I characterize equilibrium in a perfect-foresight setting and describe the dynamics of government spending commitments. Section 5 uses a linear approximation to calculate equilibria explicitly under stochastic as well as deterministic assumptions. Section 6 contains concluding observations.

2. SETTING UP THE MODEL

The analytical setting for the model is from Brock (1974). This section and the next two simplify by assuming a deterministic environment, but a stochastic extension is studied in Section 5.

An overview of the sequence of events within each discrete time period is as follows. Households and the government enter a period t holding net asset stocks dated $t-1$, along with the interest that those assets pay out at the start of the new period. Goods and asset markets then meet simultaneously. The government finances its consumption purchases and net debt retirement by printing money; at the same time, households consume and decide what level of monetary balances (dated t) to carry over to the start of period $t+1$. The equilibrium interaction of government and private decisions in period-t markets determines the overall money price level for date t and the t-dated stocks of government and household assets that are carried over to the start of period $t+1$.

2.1. Households

The economy is populated by a large fixed number of identical households that take the economy's aggregates and prices as given. A household's satisfaction depends only on its own consumption of a single composite good and on transaction services from holding real monetary balances. (Public consumption does not directly affect household utility.)

The notation uses lowercase letters for household choice variables and uppercase letters for the corresponding economywide *per-household* totals, which are averages of individual household choices. Thus, for example, m is a particular household's choice of real monetary balances, and M is total *real* monetary balances per household. When there is no risk of confusion, I refer to economywide quantities per household simply as *aggregate* quantities.

At the start of period t, households maximize

$$U_t = \sum_{\tau=t}^{\infty}(1+\rho)^{-(\tau-t)}[c_\tau + \vartheta(m_\tau)], \qquad \textbf{(1)}$$

where $\rho \in (0,1)$. The period utility function for money, $\vartheta(m)$, is twice continuously differentiable, increasing, and strictly concave on $[0,\infty)$.

Let $\boldsymbol{P}_t$ denote the economy's money price level during period t. (Throughout the paper, boldface letters denote variables with values proportional to the monetary unit.) The inflation rate from period t to $t+1$, π_{t+1}, is the tax rate on currency, given by

$$\pi_{t+1} = (\boldsymbol{P}_{t+1} - \boldsymbol{P}_t)/\boldsymbol{P}_{t+1}.$$

Its maximum value, $\pi = 1$, is the confiscatory rate.

Linearity of utility in consumption fixes the equilibrium real interest rate at ρ. By assumption, all nonmoney assets offer the rate of return ρ ex post, that is, are indexed to the price level. The household thus maximizes U_t in equation (1) subject to a given level of real wealth at the end of period $t-1$, w_{t-1}, given its real monetary balances, m_{t-1}, and an intertemporal budget constraint. The intertemporal constraint comes from integrating a sequence of period-by-period finance constraints of the form

$$w_t = (1+\rho)w_{t-1} - c_t - (\rho + \pi_t)m_{t-1}$$

and imposing the solvency condition $\lim_{t\to\infty}(1+\rho)^{-t}w_t \geq 0$.[7]

It is well known [see Brock (1974)] that for a given expected inflation rate, the optimal household choice of period-t real balances, m_t, satisfies

$$\vartheta'(m_t) = \frac{\rho + \pi_{t+1}}{1+\rho}. \tag{2}$$

Thus, m_t is a decreasing function of expected inflation between t and $t+1$.[8] Because utility is linear in consumption, moreover, the optimal m_t in (2) depends *only* on π_{t+1} (given ρ), a fact that I use below to simplify the description of Markov perfect equilibria.

2.2. Government

The government's goal is to finance at minimum welfare cost an exogenous path of aggregate public consumption purchases per household, G_t. A finance constraint links the change in government debt to the difference between government consumption and net revenue. To simplify, I assume that money creation is the only form of taxation available to the government.

The government's social welfare criterion is

$$V_t = \sum_{\tau=t}^{\infty}(1+r)^{-(\tau-t)}[C_\tau + z(M_\tau)], \tag{3}$$

where $z(M)$ is a nondecreasing, concave, twice continuously differentiable function of the representative household's real balances. The government discount rate r may equal the market rate ρ, but it could exceed ρ if, for example, the current government's rule is subject to termination on a random date. The literature on tax smoothing generally assumes $r = \rho$ to obtain its martingale prediction for tax rates (including inflation). I assume $r \geq \rho$.

The function $z(M)$ in (3) describes the government's welfare valuation of the services that households derive from real money holdings; but it does not coincide with the household utility function $\vartheta(m)$. Most important, I assume that $z'(M) = 0$ for M exceeding the level of real balances that households demand when the *expected inflation rate* is zero. As becomes clear in Section 4, this assumption, together with the inflation-cost function that I posit in equation (8), serves to pin down $\pi = 0$ unambiguously as the government's long-run target inflation rate.[9]

Let D_t denote the aggregate per-household stock of real government nonmoney debt at the end of period t. All debts (assets when negative) are consumption-indexed bonds paying the real interest rate, ρ. The government's period finance constraint is

$$D_t = (1+\rho)D_{t-1} + G_t - [\pi_t M_{t-1} + (M_t - M_{t-1})]. \tag{4}$$

The term in square brackets above is government *seigniorage* revenue in period t, the sum of (i) the inflation tax on real monetary balances carried over from period $t-1$ and (ii) households' desired increase in real balances in period t.[10]

The government's intertemporal budget constraint comes from integrating (4), assuming no Ponzi finance, $\lim_{t\to\infty}(1+\rho)^{-t}D_t \le 0$:

$$\sum_{\tau=t}^{\infty}(1+\rho)^{-(\tau-t)}G_\tau + (1+\rho)D_{t-1}$$
$$\le \sum_{\tau=t}^{\infty}(1+\rho)^{-(\tau-t)}[\pi_\tau M_{\tau-1} + (M_\tau - M_{\tau-1})]$$
$$= -(1+\rho)M_{t-1} + \sum_{\tau=t}^{\infty}(1+\rho)^{-(\tau-t)}(\rho+\pi_\tau)M_{\tau-1}. \tag{5}$$

Comparison of (5) with the household's period finance constraint, $w_t = (1+\rho)w_{t-1} - c_t - (\rho+\pi_t)m_{t-1}$, shows that private expenditures on money services *less* initial real balance holdings, M_{t-1} (a government liability and a corresponding household asset), equals the resources that government obtains from seigniorage.[11] Constraint (5) highlights a fact central to solving the model: The government's fiscal position at the start of a period t depends entirely on the two liability stocks, D_{t-1} and M_{t-1}, carried over from the previous period, and on the present discounted value of committed government purchases for period t and after.

Define real government *commitments* at the end of period $t-1$, K_{t-1}, by

$$K_{t-1} \equiv \frac{1}{(1+\rho)}\sum_{\tau=t}^{\infty}(1+\rho)^{-(\tau-t)}G_\tau + D_{t-1}. \tag{6}$$

When written in terms of this new variable, constraint (5) becomes

$$(1+\rho)(K_{t-1}+M_{t-1}) \le \sum_{\tau=t}^{\infty}(1+\rho)^{-(\tau-t)}(\rho+\pi_\tau)M_{\tau-1}. \tag{7}$$

2.3. Technology and the Output Cost of Inflation

The economy is endowed with an exogenously fixed "potential" aggregate output level, Y, but output is perishable and cannot be transformed into capital for future use. Consumption need not equal potential output, however, because the amount of output available for consumption falls as the economy's inflation rate diverges from zero. Specifically, private consumption will be

$$C_t = Y - G_t - \kappa(\pi_t) \tag{8}$$

in equilibrium, where $\kappa(0) = 0$, $\kappa'(0) = 0$, $\kappa''(\pi) > 0$ for all π, and $\kappa'(\pi)$ has the same sign as π.

The inflation-cost function $\kappa(\pi)$ in (8) is meant to capture costs distinct from the inflation-tax distortion of money demand, for example, the reduction in allocative efficiency often said to accompany a rise in inflation.[12] In the stochastic version of the model, $\kappa(\pi)$ comprises costs of unanticipated as well as anticipated inflation. The "shoe-leather" welfare costs associated with inefficiently low money demand, in contrast, are entirely due to anticipated inflation.

The assumption that $\kappa(\pi)$ has its minimum at $\pi = 0$ is somewhat arbitrary, but it is congruent with the earlier assumption that the government's period objective for private real balances, $z(M)$, reaches a maximum where $\pi = 0$. Together, these two assumptions make $\pi = 0$ the government's target inflation rate.

3. EQUILIBRIUM WITHOUT COMMITMENT: DEFINITION

The government is assumed to be unable to precommit its future monetary policy actions. (It is committed only to paying its nonmonetary debts and to following the given expenditure path $\{G_t\}$.) The government instead sets the nominal money supply $\boldsymbol{M}_t$ in every period t so as to maximize the objective function in (3). Households observe the government's choice of $\boldsymbol{M}_t$ and then choose the levels of real balances they will carry into period $t+1$.

Equilibrium paths for the economy are defined by government and household policy functions such that:

1. The government's policy function maximizes its objective (3) in any state of the economy, given the government budget constraint and the behavior of aggregate money demand induced by household decision rules.
2. The representative household policy function maximizes private utility in any state of the economy, given the government's policy function and aggregate money demand.

3.1. Inflation Rate and State Transitions

Let $\boldsymbol{M}_{t-1}$ be the aggregate *nominal* money supply (per household) at the end of period $t-1$. When markets meet in period t, the government prints $\boldsymbol{M}_t - \boldsymbol{M}_{t-1}$ currency units with which it purchases goods and assets from the public. The government's policy moves are most conveniently formulated as choices of gross growth rates for the nominal money supply, $\gamma_t \equiv \boldsymbol{M}_t/\boldsymbol{M}_{t-1}$.

The aggregate *state* of the economy when period t starts is observed by households and government and is given by the vector

$$S_t = (K_{t-1}, M_{t-1}).$$

I assume a Markov perfect equilibrium, in which players' strategies are stationary functions of the state of the economy and depend on the past history of play only through that state. However, household strategies also are functions of contemporaneous money-supply growth, which households observe before making the period's money-demand decisions. Date-t money-supply growth is informative about S_{t+1} and thus about the following period's inflation, which in turn influences date-t money demand.

To make intertemporal decisions, the government and private sector alike must understand how alternative nominal money-supply growth rates affect inflation and the economy's state. This understanding, in turn, presupposes rational beliefs about how *aggregate* (per household) demand for real balances is determined. Without loss of generality, assume that households and the government take as given the aggregate real money-demand schedule

$$M_t = L(\gamma_t, S_t). \tag{9}$$

It is shown later that a schedule of this form is consistent with optimal household and government behavior.

The interaction between aggregate real money demand M_t and the government's choice of nominal money-supply growth determines the equilibrium period-t price level $\boldsymbol{P}_t = \boldsymbol{M}_t/M_t$. Because $\boldsymbol{P}_{t-1}$ is given by history, money-supply growth also determines the realized inflation rate between periods $t-1$ and t, $\pi_t = (\boldsymbol{P}_t - \boldsymbol{P}_{t-1})/\boldsymbol{P}_t$. Players' forecasts of inflation can be expressed in terms of nominal money growth and the current state through the equation

$$\pi_t = 1 - (\boldsymbol{P}_{t-1}/\boldsymbol{P}_t) = 1 - [L(\gamma_t, S_t)/M_{t-1}] \times (1/\gamma_t), \tag{10a}$$

to be denoted by

$$\pi_t = \Pi(\gamma_t, S_t). \tag{10b}$$

Through definition (6) and equation (9), the government's perceived finance constraint (4), expressed in terms of commitments, is

$$K_t = (1+\rho)K_{t-1} - \{\pi_t M_{t-1} + [L(\gamma_t, S_t) - M_{t-1}]\}.$$

The preceding equation and (10a) together imply that

$$K_t = (1+\rho)K_{t-1} - [1 - (1/\gamma_t)]L(\gamma_t, S_t), \tag{11a}$$

to be denoted by

$$K_t = \Delta(\gamma_t, S_t). \tag{11b}$$

Equations (9) and (11b) together yield the state transition equation that agents take as given,

$$S_{t+1} = [\Delta(\gamma_t, S_t), L(\gamma_t, S_t)], \tag{12a}$$

which defines the function $\Psi\colon \mathrm{R}^3 \to \mathrm{R}^2$ such that

$$S_{t+1} = \Psi(\gamma_t, S_t). \tag{12b}$$

3.2. Government's Policy Rule

Consider first the problem faced by the government when it takes the money-demand schedule (9) as given. Let $V(S_t) = V(K_{t-1}, M_{t-1})$ be the government's value function evaluated at the start of period t, that is, the result of maximizing V_t in (3) subject to (7), (9), (10b), and (12b). It is clear from equation (7) that both of the partial derivatives $\partial V/\partial K_{t-1}$ and $\partial V/\partial M_{t-1}$ are less than or equal to zero.

The government's optimal policy choice in period t can be characterized with the help of Bellman's equation. By equations (3), (8), and (9), $V(S)$ satisfies the recursion

$$V(S_t) = \max_{\gamma_t}\left\{Y - G_t - \kappa(\pi_t) + z[L(\gamma_t, S_t)] + \frac{1}{(1+r)}V(S_{t+1})\right\}, \tag{13}$$

subject to equations (10b) and (12b). By direct substitution of the constraints, the government's optimal choice of period-t money growth γ_t maximizes

$$Y - G_t - \kappa[\Pi(\gamma_t, S_t)] + z[L(\gamma_t, S_t)] + \frac{1}{(1+r)}V\{\Delta(\gamma_t, S_t), L(\gamma_t, S_t)\}. \tag{14}$$

The maximizing value of γ_t, assumed to exist and be unique, defines the policy-choice function

$$\gamma_t = \Gamma(S_t). \tag{15}$$

3.3. Household's Decision Rule

Each household observes the government's choice of γ_t and uses this information to decide on its own period-t real balances m_t. A household strategy is represented by the policy function

$$m_t = \ell(\gamma_t, S_t).$$

The intuitive motivation for this policy function comes from the money-demand equation (2), which makes individual money demand a negative function of expected inflation. A government's incentive to inflate on date $t+1$ is higher when its real commitments at that period's start, K_t, are higher, and when aggregate real money holdings, M_t, are higher. Households thus will forecast the inflation rate π_{t+1} by calculating how the money-supply growth decision γ_t affects K_t and M_t, given K_{t-1} and M_{t-1}. Notice that household wealth does *not* enter the policy function for real balances because the marginal utility of consumption was assumed to be independent of wealth in (1). [In equilibrium, of course, $\ell(\gamma_t, S_t)$ must equal the aggregate function $L(\gamma_t, S_t)$ in (9).]

3.4. Equilibrium

Equilibrium now may be defined. By assuming that government and household choices are functions of the *minimal* sets of variables compatible with perfection, I have restricted the analysis to recursive, Markov perfect equilibria of the type studied by Fudenberg and Tirole (1986), Bernheim and Ray (1987), and Maskin and Tirole (1988), among others. The force of focusing on Markov perfect equilibria is to exclude other potential equilibria involving strategies with memory, as in the reputational models discussed in Rogoff's (1989) critical survey.[13]

DEFINITION 1. *Let the state of the economy at the start of a period t be* $S_t = (K_{t-1}, M_{t-1})$. *An* equilibrium *consists of a government policy function* $\gamma = \Gamma(S)$, *a household policy function* $m = \ell(\gamma, S)$, *and a state transition equation* $S_{\tau+1} = \Psi(\gamma_\tau, S_\tau)$, *such that for all dates t and any starting state* S_t, *the following hold*:

(i) *Government maximization. The choice* $\gamma_t = \Gamma(S_t)$ *maximizes*

$$V_t = Y - G_t - \kappa[\Pi(\gamma_t, S_t)] + z\left[L(\gamma_t, S_t)\right] + \sum_{\tau=t+1}^{\infty}(1+r)^{-(\tau-t)}\mathbf{(}Y - G_\tau - \kappa\{\Pi[\Gamma(S_\tau), S_\tau]\} + z\{L[\Gamma(S_\tau), S_\tau]\}\mathbf{)}$$

subject to the government intertemporal budget constraint

$$(1+\rho)(K_{t-1} + M_{t-1}) \le [\rho + \Pi(\gamma_t, S_t)]M_{t-1} + \sum_{\tau=t+1}^{\infty}(1+\rho)^{-(\tau-t)}\{\rho + \Pi[\Gamma(S_\tau), S_\tau]\}L[\Gamma(S_{\tau-1}), S_{\tau-1}]$$

and the transition equations

$$S_{t+1} = \Psi(\gamma_t, S_t),$$

$$S_{\tau+1} = \Psi[\Gamma(S_\tau), S_\tau], \qquad \tau > t.$$

(ii) *Household maximization. The choice* $m_t = \ell(\gamma_t, S_t)$ *satisfies equation* (2) *when each household takes the government's strategy* $\Gamma(S)$ *and those of other households as given and forecasts inflation using* (10b) *and* (12b),

$$\vartheta'[\ell(\gamma_t, S_t)] = \frac{\mathbf{(}\rho + \Pi\{\Gamma[\Psi(\gamma_t, S_t)], \Psi(\gamma_t, S_t)\}\mathbf{)}}{1+\rho}. \tag{16}$$

(iii) *Rational expectations.* $\ell(\gamma_t, S_t) = L(\gamma_t, S_t)$.

An equilibrium government strategy thus prescribes an optimal action at each date and state, given future implementation of the same strategy, and given the strategies of private actors. Equilibrium household strategies, similarly, prescribe optimal actions at each date and state, given the government's strategy and those of the other households.

The equilibrium concept just described characterizes outcomes of a dynamic game of alternating moves by the government and private sector. By construction, any equilibrium is subgame-perfect. Gale (1982, Sec. 3.4) refers to this type of equilibrium as a "perfect leader-follower equilibrium," Chari et al. (1989) call it a "time-consistent equilibrium," Chari and Kehoe (1990) call it a "sustainable plan," and Stokey (1991) names it a "credible policy." Cole and Kehoe (1996), who restrict their analysis to Markov strategies as this paper does, use the term "recursive equilibrium."

4. EQUILIBRIUM WITHOUT COMMITMENT: CHARACTERIZATION

This section presents a qualitative picture of the economy's equilibrium path. That picture turns out to be quite simple when public and private time-preference rates coincide. In that case, inflation declines to zero over time as the government builds up a large enough asset stock to finance public spending out of interest receipts alone, without seigniorage. When the government's time-preference rate exceeds the private sector's, however, the economy may follow very different routes.

Some preliminary propositions are helpful in deriving these results. I assume that in the economy's initial position the government is creating money at a nonnegative rate, so that $\gamma \geq 1$.

4.1. Preliminary Results

The first preliminary result shows that in any equilibrium, higher rates of monetary growth are associated with higher current inflation rates and lower growth rates for public commitments.

PROPOSITION 1. *In an equilibrium with nonnegative money-supply growth,* $\partial\Pi/\partial\gamma_t > 0$ *and* $\partial\Delta/\partial\gamma_t < 0$*. That is, higher money-supply growth raises contemporaneous inflation and lowers the end-period stock of public commitments.*

Proof 1. Equations (10a) and (11a) show that, for all t,

$$\frac{\partial\Pi}{\partial\gamma_t} = \left(\frac{M_t}{\gamma_t^2 M_{t-1}}\right)\left[1 - \left(\frac{\gamma_t}{M_t}\frac{\partial L}{\partial\gamma_t}\right)\right], \tag{17}$$

$$\frac{\partial\Delta}{\partial\gamma_t} = -\left(1 - \frac{1}{\gamma_t}\right)\frac{\partial L}{\partial\gamma_t} - \frac{M_t}{\gamma_t^2}. \tag{18}$$

There are two cases to consider:

(i) If $\partial L/\partial\gamma_t \geq 0$, then (18) implies $\partial\Delta/\partial\gamma_t < 0$ (because $\gamma_t \geq 1$). That conclusion shows, however, that in equilibrium the government will always choose a money-growth rate such that $\partial\Pi/\partial\gamma_t > 0$. If $\partial\Pi/\partial\gamma_t > 0$ didn't hold, the government would have an incentive to *raise* monetary growth, thereby lowering end-of-period commitments without raising inflation. So, if $\partial L/\partial\gamma_t \geq 0$, then by (17), the government will always choose a point of the aggregate money-demand schedule where the elasticity of real money demand with respect to nominal money growth is below unity.

(ii) What if, instead, $\partial L/\partial\gamma_t < 0$? This case automatically would entail $\partial\Pi/\partial\gamma_t > 0$ [by (17)], so that, at an optimum for the government, $\partial\Delta/\partial\gamma_t$ is necessarily negative once again. If it were not, the government would wish to lower monetary growth, thereby lowering inflation without raising end-of-period commitments. ■

The next result simplifies the interpretation of equilibria by showing that any equilibrium aggregate real money-demand schedule $L(\gamma_t, S_t)$ can be written as a function of a single variable, the end-of-period commitment stock, K_t.

PROPOSITION 2. *In equilibrium,* $L(\gamma_t, S_t)$ *is of the form*

$$L(\gamma_t, S_t) = \hat{L}[\Delta(\gamma_t, S_t)] = \hat{L}(K_t). \tag{19}$$

Proof. Equation (16) shows that, in equilibrium,

$$L(\gamma_t, S_t) = (\vartheta')^{-1}\left\{\frac{\rho + \Pi[\Gamma(S_{t+1}), S_{t+1}]}{1+\rho}\right\}. \tag{20}$$

Along an equilibrium path, however, $S_{t+1} = (K_t, M_t) = [K_t, L(\gamma_t, S_t)]$, and so, equation (20) gives $L(\gamma_t, S_t)$ implicitly as a function $\hat{L}(K_t)$ of K_t alone. ■

The preceding finding allows us to think of the private sector's equilibrium forecast of inflation between periods t and $t+1$ as depending only on its forecast of the beginning-date $t+1$ stock of public-sector commitments. Intuitively, one would guess that $\hat{L}'(K_t) < 0$ in any equilibrium: People reduce their real balances when they know the end-of-period stock of public commitments is higher. That conjecture is verified below by considering the government's intertemporal Euler condition. In analyzing that condition, the next proposition is helpful.

PROPOSITION 3. *In an equilibrium with nonnegative money-supply growth,*

$$1 + [1 - (1/\gamma_t)]\hat{L}'(K_t) > 0, \tag{21}$$

$$\frac{\partial\Delta}{\partial\gamma_t} = \frac{-M_t/\gamma_t^2}{1 + [1 - (1/\gamma_t)]\hat{L}'(K_t)} < 0, \tag{22}$$

$$\frac{\partial\Delta}{\partial K_{t-1}} = \frac{(1+\rho)}{1 + [1 - (1/\gamma_t)]\hat{L}'(K_t)} > 0. \tag{23}$$

Furthermore, if $\partial L/\partial\gamma_t > 0$,

$$1 + \hat{L}'(K_t) > 0. \tag{24}$$

Proof. To compute the derivatives in (22) and (23), use (11a) and (11b), substituting $\hat{L}(K_t)$ for $L(\gamma_t, S_t)$ and applying the chain rule. Inequality (21) then follows from (22) and Proposition 1 (which established that $\partial\Delta/\partial\gamma_t < 0$). To prove (24) when $\partial L/\partial\gamma_t > 0$, apply the chain rule to (19) and use (11b) to derive

$$\frac{\partial L}{\partial\gamma_t} = \hat{L}'(K_t)\frac{\partial\Delta}{\partial\gamma_t}. \tag{25}$$

Combining (25) with (18) gives

$$\hat{L}'(K_t) = -1\Bigg/\left\{1 + (1/\gamma_t)\left[\left(\frac{\gamma_t}{M_t}\frac{\partial L}{\partial\gamma_t}\right)^{-1} - 1\right]\right\}.$$

However, Proposition 1 implies [via equation (17)] that $(\gamma_t/M_t)\partial L/\partial\gamma_t < 1$ when the government is optimizing. Inequality (24) follows immediately if $\partial L/\partial\gamma_t > 0$. ■

The strictly positive term in the denominator of (22) and (23) reflects a multiplier effect that influences commitment accumulation because period-t money demand depends on K_t itself. A unit rise in K_{t-1}, for example, has a direct positive effect of $1+\rho$ on K_t, but it has an additional indirect effect on K_t by changing $\hat{L}(K_t)$ as well. The total result is given by (23).

4.2. Government Optimality Conditions

To derive first-order necessary conditions for an optimal money-growth path, differentiate (14) with respect to γ_t. At an interior maximum [recall equation (12a)],

$$\kappa'(\pi_t)\frac{\partial \Pi}{\partial \gamma_t} = z'[L(\gamma_t, S_t)]\frac{\partial L}{\partial \gamma_t} + \frac{1}{(1+r)}\left(\frac{\partial V}{\partial K_t}\frac{\partial \Delta}{\partial \gamma_t} + \frac{\partial V}{\partial M_t}\frac{\partial L}{\partial \gamma_t}\right). \tag{26}$$

It is helpful to rewrite this condition in terms of the reduced-form money-demand schedule of equation (19), which depends on K_t only. Substitution of (9) and (25) into (26) gives

$$\kappa'(\pi_t)\frac{\partial \Pi}{\partial \gamma_t} = \frac{\partial \Delta}{\partial \gamma_t}\left\{z'(M_t)\hat{L}'(K_t) + \frac{1}{(1+r)}\left[\frac{\partial V}{\partial K_t} + \frac{\partial V}{\partial M_t}\hat{L}'(K_t)\right]\right\}, \tag{27}$$

which is the same Euler equation that would have resulted from substitution of $\hat{L}(K_t)$ for $L(\gamma_t, S_t)$ in (14) prior to maximization.

The interpretation of Euler equation (27) is standard.[14] The left-hand side is the output cost of incrementally higher period-t money growth—the product of the marginal cost of current inflation and the marginal inflation effect of money growth. The right-hand side is the marginal value of higher period-t money growth—the product of the reduction in K_t due to a higher γ_t and the marginal value to the government of lower end-of-t-commitments. A lower K_t, in turn, affects social welfare both by raising real money demand, M_t, and by changing discounted period $t+1$ value V_{t+1}, which depends on the end-of-t stocks K_t and M_t.

A further definition helps to clarify the economic implications of (27). Define the *shadow price of public commitments* at the start of period t, λ_t, by

$$\lambda_t \equiv \frac{\left\{\left[z'(M_t) + \dfrac{\kappa'(\pi_t)}{\gamma_t M_{t-1}}\right]\hat{L}'(K_t) + \dfrac{1}{(1+r)}\left[\dfrac{\partial V}{\partial K_t} + \dfrac{\partial V}{\partial M_t}\hat{L}'(K_t)\right]\right\}}{1 + \left(1 - \dfrac{1}{\gamma_t}\right)\hat{L}'(K_t)}. \tag{28}$$

The price λ_t is the marginal value to the government of having an additional unit of resources in private rather than public hands. Recall that the maximized value of (14) is the government's value function, $V(S_t) = V(K_{t-1}, M_{t-1})$. An envelope argument that uses (23) establishes the equality:

$$\lambda_t = \frac{1}{(1+\rho)}\frac{\partial V}{\partial K_{t-1}} \leq 0, \qquad \forall\, t. \tag{29}$$

That is, λ_t is the effect on social welfare of a unit increase in government commitments at the start of period t. [A unit rise in K_{t-1} raises beginning-of-t government commitments by $1+\rho$ units, not by 1 unit, which explains the discounting in equation (29)]. Another envelope argument leads to [recall equation (10a)]

$$\frac{\partial V}{\partial M_{t-1}} = -\kappa'(\pi_t)\frac{\partial \Pi}{\partial M_{t-1}} = -\kappa'(\pi_t)\frac{M_t}{\gamma_t M_{t-1}^2} \leq 0, \qquad \forall\, t. \tag{30}$$

Now, use (17), (19), (22), and (28)–(30) to express the first-order condition (27) in terms of λ_t and λ_{t+1}. The result (after some algebra) is the pair of conditions,

$$\kappa'(\pi_t) = -\lambda_t M_{t-1}, \tag{31}$$

$$\lambda_t = \frac{z'(M_t)\hat{L}'(K_t)}{1+\hat{L}'(K_t)} + \frac{\lambda_{t+1}}{(1+r)}\left[\frac{(1+\rho)+(1-\pi_{t+1})\hat{L}'(K_t)}{1+\hat{L}'(K_t)}\right]. \tag{32}$$

The meanings of these two equations are grasped most easily by thinking of the government's move as a direct choice of the inflation rate, π_t, rather than a choice of the contemporaneous money-supply growth rate, γ_t.[15] Condition (31) simply equates the marginal current benefit from a fall in inflation to the marginal value of the resources the government would thereby forgo.

Condition (32) rules out any welfare gain from a perturbation in the path of public commitments that lowers K_t incrementally (say) but leaves commitments unchanged on all other dates. To understand (32), let us assume provisionally that $\partial L/\partial\gamma_t \geq 0$, so that inequality (24) holds (see Proposition 3). (The provisional assumption is confirmed below.) The intertemporal trade-off involved in the choice of an inflation rate is embodied in (11a), which can be expressed as

$$K_t + \hat{L}(K_t) = (1+\rho)K_{t-1} + (1-\pi_t)M_{t-1}. \tag{33}$$

Because the commitment multiplier implied by (33) is $1/[1+\hat{L}'(K_t)]$ [a positive number, if inequality (24) holds], the period-t inflation increase that changes K_t by the infinitesimal amount $\mathrm{d}K_t < 0$ reduces social welfare by $\lambda_t[1+\hat{L}'(K_t)]\,\mathrm{d}K_t = [-\kappa'(\pi_t)/M_{t-1}][1+\hat{L}'(K_t)]\,\mathrm{d}K_t$. At an optimum, however, this welfare cost just equals the benefits of a one-unit commitment reduction lasting one period: an immediate rise in household money demand—worth $z'(M_t)\hat{L}'(K_t)\,\mathrm{d}K_t$ in current welfare terms—plus the present marginal value of the period $t+1$ inflation reduction that returns commitments to their initial path—which is worth $(1+r)^{-1}\lambda_{t+1}[(1+\rho)+(1-\pi_{t+1})\hat{L}'(K_t)]\,\mathrm{d}K_t = (1+r)^{-1}[-\kappa'(\pi_{t+1})/M_t][(1+\rho)+(1-\pi_{t+1})\hat{L}'(K_t)]\,\mathrm{d}K_t$.

4.3. Slope of Reduced-Form Money-Demand Schedule

The following result is central to a characterization of equilibrium dynamics.

PROPOSITION 4. *In an equilibrium such that* (31) *and* (32) *hold,*

$$\hat{L}'(K) \leq 0.$$

Proof. Suppose instead that $\hat{L}'(K) > 0$. Assume that inflation initially is positive. Because $r \geq \rho$ by assumption, λ_t, which is a negative number, must fall over time (i.e., become more negative) according to condition (32). Condition (31) therefore implies that inflation must rise over time, equation (2) that money demand must fall over time, and the assumption $\hat{L}'(K) > 0$ that commitments also must fall. However, the government wouldn't find it optimal to play the strategy the private sector expects along the path just described. By slightly *lowering* inflation on any date and maintaining inflation at that level forever, the government could freeze its commitments, thereby reaching a higher welfare level on that date and on every future date while respecting intertemporal budget balance. Thus, the paths that (31) and (32) generate when $\hat{L}'(K) > 0$ are not equilibrium paths. ■

COROLLARY. *In an equilibrium with nonnegative money-supply growth,* $\partial L/\partial \gamma_t \geq 0$ *and* $\partial L/\partial K_{t-1} \leq 0$.

Proof. Apply the chain rule to (19) and use Proposition 1 and inequality (23). ■

4.4. Stationary States

Equations (31) and (32) together summarize the dynamics of the model. The first dynamic implication concerns stationary-state equilibria, equilibria in which public commitments, their shadow price, and inflation all remain constant over time.

One stationary state is described by $\bar{K} = \bar{\lambda} = \bar{\pi} = 0$. These values satisfy (31) and (32) because $\kappa'(0) = 0$ and $z'(\bar{M}) = 0$ at the real-balance level $\bar{M}$ that households demand when expected inflation in zero {that is, at $\bar{M} \equiv \vartheta^{-1}[\rho/(1+\rho]\}$. To see that the government budget constraint (7) is satisfied in this stationary state, suppose that $M_{t-1} = \bar{M}$. By (6), $K_{t-1} = 0$ implies that the government holds a *negative* debt D_{t-1} given by

$$-D_{t-1} = \frac{1}{(1+\rho)} \sum_{\tau=t}^{\infty} (1+\rho)^{-(\tau-t)} G_\tau,$$

so that it can finance all current and future purchases out of asset income, without ever resorting to inflation. Thus, M_t will remain at $\bar{M}$ and K_t at $\bar{K} = 0$. Because the government never needs to levy distorting inflation taxes, λ_t, the marginal inflation-tax distortion, remains steady at $\bar{\lambda} = 0$.

In this zero-inflation stationary state, the budget need not be balanced on a period-by-period basis: Deficits will be run when G_t is unusually high, surpluses when it is unusually low. What *is* true is that government assets always equal the present value of future public spending, and so, there is never a need to supplement the budget with seigniorage revenue.

Because λ cannot take positive values, the zero-inflation stationary state is the *only* one when $\hat{L}'(K) < 0$ and $r \leq \rho$ (the government discount rate is no greater than the market rate).[16] When $r > \rho$, as allowed above, however, steady states with $\lambda < 0$ may arise. These are characterized by constant levels of π and K.

A very special case arises when $\hat{L}'(K) \equiv 0$. In general, this condition can hold in a Markov perfect equilibrium only when household money demand is completely insensitive to the nominal interest rate. Under this assumption, (32) reduces to

$$\lambda_t = \frac{(1+\rho)}{(1+r)}\lambda_{t+1},$$

a familiar condition for intertemporal optimization in dynamic fiscal-policy models in which precommitment is possible or irrelevant. For $r = \rho$, this condition becomes $\lambda_t = \lambda_{t+1}$, in which case (31) delivers the prediction that inflation will be the same on all dates. This is the intertemporal tax-smoothing formula applied to inflation by Mankiw (1987) and Grilli (1988, 1989). Every level of K corresponds to a distinct stationary state when $r = \rho$, and the associated constant inflation rate keeps K constant.

Inflation generally isn't constant when money demand is interest-sensitive because the government knows that its budgetary position affects inflation expectations and, with them, private money demand. I assume below that $\hat{L}'(K) < 0$.

4.5. Equilibrium Dynamics

Consider first the case $r = \rho$ (assumed in most of the tax-smoothing literature). Because $\hat{L}'(K_t) < 0$, equation (32) shows that the inequalities $\lambda_t < \lambda_{t+1} \leq 0$ must hold in this case. So, λ_t converges over time to $\bar{\lambda} = 0$, the unique stationary value, as K_t converges to $\bar{K} = 0$ and π_t converges to $\bar{\pi} = 0$ [see equation (31)].

The interaction of government policy and rational private expectations thus drives the economy to a noninflationary long-run equilibrium when $r = \rho$. That result hinges crucially on the equilibrium relationship between public commitments and private expectations of inflation. As noted above, when $\hat{L}'(K) \equiv 0$—in which case money demand is not responsive to the government's incentives to inflate—the path of inflation is flat and K is constant. The government has no reason to change K because the gross return on asset accumulation, $1 + \rho$, is then exactly offset by the government's discount factor, $1/(1+r) = 1/(1+\rho)$. In the equilibrium constructed above, in contrast, additional government saving yields the gross return $1 + \rho$ *plus* the extra benefits from the induced increase in money demand [see (32)]. Because the government's discount factor is just $1/(1+\rho)$, the government will reduce its spending commitments, K, over time, by always setting monetary growth and inflation higher than the level that would be consistent with unchanging commitments.

These conclusions about the economy's equilibrium path would be qualitatively unchanged if r were below ρ, or if r were greater than ρ, but not by enough to produce a second stationary state. Once a second stationary equilibrium appears, however, it becomes difficult to analyze stability without more detailed information on inflation costs and on government and household preferences. It is possible (for r sufficiently high relative to ρ) that there is a stable inflationary long-run equilibrium, and that the $\bar{\pi} = 0$ equilibrium is unstable. A sudden rise in the government's discount rate (the result of increased political instability, say), could turn a stable zero-inflation equilibrium into an unstable one, thereby allowing small disturbances to propel the economy into high and persistent inflation. The linear examples in Section 5 illustrate some of these possibilities.[17]

As the examples also show, there is no general guarantee that equilibrium is unique. For given fundamentals, there can be several equilibrium paths for the economy, possibly converging to different stationary states.

5. SOME LINEAR EXAMPLES

Closed-form linear-quadratic examples illustrate some characteristics of the equilibria defined and analyzed above.[18] In the examples, I assume that $z(M) \equiv 0$ in (3), so that inflation reduces welfare only through its negative current-output effect.

An advantage of the linear-quadratic setup is that it allows an easy analysis of the model's equilibrium when agents face specific types of uncertainty. I therefore allow for the possibility that government spending, G_t, is a random variable generated by an exogenous first-order Markov process. (Additional assumptions on that process are introduced below.) The realization of G_t is revealed in period t *before* the government implements its period-t policy action. As a result, households generally will make unsystematic forecast errors in a stochastic equilibrium. I assume that, despite the stochastic environment, government debt payments are not indexed to the realized state of nature.

The key strategem delivering linearity is a redefinition of the model in terms of aggregate *inflation-tax* payments, μ_t, where

$$\mu_t \equiv \pi_t M_{t-1}.$$

In line with this approach, I assume that a household's demand for real balances is a linear decreasing function of the inflation-tax revenue it expects the government to collect next period,

$$m_t = \bar{m} - \delta E_t(\mu_{t+1}), \tag{34}$$

and that the output cost of inflation is given by the function $\kappa(\mu_t) = (\frac{1}{2})\mu_t^2$. The government thus is assumed to maximize

$$V_t = -\frac{1}{2} E_t \left[\sum_{\tau=t}^{\infty} (1+r)^{-(\tau-t)} \mu_\tau^2 \right]. \tag{35}$$

In (34) and (35), $E_t[\cdot]$ denotes a rational expectation conditional on the vector of economic state variables known at the start of period t, (G_t, D_{t-1}, M_{t-1}). An optimal government policy rule will take the form of a deterministic linear function of this state vector. The resulting sequence of contingency plans must satisfy intertemporal budget constraint (5) with probability 1.

The money-demand specification (34) is plausible (at least as an approximation) if the elasticity of household money demand with respect to expected inflation is low enough that inflation-tax proceeds and inflation move together.[19] A further parameter restriction necessary for equilibrium is

$$\delta < 1/\rho. \tag{36}$$

Condition (36) requires the elasticity of aggregate money demand with respect to $[\rho + E_t(\pi_{t+1})]/(1 + \rho)$, the opportunity cost of holding money, to be less than unity.

5.1. Deterministic Case

If government spending follows a known exogenous path, then in each period t, the government maximizes $-(\frac{1}{2})\mu_t^2 + (1+r)^{-1}V(K_t, M_t)$ subject to equation (33), written as $K_t + \hat{L}(K_t) = (1+\rho)K_{t-1} - \mu_t + M_{t-1}$, with K_{t-1} and M_{t-1} given. Without loss of generality, the government's period-t action can be viewed as a direct choice of μ_t. My working conjectures are that the aggregate reduced-form money-demand relationship takes the form

$$M_t = \hat{L}(K_t) = \bar{M} - \beta K_t, \tag{37}$$

and that the government's optimal policy function is of the form

$$\mu_t = \varphi_0 + \varphi_1 K_{t-1} + \varphi_2 M_{t-1}. \tag{38}$$

On an equilibrium path with $K_t = 0$, it must be true that $\mu_t = 0$ as well, and so, (37) and (38) together imply the restriction

$$\varphi_0 + \varphi_2 \bar{M} = 0. \tag{39}$$

I now show that the functions (37) and (38) characterize equilibria for appropriate coefficient values.

Suppose the government is choosing its period-t action, μ_t. The government takes as given that aggregate money demand obeys (37) in all periods $\tau \geq t$. Equation (33) then implies that

$$K_t = \frac{(1+\rho)K_{t-1} + M_{t-1} - \bar{M} - \mu_t}{1-\beta}. \tag{40}$$

[Notice that M_{t-1} in (40) could be any arbitrary value, and is not necessarily related to K_{t-1} by (37).] If the government follows policy rule (38) from period $t+1$ on, its end-of-period commitments starting in $t+1$ are given by

$$K_\tau = \psi K_{\tau-1}, \qquad \forall \tau \geq t+1, \tag{41}$$

where

$$\psi \equiv \frac{[1+\rho-\varphi_1-(1-\varphi_2)\beta]}{1-\beta}. \tag{42}$$

(I check later in specific cases that $0 < \beta, \psi < 1$ in equilibrium.) Equation (41) implies that under (37) and (38), the government's value function for period $t+1$ therefore is

$$V(S_t) = V[K_t, \hat{L}(K_t)] = -\left(\frac{1}{2}\right)\frac{(\varphi_1-\varphi_2\beta)^2}{1-\psi^2/(1+r)}K_t^2.$$

Bellman's principle implies that the optimal period-t policy μ_t necessarily maximizes $V_t = -(\frac{1}{2})\mu_t^2 + (1+r)^{-1}V(K_t, M_t)$ subject to (37) and (40); that is, it satisfies

$$\varphi_0+\varphi_1 K_{t-1}+\varphi_2 M_{t-1}$$

$$= \arg\max_{\mu_t}\boldsymbol{\Bigg(}-\frac{1}{2}\left\{\mu_t^2+\frac{(\varphi_1-\varphi_2\beta)^2}{1+r-\psi^2}\left[\frac{(1+\rho)K_{t-1}+M_{t-1}-\bar{M}-\mu_t}{1-\beta}\right]^2\right\}\boldsymbol{\Bigg)}. \tag{43}$$

By differentiating the term in boldface parentheses in (43) and equating coefficients with (38), one finds that

$$\varphi_1 = \frac{(1+\rho)(\varphi_1-\varphi_2\beta)^2}{(\varphi_1-\varphi_2\beta)^2+(1-\beta)^2(1+r-\psi)^2}, \qquad \varphi_2 = \frac{\varphi_1}{1+\rho}, \qquad \varphi_0 = -\varphi_2\bar{M}. \tag{44}$$

[The last equality is (39).] Definition (42) now gives the optimal value of φ_1 as a function of the parameter β in (37):

$$\varphi_1 = (1+\rho)\left[1-\frac{(1+r)(1-\beta)^2}{(1+\rho-\beta)^2}\right]. \tag{45}$$

The optimal policy coefficients φ_0 and φ_2 follow immediately.

The exercise is still incomplete, however: It remains to ensure that (37) is the result of optimal household behavior when households predict on the basis of (37) and (45). That equality holds only when $\bar{M}$ and β are related in a specific way to the parameters $\bar{m}$ and δ in the household money-demand equation (34).

To find the necessary relationship, observe that, given (37), a household's rational period-t forecast of μ_{t+1} is

$$\mu_{t+1} = \varphi_0+\varphi_1 K_t+\varphi_2(\bar{M}-\beta K_t) = \varphi_1\left(1-\frac{\beta}{1+\rho}\right)K_t$$

[by (37) and (44)]. Thus, by (34), each household will demand real balances $\bar{m}-\delta\varphi_1\{1-[\beta/(1+\rho)]\}K_t$. In an equilibrium, this demand function must coincide with (37), which requires that $\bar{m} = \bar{M}$ and

$$\beta = \delta\varphi_1\left(1 - \frac{\beta}{1+\rho}\right) \Leftrightarrow \varphi_1 = \frac{(1+\rho)\beta}{\delta(1+\rho-\beta)}. \tag{46}$$

When combined, (45) and (46) lead to a quadratic equation that any equilibrium value of β, β^*, must satisfy:

$$(1 - r\delta)\beta^2 + [2(r-\rho)\delta - (1+\rho)]\beta + \delta[\rho(1+\rho) + (\rho - r)] = 0. \tag{47}$$

Rather than presenting a general analysis of solutions to (47), I concentrate on two special cases of interest.

Case 1. $r = \rho$. In this case the solutions to (47) are both positive and real. They are

$$\beta = \frac{(1+\rho) \pm \sqrt{(1+\rho)^2 - 4\rho\delta(1+\rho)(1-\rho\delta)}}{2(1-\rho\delta)}. \tag{48}$$

The Appendix shows a proof that the larger of these two solutions exceeds 1, and thus cannot be the equilibrium value β^* [because $\beta^* = \hat{L}'(K)$; see inequality (24) above]. The smaller solution in (48) is β^*, and the Appendix shows that $\rho\delta < \beta^* < 1$.

The inequality $\beta^* > \rho\delta$ implies that public commitments will decline over time to the stationary state $\bar{K} = 0$. These dynamics follow from (41), because the equilibrium ψ^* can be expressed as

$$\psi^* = 1 - \frac{(\beta^* - \rho\delta)}{\delta(1-\beta^*)} \tag{49}$$

with the help of (42), (44), and (46). (The Appendix shows that $\psi^* > 0$.) In the case $r = \rho$, we therefore have a unique equilibrium with the features described in Section 4.

Case 2. $r = \rho/(1-\rho\delta)$. This is a case in which the government discount rate exceeds that of the private sector. A direct check using (47) shows that $\beta^* = \rho\delta < 1$ defines an equilibrium.

In this case, however, (49) implies that $\psi^* = 1$, so that public commitments will follow $K_t = K_{t-1}$ along the economy's equilibrium path. In other words, there is an equilibrium with perfect inflation smoothing despite the government's awareness that the level of government commitments influences household expectations. When $r = \rho/(1-\rho\delta)$, the gap between government and household discount rates just offsets the additional government saving incentives due to market expectations. As a result, any initial value of government commitments will be maintained indefinitely if the economy starts from a position on the equilibrium path (that is, with initial real balances related to initial commitments by $M_{t-1} = \bar{m} - \beta^* K_{t-1}$). The policy function (38) is just $\mu_t = \rho K_{t-1}$ along equilibrium paths.

When $1-2\rho\delta > 0$, the equilibrium solution $\beta^* = \rho\delta$ is unique. When $1-2\rho\delta < 0$, however, there may be a second equilibrium $\beta^{**} \in (\rho\delta, 1)$, where $\beta^{**} = 1 - \rho[(1-\rho\delta)/(2\rho\delta-1)]$. (See Section 5.2 for a numerical example.) Even in a linear-quadratic setting, therefore, multiple equilibria appear to be possible for $r > \rho$. A second equilibrium arises in the present case when households' expectation that lower public commitments will lead to lower inflation provides just the incentive the government needs to induce a paring down of public commitments over time.

5.2. Stochastic Case

Now assume that government spending is a random variable that follows a first-order Markov process. Recall that the period-t realization G_t is revealed at the start of period t, before the government chooses μ_t but after the public has chosen the previous period's real balances, M_{t-1}.

It is convenient to redefine the stock of public commitments at the end of period $t-1$ in terms of expected values as

$$K_{t-1} \equiv \frac{1}{(1+\rho)} E_{t-1}\left[\sum_{\tau=t}^{\infty}(1+\rho)^{-(\tau-t)}G_\tau\right] + D_{t-1}.$$

We now need to distinguish, however, the end-of-$(t-1)$ commitment measure K_{t-1}, on which households' choice of M_{t-1} is based, from the *start-of-t* commitment measure on which the government bases its choice of μ_t. The difference between the two depends on the unanticipated component of G_t. Define the expectational revision at the start of period t, ε_t, by

$$\varepsilon_t \equiv \frac{1}{(1+\rho)}\left\{E_t\left[\sum_{\tau=t}^{\infty}(1+\rho)^{-(\tau-t)}G_\tau\right] - E_{t-1}\left[\sum_{\tau=t}^{\infty}(1+\rho)^{-(\tau-t)}G_\tau\right]\right\}.$$

The commitment variable relevant for the government's period-t decisions is then

$$\tilde{K}_t \equiv K_{t-1} + \varepsilon_t = \frac{1}{(1+\rho)} E_t\left[\sum_{\tau=t}^{\infty}(1+\rho)^{-(\tau-t)}G_\tau\right] + D_{t-1},$$

and the government finance constraint becomes

$$\tilde{K}_{t+1} = (1+\rho)\tilde{K}_t - (M_t - M_{t-1}) - \mu_t + \varepsilon_{t+1}. \tag{50}$$

I assume that ε_t, which has a mean of zero conditional on information known in period $t-1$, has a finite variance and is distributed independently of period-$(t-1)$ information.

Because the realization of G_{t+1} is not known by households in period t, the stochastic analogue of (37) has the form

$$M_t = \hat{L}[E_t(\tilde{K}_{t+1})] = \bar{M} - \beta E_t(\tilde{K}_{t+1}) = \bar{M} - \beta K_t. \tag{51}$$

When combined, (50) and (51) give the two dynamic equations

$$\tilde{K}_{t+1} = \frac{(1+\rho)\tilde{K}_t + M_{t-1} - \bar{M} - \mu_t}{1-\beta} + \varepsilon_{t+1}, \tag{52}$$

$$M_t = \bar{M} - \beta\left[\frac{(1+\rho)\tilde{K}_t + M_{t-1} - \bar{M} - \mu_t}{1-\beta}\right]. \tag{53}$$

The problem of maximizing (35) subject to (52) and (53) was solved in Section 5.1 with the stochastic shock ε suppressed and with $\tilde{K}_t$ formally labeled as K_{t-1}. The optimal policy rule in the present stochastic case is, however, the *same* function of the state variables as in the deterministic case [Sargent (1987, p. 37)]. Thus, the optimal policy rule [given (51)] is of the form (38), with $\tilde{K}_t$ in place of K_{t-1} and with coefficients again described by (44) and (45). Because

$$E_t(\mu_{t+1}) = E_t\{\varphi_0 + \varphi_1\tilde{K}_{t+1} + \varphi_2[\bar{M} - \beta E_t(\tilde{K}_{t+1})]\} = \varphi_1\left(1 - \frac{\beta}{1+\rho}\right)K_t,$$

condition (46) remains necessary for equilibrium. An equilibrium value of β, β^*, is thus a root of the quadratic equation (47).

Some calculation shows that along the economy's equilibrium path, beginning-of-period (resp. end-of-period) public commitments follow an ARMA (1, 1,) [resp. AR(1)] process

$$\tilde{K}_{t+1} = \psi^*\tilde{K}_t + \varepsilon_{t+1} + \theta\varepsilon_t \Leftrightarrow K_t = \psi^* K_{t-1} + (\psi^* + \theta)\varepsilon_t,$$

where ψ^* is given by the formula in (49) and $\theta = \beta^*(1-\varphi_2^*)/(1-\beta^*)$. Inflation-tax revenue is generated by the AR(1) process

$$\mu_t = \psi^*\mu_{t-1} + \varphi_1^*\varepsilon_t.$$

In the case $r = \rho$, $\psi^* \in (0, 1)$, and so, both the stock of commitments and inflation-tax revenue follow stationary stochastic processes with long-run distributions centered on zero. As a numerical example, suppose that $\rho = r = 0.04$ and $\delta = 12$ (so that the elasticity of real money demand with respect to interest cost is 0.48). Then, $\beta^* = 0.798$ [by (48)] and $\psi^* = 0.869$ [by (49)]. Only in the constant-velocity case, $\delta = 0$, do (42), (45), and (48) lead to $\psi^* = 1$ and the martingale property for K_t and μ_t.

In the case $r = \rho/(1-\rho\delta)$, both the K_t and μ_t processes may be martingales even for $\delta > 0$, because $\beta^* = \rho\delta$ is one equilibrium.[20] For the specific parameter assignments of the last paragraph, which imply $r = 0.077$, the martingale equilibrium is the only one.

Suppose, however, that $\rho = 0.04$ once again but that $\delta = 20$, giving $r = \rho/(1-\rho\delta) = 0.04/(1-0.8) = 0.2$. There is still an equilibrium with $\beta^* = \rho\delta$, but there is also a second, convergent equilibrium in which $\beta^{**} = 0.987$ and $\psi^{**} = 0.3$. A higher interest elasticity of money demand makes possible an equilibrium in which money demand responds so strongly to public debt reduction that the government finds it optimal to accumulate wealth over time despite its high rate of time preference.

6. CONCLUSION

This paper has explored the intertemporal behavior of seigniorage and government spending commitments in a dynamic game-theoretic model that determines the path of a key endogenous state variable, the public debt. When government and private-sector discount rates are the same, as intertemporal tax-smoothing analyses typically assume, a Markov perfect equilibrium requires declining paths of inflation and government commitments. In long-run equilibrium, the government holds an asset stock sufficient to finance future government expenditures without the need for inflation (or, for that matter, other distorting taxes).

When the government's discount rate exceeds the market's, however—perhaps as a result of finite political lifetimes—alternative Markov perfect paths for inflation and budgetary commitments are possible, including inflationary steady states. There is no general guarantee of a unique equilibrium.

Although the model yields predictions broadly consistent with the apparent long-term behavior of prices in many countries, it is less clear that it can capture the great disparities in budgetary and inflationary experiences across economies and epochs. Some government-caused inflation is not motivated by seigniorage needs, official preferences change over time, and measured inflation is subject to serially correlated shocks beyond government control. Income-distribution and employment goals, two factors absent from the paper's model, are particularly important. Political uncertainly has been introduced into the model in a rudimentary way, but it would plainly be desirable to build explicitly on the social and economic tensions underlying political theories of budget processes [see Alesina and Perotti (1995) for a survey]. Such an extension might explain why the zero-tax stationary equilibrium predicted by some versions of the model is literally never observed in reality.[21]

Despite its strong simplifying assumptions, the model does capture forces that influence fiscal and monetary policy formulation even in countries where inflation seems most deeply rooted. The model helps explain, for example, why governments in budgetary crisis often sharply devalue their currencies at the outset of stabilization, thereby spurring domestic inflation temporarily but (hopefully) promoting increases in official exchange foreign reserves. A partial rationale for devaluation is that it may lower *future* inflation by objectively improving the budgetary situation—just as in the account offered above. The model also throws light on the current European exercise in fiscal retrenchment in preparation for economic and monetary union.

NOTES

1. In this paper, my focus is on the seigniorage motive for inflation. A number of authors, starting with Kydland and Prescott (1977), show how excessive inflation can result from the time-inconsistency

problem of a government that wishes to raise employment above some "natural" rate. Without a more detailed account of why governments may want to do this, it is difficult to relate the literature on the employment motive for inflation to the budgetary issues that concern me below. Any such account is likely to involve budgetary incentives, however (for example, a government's desire to raise income-tax revenue while cutting public transfer payments to the unemployed).

2. Chari et al. (1996) show in a variety of models that even when all other conventional taxes distort, the optimal precommitment path for inflation follows Milton Friedman's "optimal quantity of money" rule (such that the nominal interest rate is zero). In the model of the present paper, inflation is the only tax. That assumption may be viewed as a reflection of political obstacles to setting conventional taxes at Ramsey-optimal levels.

3. Evidence on the inflation-smoothing approach (in totality rather unsupportive) is discussed by Mankiw (1987), Grilli (1988, 1989), Judd (1989), Bizer and Durlauf (1990), Poterba and Rotemberg (1990), and Calvo and Leiderman (1992).

4. In the absence of a government precommitment capability, Ramsey plans sometimes can be supported as equilibria through intricate government debt-management strategies [Lucas and Stokey (1983) and Persson et al. (1987)] or in specific self-fulfilling trigger-strategy or reputational equilibria [Chari and Kehoe (1990), Rogoff (1989), and Stokey (1991)]. Debt-management strategies are known to be effective only in very special circumstances, however. Calvo and Obstfeld (1990) show that the prescriptions of Persson et al. (1987) are not generally valid, suggesting that the problem of dynamic inconsistency underlying the present paper's analysis need not disappear when the government can hold nominal assets. Additionally, as Rogoff's (1989) discussion indicates, the empirical relevance of reputational equilibria remains controversial. In previous work [Obstfeld (1991)], I derive the Ramsey solution for a planning problem similar to that of Section 2 and discuss its dynamic inconsistency in detail.

5. According to the model, a multicountry cross-sectional study thus would find a stronger impact of government debt levels on inflation than a Ramsey tax-smoothing rule would predict. That *some* significant positive cross-sectional link between debt levels and inflation exists is confirmed by Campillo and Miron (1997).

6. Judd (1989) independently reaches this conclusion, based on simulations of a stochastic model of capital, labor, and money taxation. Bohn (1988) and Poterba and Rotemberg (1990) take approaches similar to mine in modeling optimal inflation. Their analyses, however, do not consider equilibrium dynamics in any detail.

7. The preceding constraint reflects the household's loss during period t of $[1-(\boldsymbol{P}_{t-1}/\boldsymbol{P}_t)]m_{t-1} = \pi_t m_{t-1}$ on real balances carried over from period $t-1$.

8. In equilibrium, households are indifferent between alternative feasible intertemporal consumption allocations. Each period, the representative household chooses to consume aggregate output (net of inflation costs) less government consumption [see equation (8)].

9. In Obstfeld (1991) I examine the case in which $z(m) = \vartheta(m)$ and $r = \rho$, so that the government maximizes the representative household's utility. In that case, $\pi = -\rho$ is the economy's unique stationary point.

10. This sum equals $(\boldsymbol{M}_t - \boldsymbol{M}_{t-1})/\boldsymbol{P}_t$, where $\boldsymbol{M}$ denotes nominal money holdings per household. Thus, when $\boldsymbol{P}_t$ is the equilibrium price level, seigniorage equals the real resources that the government is able to purchase from each household in exchange for money. To work in terms of present values, below, I assume a transversality condition on equilibrium household real money balances, $\lim_{t\to\infty}(1+\rho)^{-t}m_t = 0$.

11. See Auernheimer (1974).

12. Driffill et al. (1990) survey the literature on the costs of inflation.

13. In related models, Chang (1996) and Phelan and Stacchetti (1996) describe algorithmic methods for characterizing all equilibria, not just the Markovian equilibria.

14. In working with (27), I am assuming that it indeed characterizes the government's optimum. Section 5 presents a linear approximation to the model in which equilibrium exists and a counterpart of (27) characterizes it.

15. There is no loss of generality in taking this approach. Equation (10a) and Proposition 1 imply that in equilibrium, π_t and γ_t are linked by an invertible relationship.

16. The proof is immediate from (32). Because $\pi \leq 1$, by definition,

$$\frac{(1+\rho)+(1-\pi_{t+1})\hat{L}'(K_t)}{1+\hat{L}'(K_t)} > (1+\rho)$$

(provided $\pi > -\rho$, which I am assuming). So, no constant (negative) value of λ can satisfy (32).

17. See Obstfeld (1991) for a diagrammatic exposition.

18. For similar calculations in a deterministic model, see Cohen and Michel (1988).

19. Notice that there are limits on the maximum feasible value of μ and on the minimum value consistent with equilibrium. The first-order conditions that I work with below will not hold when one of these constraints on μ binds. It may be pushing the linear specification too far to apply it in a stochastic setting, where constraints on μ could come into force at some point. In my view, the interior results obtained still provide a useful starting point for analyzing the model's empirical implications.

20. When $\psi^* = 1$ and when government spending follows the martingale process $G_t = G_{t-1} + \eta_t$, government *debt* follows the martingale process $D_t = D_{t-1} + \theta\varepsilon_t$ (where $\varepsilon_t = \eta_t/\rho$).

21. Governments might be reluctant, for example, to leave a possibly hostile successor with a large bequest of public assets.

REFERENCES

Alesina, A. & R. Perotti (1995) The political economy of budget deficits. *International Monetary Fund Staff Papers* 42, 1–31.

Auernheimer, L. (1974) The honest government's guide to the revenue from the creation of money. *Journal of Political Economy* 82, 598–606.

Barro, R.J. (1979) On the determination of the public debt. *Journal of Political Economy* 87, 940–971.

Barro, R.J. (1983) Inflationary finance under discretion and rules. *Canadian Journal of Economics* 16, 1–16.

Bernheim, B.D. & D. Ray (1987) Economic growth with intergenerational altruism. *Review of Economic Studies* 54, 227–241.

Bizer, D.S. & S.N. Durlauf (1990) Testing the positive theory of government finance. *Journal of Monetary Economics* 26, 123–141.

Bohn, H. (1988) Why do we have nominal government debt? *Journal of Monetary Economics* 21, 127–140.

Brock, W.A. (1974) Money and growth: The case of long-run perfect foresight. *International Economic Review* 15, 750–777.

Calvo, G.A. (1978) On the time consistency of optimal policy in a monetary economy. *Econometrica* 46, 1411–1428.

Calvo, G.A. & L. Leiderman (1992) Optimal inflation tax under precommitment: Theory and evidence. *American Economic Review* 82, 179–194.

Calvo, G.A. & M. Obstfeld (1990) Time consistency of fiscal and monetary policy: A comment. *Econometrica* 58, 1245–1247.

Campillo, M. & J.A. Miron (1997) Why does inflation differ across countries? In C.D. Romer & D.H. Romer (eds.), *Reducing Inflation: Motivation and Strategy*, pp. 53–79. Chicago: University of Chicago Press.

Chang, R. (1996) Credible Monetary Policy with Long Lived Agents: Recursive Approaches. Mimeo, Federal Reserve Bank of Atlanta.

Chari, V.V., & P.J. Kehoe (1990) Sustainable plans. *Journal of Political Economy* 98, 783–802.

Chari, V.V., P.J. Kehoe & E.C. Prescott (1989) Time consistency and policy. In R.J. Barro (ed.), *Modern Business Cycle Theory*, pp. 265–305. Cambridge, MA: Harvard University Press.

Chari, V.V., L.J. Christiano & P.J. Kehoe (1996) Optimality of the Friedman rule in economies with distorting taxes. *Journal of Monetary Economics* 37, 203–223.

Cohen, D. & P. Michel (1988) How should control theory be used to calculate a time-consistent government policy? *Review of Economic Studies* 55, 263–274.

Cole, H.L. & T.J. Kehoe (1996) A self-fulfilling model of Mexico's 1994–1995 debt crisis. *Journal of International Economics* 41, 309–330.

Driffill, J., G.E. Mizon & A. Ulph (1990) Costs of inflation. In B.M. Friedman & F.H. Hahn (eds.), *Handbook of Monetary Economics*, Vol. 2, pp. 1013–1066. Amsterdam: North-Holland.

Fischer, S. (1986) Time Consistent Monetary and Fiscal Policies: A Survey. Mimeo, Massachusetts Institute of Technology.

Fudenberg, D. & J. Tirole (1986) *Dynamic Models of Oligopoly*, Fundamentals of Pure and Applied Economics 3. Chur, Switzerland: Harwood Academic.

Gale, D. (1982) *Money*: *In Equilibrium*, Cambridge Economic Handbooks. Welwyn and Cambridge, UK: Nisbet and Cambridge University Press.

Grilli, V. (1988) Fiscal Policies and the Dollar/Pound Exchange Rate: 1870-1984. Mimeo, Yale University.

Grilli, V. (1989) Seigniorage in Europe. In M. de Cecco & A. Giovannini (eds.), *A European Central Bank? Perspectives on Monetary Unification After Ten Years of the EMS*, pp. 53–79. Cambridge, UK: Cambridge University Press.

Judd, K.L. (1989) Optimal Taxation in Dynamic Stochastic Economies. Mimeo, Hoover Institution.

Kydland, F.E. & E.C. Prescott (1977) Rules rather than discretion: The inconsistency of optimal plans. *Journal of Political Economy* 85, 473–492.

Lucas, R.E., Jr. & N.L. Stokey (1983) Optimal fiscal and monetary policy in an economy without capital. *Journal of Monetary Economics* 12, 55–93.

Mankiw, N.G. (1987) The optimal collection of seigniorage: Theory and evidence. *Journal of Monetary Economics* 20, 327–341.

Maskin, E. & J. Tirole (1988) A theory of dynamic oligopoly, I: Overview and quantity competition with large fixed costs. *Econometrica* 56, 549–569.

Obstfeld, M. (1991) A model of currency depreciation and the debt-inflation spiral. *Journal of Economic Dynamics and Control* 15, 151–177.

Persson, M., T. Persson & L.E.O. Svensson (1987) Time consistency of fiscal and monetary policy. *Econometrica* 55, 1419–1431.

Phelan, C. & E. Stacchetti (1996) Capital Taxation as a Recursive Game. Mimeo, Northwestern University.

Poterba, J.M. & J.J. Rotemberg (1990) Inflation and taxation with optimizing governments. *Journal of Money, Credit and Banking* 22, 1–18.

Rogoff, K. (1989) Reputation, coordination, and monetary policy. In R.J. Barro (ed.), *Modern Business Cycle Theory*, pp. 236–264. Cambridge, MA: Harvard University Press.

Sargent, T.J. (1987) *Dynamic Macroeconomic Theory*. Cambridge, MA: Harvard University Press.

Stokey, N.L. (1991) Credible public policy. *Journal of Economic Dynamics and Control* 15, 627–656.

APPENDIX

This Appendix takes care of some unfinished details from Section 5. Let β^* be the smaller of the roots given by (48), $\beta^{*\prime}$ the larger. Proof is given here that when $r = \rho$,

(i) $\beta^* \in (\rho\delta, 1)$ and $\beta^{*\prime} \in (1, \infty)$.
(ii) $\psi^* > 0$ [where ψ^* is defined by (49)].

Proof of (i). First notice that both β^* and $\beta^{*\prime}$ are real, because [see (48)] $\rho\delta(1 - \rho\delta)$ has its maximum at $\rho\delta = \frac{1}{2}$, and $\rho > 0$. The roots β^* and $\beta^{*\prime}$ are the zeroes of the polynomial

$$\zeta(\beta) = (1 - \rho\delta)\beta^2 - (1 + \rho)\beta + \rho\delta(1 + \rho) \quad \textbf{(A.1)}$$

[the left-hand side of (47) with $r = \rho$], which has the derivative

$$\zeta'(\beta) = 2(1 - \rho\delta)\beta - (1 + \rho). \quad \textbf{(A.2)}$$

Because $1 > \rho\delta$ according to (36), $\zeta(\rho\delta) > 0$ and $\zeta'(\rho\delta) < 0$; moreover, $\zeta'(\beta) < 0$ for all $\beta < \rho\delta$. So, necessarily, $\beta^* > \rho\delta$. However, $\zeta(1) = \rho(\rho\delta - 1) < 0$, so $\beta^* < 1$ and $\beta^{*\prime} > 1$. ■

Proof of (ii). With the help of (48) and (49), $\psi^* > 0$ can be shown, after much tedious algebra, to be equivalent to $\rho\delta(2 - \rho\delta) < 1$. The function $\rho\delta(2 - \rho\delta)$ reaches its maximum of 1 when $\rho\delta = 1$, however; so, assumption (36) $\Rightarrow \psi^* > 0$. ■