Default and the Maturity Structure in Sovereign Bonds*

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Abstract

This paper studies the maturity composition and the term structure of interest rate spreads of government debt in emerging markets. In the data, when interest rate spreads rise, debt maturity shortens and the spread on short-term bonds is higher than on long-term bonds. To account for this pattern, we build a dynamic model of international borrowing with endogenous default and multiple maturities of debt. Short-term debt can deliver higher immediate consumption than long-term debt; large long-term loans are not available because the borrower cannot commit to save in the near future towards repayment in the far future. However, issuing long-term debt can insure against the need to roll-over short-term debt at high interest rate spreads. The trade-off between these two benefits is quantitatively important for understanding the maturity composition in emerging markets. When calibrated to data from Brazil, the model matches the dynamics in the maturity of debt issuances and its comovement with the level of spreads across maturities.

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1 Introduction

Emerging markets face recurrent and costly financial crises that are characterized by limited access to credit and high interest rates on foreign debt. As crises approach, not only is debt limited but also the maturity of debt shortens, as documented by Broner, Lorenzoni, and Schmukler (2007).\footnote{Calvo and Mendoza (1996) document in detail how in Mexico during 1994, most of the public debt was converted to 91-day Tesobonos. Bevilaqua and Garcia (2000) document a similar rise in short-term government debt in Brazil during the 1999 crisis.} During these periods, however, the interest rate spread on short-term bonds rises more than the spread on long-term bonds. Why do countries shorten their debt maturity during crises even though spreads appear higher for shorter maturity debt? To answer this question, this paper develops a dynamic model of the maturity composition in which debt prices reflect endogenous default risk and debt maturity responds to the prices of short- and long-term debt contracts. Our model can rationalize shorter debt maturity during crises as the result of a liquidity advantage in short-term debt contracts; although these contracts carry higher spreads than longer term debt, they can deliver larger resources to the country in times of high default risk.

We first analyze the dynamics of the maturity composition of international bonds and the term structure of interest rate spreads for four emerging market countries: Argentina, Brazil, Mexico, and Russia. We use data on prices and issuances of foreign-currency denominated bonds to estimate spread curves – interest rate spreads over U.S. Treasury bonds across maturity – as well as duration, a measure of the average time to maturity of payments on coupon paying bonds. We find that governments issue short-term debt more heavily when spreads are high and spread curves are downward sloping, and they issue long-term debt more heavily when spreads are low and spread curves are upward sloping. Across these four countries, within periods in which 2-year spreads are below their 25th percentile, the average duration of new debt is 7.1 years, and the average difference between the 10-year spread and the 2-year spread is 2.3 percentage points. But when the 2-year spreads are above their 75th percentile, the average duration shortens to 5.7 years, while the average difference between the 10-year spread and the 2-year spread is $-0.5$ percentage points. From this evidence we conclude that the maturity of debt shortens in times of high spreads and downward-sloping spread curves.

We then develop a dynamic model with defaultable bonds to study the choice of debt maturity and its covariation with the term structure of spreads. In our model, a risk averse borrower faces persistent income shocks and can issue long and short duration bonds. The borrower can default on debt at any point in time, but faces costs of doing so. Default occurs...
in equilibrium in low-income, high-debt times because the cost of coupon payments outweighs
the costs of default when consumption is low. Interest rate spreads on long and short bonds
compensate foreign lenders for the risk-adjusted expected loss from future defaults. Thus,
the supply of credit is more stringent in times of low income and high outstanding debt,
because the probability of default is high.

The model generates the observed dynamics of spread curves because the endogenous
probability of default is persistent, yet mean reverting, as a result of the dynamics of debt
and income. When debt is low and income is high, default is unlikely in the near future, so
spreads are low. However, long-terms spreads are higher than short-term spreads because
default may become likely in the far future if the borrower receives a sequence of bad shocks
and accumulates debt. On the other hand, when income is low and debt is high, default is
likely in the near future, so spreads are high. Long-term spreads, however, increase by less
than short-term spreads because the borrower’s likelihood of repaying may rise if it receives
a sequence of good shocks and reduces its debt. Although cumulative default probabilities
on long-term debt are always larger than on short-term debt, the long spread can be lower
than the short spread because it reflects a lower average future default probability.

The model can rationalize the covariation observed in the data between the maturity
structure of debt issuances and the term structure of spreads as reflecting a trade-off between
insurance benefits of long-term debt and liquidity benefits of short-term debt, both due to
the presence of default. Long-term debt provides insurance against the uncertainty of short-
term interest rate spreads. Since short-term spreads rise during periods of low income, when
default risk is high, issuing long-term debt allows the borrower to avoid rolling over short-
term debt at high spreads in states when consumption is low. Moreover, long-term debt
insures against future periods of limited credit availability; in particular, the borrower can
avoid capital outflows in recessions by issuing long-term debt.

Even though long debt dominates short debt in terms of insurance, it is not as effective in
delivering high immediate consumption; hence the liquidity benefit of short-term debt. Short-
term debt allows the borrower to pledge more of his future income toward debt repayment
because in each subsequent period the threat of default punishment gives him incentives for
repayment before any further short debt is issued. Long-term debt contracts do not allow
such large transfers because the borrower is unable to commit to saving in the near future
toward repayment in the further future. Effectively, the threat of default punishment is lower
with long-term debt given that it will be relevant only in the future, when the long-term debt
is due. This greater efficacy of short-term debt in alleviating commitment problems for debt
repayment is reflected in more lenient price schedules and smaller drops in short-term prices
with increases in the level of debt issues. In this sense, short debt is a more liquid asset, and consumption can always be marginally increased by more with short-term debt than with long-term debt.

The time-varying maturity structure responds to a time-varying valuation of the insurance benefit of long-term debt and the liquidity benefit of short-term debt. Periods of low default probabilities and upward spread curves correspond to states when the borrower is wealthy and values insurance. Thus, the portfolio is shifted toward long debt. Periods of high default probabilities and inverted spread curves correspond to states when the borrower is poor and credit is limited. These are times when liquidity is most valuable, and thus the portfolio is shifted toward shorter-term debt. We can therefore rationalize higher short-term debt positions in times of crises as an optimal response to the illiquidity of long-term debt, and the tighter availability of its supply.

When calibrated to Brazilian data, the model quantitatively matches the dynamics of the maturity composition of new debt issuances and its covariation with spreads observed in the data. Our findings indicate that the insurance benefits of long-term debt and the liquidity benefits of short-term debt are quantitatively important in understanding the dynamics of the maturity structure observed in Brazil. Importantly, the maturity structure in the model responds to the underlying dynamics of default probabilities reflected in spread curves, which match the data well.

**Related Literature**

This paper is related to the literature on the optimal maturity structure of government debt. Angeletos (2002), Buera and Nicolini (2004) and Shin (2007) show that, when debt is not state contingent, a rich maturity structure of government bonds can be used to replicate the allocations obtained with state-contingent debt in economies with distortionary taxes as in Lucas and Stokey (1983). In these closed economy models, short- and long-term interest rate dynamics reflect the variation in the representative agent’s marginal rate of substitution, which changes with the state of the economy. Thus, having a rich enough maturity structure is equivalent to having assets with state-contingent payoffs. Our paper shares with these papers the message that managing the maturity composition of debt can provide benefits to the government because of uncertainty over future interest rates. The message is particularly relevant for the case of emerging market economies. As Neumeyer and Perri (2005) have

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2 Lustig, Sleet, and Yeltekin (2006) develop a general equilibrium model with uninsurable nominal frictions to study the optimal maturity of government debt. They find that higher interest rates on long-term debt relative to short-term debt reflect an insurance premium paid by the government, for the benefits long-term debt provides in hedging against future shocks.
shown, fluctuations in country specific interest rate spreads play a major role in accounting for the large business cycle fluctuations in emerging markets. The lesson that our paper provides in this context is that the volatility of the maturity composition of debt in these countries is an optimal response to these interest rate fluctuations. However, in contrast to these papers, the fluctuations in interest rates in our model reflect time variation in the endogenous country’s own probability of default.3

The maturity of debt in emerging countries is also of interest because of the general view that countries could alleviate their vulnerability to very costly crises by choosing the appropriate maturity structure. For example, Cole and Kehoe (1996) argue that the 1994 Mexican debt crisis could have been avoided if the maturity of government debt had been longer. Longer maturity debt would allow countries to better manage external shocks and sudden stops. Broner, Lorenzoni, and Schmukler (2007) formalize this idea in a model where the government can avoid a crisis in the short term by issuing long-term debt. In their model, with risk averse lenders who face liquidity shocks, long-term debt is more expensive, so the maturity composition is the result of a trade-off between safer long-term debt and cheaper short-term debt. In line with their paper, we also find that short-term debt provides larger liquidity benefits. In contrast to Broner, Lorenzoni, and Schmukler, in our model the time-varying availability of short- and long-term debt is an equilibrium response to compensate for the economy's default risk, rather than to compensate for foreign lenders' shocks. Moreover, our paper is the first to develop a dynamic framework with defaultable debt and multiple maturities with which these questions can be analyzed and assessed quantitatively.

The larger liquidity benefits of short-term debt relative to long-term debt arise in our model because short-term contracts are more effective in solving the commitment problem of the borrower in terms of future debt and default policies. In this regard, our paper is related to Jeanne’s (2004) model where short-term debt gives more incentives for the government to implement better policies. When short-term debt needs to be rolled over, creditors can discipline the government by rolling over the debt only after desired policies are implemented.4 Moreover, when defaulted debt is renegotiated, Bi (2007) shows that long-term debt is more expensive also to compensate for debt dilution. Absent explicit seniority clauses, issuing short-term debt can dilute the recovery of long-term debt in case of default.

The theoretical model in this paper builds on the work of Aguiar and Gopinath (2006) and Arellano (2008), who model equilibrium default with incomplete markets, as in the seminal

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3The idea that credit risk makes longer term debt attractive is also present in Diamond (1991) in a three period model of corporate debt where firms have private information about their future credit rating.

4Commitment problems have been shown to reduce the level of sustainable debt in the literature of optimal policy without commitment, as in Krusell, Martin, and Rios-Rull (2006).
paper on sovereign debt by Eaton and Gersovitz (1981). This paper extends this framework to incorporate long debt of multiple maturities. In connecting our model to the data, a contribution of this paper is to develop a tractable framework with multiple bonds that have empirically relevant duration, rather than one-period debt, as in these earlier papers. In recent work, Chatterjee and Eyigungor (2009) and Hatchondo and Martinez (2009) show that models with a single long-term defaultable bond allow a better fit of emerging market data in terms of the volatility and mean of the country spread as well as debt levels. Bonds in our model are perpetuity contracts of the sort considered in Hatchondo and Martinez (2009), with non-state-contingent coupon payments that decay at different rates. Bonds with payments that decay quickly have more of their value paid early, and so have short duration. This gives a recursive structure to debt accumulation that allows the model to be characterized in terms of a small number of state variables although decisions at any date are contingent on a long sequence of future expected payments.

Models with endogenous default generate a time-varying probability of default that is linked to the dynamics of debt and income. With multiple maturities of debt, the dynamics of the spread curve in our model reflect the time-varying default probability, in the same way that Merton (1974) derived for credit spread curves on defaultable corporate bonds. In Merton’s model, when the exogenous default probability is low, the credit spread curve is upward sloping, and when the default probability is high, credit spread curves are downward sloping or hump shaped. The spread curve dynamics in this paper follow Merton’s results. However, our framework differs from Merton’s in that the probability of default and the level and maturity composition of debt issuances are endogenous variables.

The outline of the paper is as follows. Section 2 documents the dynamics of the spread curve and maturity composition for four emerging markets: Argentina, Brazil, Mexico, and Russia. Section 3 presents the theoretical model. Section 4 presents some examples to illustrate the mechanism for the optimal debt portfolio. Section 5 presents all the quantitative results, and Section 6 concludes.

2 Emerging Markets Bond Data

We examine data on sovereign bonds issued in international financial markets by four emerging-market countries: Argentina, Brazil, Mexico, and Russia. We look at the behavior of the interest rate spreads over default-free bonds, across different maturities, and at the way the maturity of new debt issued covaries with spreads. We find that when spreads are low, governments issue long-term bonds more heavily and long-term spreads are higher than short-term
spreads. When spreads rise, the maturity of bond issuances shortens and short-term spreads are higher than long-term spreads. Our findings also confirm the earlier results of Broner, Lorenzoni, and Schmukler (2007), who showed in a sample of eight emerging economies that debt maturity shortens when spreads are very high.5

2.1 Spread Curves

We define the $n$-year spread for an emerging market country as the difference between the yield on a defaultable, zero-coupon bond maturing in $n$ years issued by the country and on a zero-coupon bond of the same maturity with negligible default risk (for example, a U.S. Treasury note). The spread is the implicit interest rate premium required by investors to be willing to purchase a defaultable bond of a given maturity. The spread curve depicts spreads as a function of maturity.

We denote the continuously compounded yield at date $t$ on a zero-coupon bond issued by country $i$, maturing in $n$ years, as $r^n_{t,i}$. The yield is related to the price $p^n_{t,i}$ of an $n$-year zero-coupon bond, with face value 1, through

$$p^n_{t,i} = \exp(-n \times r^n_{t,i}).$$

We define country $i$’s $n$-year spread as the difference in zero-coupon yields between a bond issued by country $i$ relative to a default-free bond. The $n$-year spread for country $i$ at date $t$ is given by: $s^n_{t,i} = r^n_{t,i} - r^n_{t,r,f}$, where $r^n_{t,r,f}$ is the yield of a $n$-year default-free bond.6

Since governments do not issue zero-coupon bonds in a wide range of maturities, we estimate a country’s spread curve by using secondary market data on the prices at which coupon-bearing bonds trade. The estimation procedure consists of choosing a functional form for the spread curve to fit the discounted value of coupon payments to prices, following Svensson (1994) and Broner, Lorenzoni, and Schmukler (2007). We describe this procedure further in the Appendix and illustrate that the resulting pricing errors are small.

We compute spreads starting in March 1996 at the earliest and ending in May 2004 at the latest, depending on the availability of data for each country. Figure 1 displays the estimated spreads for 2-year and 10-year bonds for Argentina, Brazil, Mexico, and Russia.

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5Broner, Lorenzoni and Schmukler (2007) focus on the relationship between the term structure of risk premia (compensation for risk aversion) and the average maturity of debt. In this section we construct measures of the term structure of yield spreads and the average duration of debt because these statistics provide the basis for the quantitative assessment of our model.

6Our data include bonds denominated in U.S. dollars and European currencies, so we take U.S. and Euro-area government bond yields as default-free.
Spreads are very volatile, and the difference between long-term and short-term spreads varies substantially over time. When spreads are low, long-term spreads are generally higher than short-term spreads. However, when the level of spreads rises, the gap between long and short-term spreads tends to narrow and sometimes reverses; the spread curve is flatter or inverted. The time series in Figure 1 show sharp increases in interest rate spreads associated with Russia’s default in 1998, Argentina’s default in 2001, and Brazil’s financial crisis in 2002. The expectation that the countries would default in these episodes is reflected in the high spreads charged on defaultable bonds.

To emphasize the pattern observed in the time series that short-term spreads tend to rise more than long-term spreads, in Figure 2 we display spread curves averaged across different

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7 For Argentina and Russia, we do not report spreads after default on external debt, unless a restructuring agreement was largely completed at a later date. We use dates taken from Sturzenegger and Zettelmeyer (2005). For Argentina, we report spreads until the last week of December 2001, when the country defaulted. The restructuring agreement for external debt was not offered until 2005. For Russia, we report spreads until the second week of August 1998 and beginning again after August 2000 when 75% of external debt had been restructured.
time periods for each country: the overall average, the average within periods with the 2-year spread below its 10th percentile, and the average within periods with the 2-year spread above its 90th percentile. When spreads are low, the spread curve is upward sloping: long-term spreads are higher than short-term spreads. When spreads are high, short-term spreads rise more than long-term spreads. For Argentina, Brazil, and Russia, the spread curve becomes downward sloping in these times. For Mexico, which had relatively smaller increases in spreads during this time period, the spread curve flattens as short spreads rise more than long spreads.\footnote{The findings are similar to empirical findings on spread curves in corporate debt markets. Sarig and Warga (1989), for example, find that highly rated corporate bonds have low levels of spreads, and spread curves that are flat or upward-sloping, while low-grade corporate bonds have high levels of spreads, and average spread curves that are hump-shaped or downward-sloping.}

2.2 The Maturity Composition of Debt and Spreads

We now examine the maturity of new debt issued by the four emerging market economies during the sample period, and relate the changes in the maturity of debt to changes in spreads.\footnote{In addition to external bond debt, emerging countries also have debt obligations with multilateral institutions and foreign banks. However, marketable debt constitutes a large fraction of the external debt. The average marketable debt from 1996 to 2004 is 56\% of total external debt in Argentina, 59\% in Brazil, and 58\% in Mexico (Cowan et al. 2006).}

In each week in the sample, we measure the maturity of debt as a quantity-weighted average maturity of bonds issued that week. We measure the maturity of a bond using two alternative statistics. The first is simply the number of years from the issue date until the maturity date. The second is the bond’s duration, defined in Macaulay (1938) as a weighted average of the number of years until each of the bond’s future payments. A bond issued at date \( t \) by country \( i \), paying annual coupon \( c \) at dates \( n_1, n_2, \ldots, n_J \) years into the future, and face value of 1 has duration \( d_{t,i}(c) \) defined by

\[
d_{t,i}(c) = \frac{1}{p_{t,i}(c)} \left( \sum_{j=1}^{J} \exp(-n_j r_{t,i}^{n_j}) n_j c + \exp(-n_J r_{t,i}^{n_J}) n_J \right),
\]

where \( p_{t,i}(c) \) is the coupon bond’s price, and \( r_{t,i}^{n} \) is the zero-coupon yield curve. The time until each future payment is weighted by the discounted value of that payment relative to the price of the bond. A zero-coupon bond has duration equal to the number of years until its maturity date, but a coupon-paying bond maturing on the same date has shorter duration. We consider duration as a measure of maturity because it is more comparable across bonds.
Figure 2: Average spread curves: overall, and within periods in the highest and lowest deciles of the 2-year spread.

with different coupon rates.

We calculate the average maturity and average duration of new bonds issued in each week by each country. Table 1 displays each country’s averages of these weekly maturity and duration series within periods of high (above median) and low (below median) 2-year spreads.

First, the table shows that duration tends to be much shorter than maturity. Because the yield on an emerging market bond is typically high, the principal payment at the maturity date is severely discounted, and much of the bond’s value comes from coupon payments made sooner in the future. This weight on coupon payments shortens the duration measure relative
Table 1: Average Maturity and Duration of New Debt

<table>
<thead>
<tr>
<th>2-year spread:</th>
<th>Maturity (years)</th>
<th>Duration (years)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>&lt; median</td>
<td>≥ median</td>
</tr>
<tr>
<td>Argentina</td>
<td>10.25</td>
<td>7.79</td>
</tr>
<tr>
<td>Brazil</td>
<td>13.66</td>
<td>7.06</td>
</tr>
<tr>
<td>Mexico</td>
<td>13.24</td>
<td>10.32</td>
</tr>
<tr>
<td>Russia</td>
<td>8.41</td>
<td>11.78</td>
</tr>
</tbody>
</table>

to the time-to-maturity measure.

Second, the average duration of debt is shorter when spreads are high than when they are low. Mexico, for example, issues debt that averages about 1.2 years longer in duration when the 2-year spread is below its median than when it is above its median. For all countries except Russia, this pattern also holds for the average time-to-maturity of bonds issued during periods of high spreads compared to low spreads: Mexico issues bonds that mature about 3 years sooner when spreads are high.

In Table 2, we report the results of several univariate panel regressions of duration on spread measures. We pool the data for the four countries and include country fixed effects. All coefficients are significant at the 1% level, and robust standard errors are reported in parentheses. Column I reports the effect of the 2-year spread on the duration of new issuances. The coefficient means that a one percentage point increase in the 2-year spread is associated with a decrease in the duration of new debt issued by just under half a year. Column 2 shows a similar effect for the 10-year spread. These figures indicate that the covariation between the duration of new debt issuance and interest rate spreads is both economically large and statistically significant.10

In Table 3, we emphasize the relationship between the spread curve slopes and average duration. The slope of the spread curve, defined here as the difference between the 10-year (long-term) and 2-year (short-term) spread, falls when the 2-year spread is high – the numbers in column 4 of Table 3 are smaller than those in column 3. During these times, however, the countries shift toward short-term debt, even though the spreads on long-term debt rise

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10Our estimates mirror the findings in Broner, Lorenzoni, and Schmukler (2007). They show that a high spread level is a statistically significant determinant for a shorter maturity of debt issuances even after controlling for selection effects due the fact that the timing of debt issuances is very irregular. Their empirical work treats the issuance of short or long term debt as a discrete variable, whereas we use the continuous variable of duration as a measurement of maturity.
Table 2: Regressions of Duration of New Issuances on Spreads

<table>
<thead>
<tr>
<th>Variable</th>
<th>I</th>
<th>II</th>
<th>III</th>
</tr>
</thead>
<tbody>
<tr>
<td>2-year spread</td>
<td>−0.446</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.116)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10-year spread</td>
<td></td>
<td>−0.394</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.102)</td>
<td></td>
</tr>
<tr>
<td>10/2 ratio</td>
<td></td>
<td></td>
<td>0.543</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>(0.162)</td>
</tr>
<tr>
<td>Constant</td>
<td>7.241</td>
<td>7.836</td>
<td>4.916</td>
</tr>
<tr>
<td></td>
<td>(0.354)</td>
<td>(0.516)</td>
<td>(0.350)</td>
</tr>
<tr>
<td>$R^2$</td>
<td>0.201</td>
<td>0.195</td>
<td>0.157</td>
</tr>
<tr>
<td>No. of obs.</td>
<td>151</td>
<td>151</td>
<td>151</td>
</tr>
</tbody>
</table>

Table 3: Slope of Spread Curve and Average Duration of Issuances

<table>
<thead>
<tr>
<th>Spread curve slope (%)</th>
<th>Duration (years)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s_{10} - s^2$</td>
<td>short spread:</td>
</tr>
<tr>
<td></td>
<td>&lt; 25th pct</td>
</tr>
<tr>
<td></td>
<td>≥ 75th pct</td>
</tr>
<tr>
<td>Argentina</td>
<td>6.33</td>
</tr>
<tr>
<td>Brazil</td>
<td>6.94</td>
</tr>
<tr>
<td>Mexico</td>
<td>8.79</td>
</tr>
<tr>
<td>Russia</td>
<td>6.46</td>
</tr>
</tbody>
</table>

less than for short-term debt. In Brazil, for example, while the spread curve changes from depicting a 10-year spread that is about 3 percentage points above the 2-year spread to one that is 1.41 percentage points below the 2-year spread, the average duration of newly issued debt reduces by more than 2 years.

2.3 Summary

The message of this section is that the spread curve and the maturity of bond issuances in emerging markets are time-varying. In particular, the level of spreads covaries negatively
with the maturity of new debt, and with the slope of the spread curve: when spreads are low, the slope of the spread curve is higher, and the maturity of new debt is longer, than when short-term spreads are high.

Standard asset pricing arguments would equate the price of a sovereign bond to the present discounted value of payments the bondholder expects to receive, adjusted for risk. Variation in interest rate spreads in this context, across maturity and over time, can come from a variety of sources, in particular, changes in the expected probability of default; changes in the amount lenders can recover in case of default; and variation in the lenders’ compensation for risk, or risk premium. To understand the determinants of the maturity composition of debt, it is important to disentangle these three factors, since they may have different effects on the incentives to accumulate short- and long-term debt. Since we do not observe the time paths of the probability of default, the expected recovery rate, or the risk premium in the data, in the following sections we build a model in which the quantity and maturity of debt are endogenous, and these three factors play a role in pricing it. When we calibrate this model to data for an emerging market economy, we can quantify the importance of these factors in explaining the relationship between spreads and debt maturity.

3 The Model

We consider a dynamic model of defaultable debt that includes bonds of short and long duration. A small open economy receives a stochastic stream of output, $y$, of a tradable good. The output shock follows a Markov process with compact support and transition function $f(y_t, y_{t+1})$. The economy trades two bonds of different duration with international lenders. Financial contracts are unenforceable, so the economy can default on its debt at any time. If the economy defaults, it temporarily loses access to international financial markets and also incurs direct output costs.

The representative agent in the small open economy (henceforth, the “borrower”) receives utility from consumption $c_t$ and has preferences given by

$$E \sum_{t=0}^{\infty} \beta^t u(c_t),$$

where $0 < \beta < 1$ is the time discount factor and $u(\cdot)$ is increasing and concave.

The borrower issues debt in the form of two types of perpetuity contracts with coupon payments that decay geometrically. We let $\{\delta_S, \delta_L\} \in [0, 1]$ denote the “decay factors” of the payments for the two bonds. A perpetuity with decay factor $\delta_m$ is a contract that specifies
a price $q_t^m$ and a loan face value $\ell_t^m$ such that the borrower receives $q_t^m \ell_t^m$ units of goods in period $t$ and promises to pay, conditional on not defaulting, $\delta_t^{m-1} \ell_t^m$ units of goods in every future period $t + n$. The decay of each perpetuity is related to its duration: a bond of this type with rapidly declining payments has a larger proportion of its value paid early on, and therefore a shorter duration, than a bond with more slowly declining payments. We let $\delta_S < \delta_L$, so that $\delta_S$ is the decay of the perpetuity with short duration and $\delta_L$ is the decay of the perpetuity with long duration. We will refer to the perpetuities with decay factors $\delta_S$ and $\delta_L$ throughout as short and long bonds, respectively.\(^{11}\)

At every time $t$ the economy has outstanding all past bond issuances. Define $b_t^m$, the stock of bonds of duration $m$ at time $t$, as the total payments due in period $t$ on all past issuances of type $m$, conditional on not defaulting:

$$b_t^m = \sum_{j=1}^{t} \delta_{m-j} \ell_{t-j}^m = \ell_{t-1}^m + \delta_m \ell_{t-2}^m + \delta_m^2 \ell_{t-3}^m + \ldots + \delta_m^{t-1} \ell_0^m + b_0^m,$$

where $b_0^m$ is given. Thus, the accumulation for the stocks of short and long perpetuities can be written recursively by the following laws of motion:

$$b_{t+1}^S = \delta_S b_t^S + \ell_t^S$$
$$b_{t+1}^L = \delta_L b_t^L + \ell_t^L$$

With these definitions, we can compactly write the borrower’s budget constraint conditional on not defaulting. Purchases of consumption are constrained by the endowment less payments on outstanding debt, $b_t^S + b_t^L$, plus the issues of new perpetuities of short duration $\ell_t^S$ at price $q_t^S$ and long duration $\ell_t^L$ at a price $q_t^L$:

$$c_t = y_t - b_t^S - b_t^L + q_t^S \ell_t^S + q_t^L \ell_t^L$$

The borrower chooses new issuances of perpetuities from a menu of contracts where prices $q_t^S$ and $q_t^L$ for are quoted for each pair $(b_{t+1}^S, b_{t+1}^L)$.

If the borrower defaults, we assume that all outstanding debts and assets $(b_t^S + b_t^L)$ are erased from the budget constraint, and the economy cannot borrow or save, so that con-

\(^{11}\)Each one of our bonds resembles the long-duration bond in Hatchondo and Martinez (2009) or Rudebusch and Swanson (2008). By having two such assets, we allow the borrower to choose the maturity composition of debt each period.
consumption equals output. In addition, the country incurs output costs:

\[ y_t^{\text{def}} = \begin{cases} y_t & \text{if } y_t \leq (1 - \lambda)\bar{y} \\ (1 - \lambda)\bar{y} & \text{if } y_t > (1 - \lambda)\bar{y} \end{cases}, \]

where \( \bar{y} \) is the mean level of output. This specification, following Arellano (2008), assumes that borrowers lose a fraction \( \lambda \) of output if output is above a threshold.

### 3.1 Recursive Problem

We now represent the borrower’s infinite horizon decision problem as a recursive dynamic programming problem. The model has two endogenous states, which are the stocks of each type of debt, \( b^S_t \) and \( b^L_t \), and one exogenous state, the output of the economy, \( y_t \). The state of the economy at date \( t \) is then given by \( (b^S_t, b^L_t, y_t) \equiv (b^S_t, b^L_t, y_t) \).

At any given state, the value of the option to default is given by

\[ v^c(b^S_t, b^L_t, y_t) = \max_{c,d} \left\{ v^c(b^S_t, b^L_t, y_t), v^d(y) \right\}, \tag{6} \]

where \( v^c(b^S_t, b^L_t, y_t) \) is the value associated with not defaulting and staying in the contract and \( v^d(y) \) is the value associated with default.

Since we assume that default costs are incurred whenever the borrower fails to repay its obligations in full, the model will only generate complete default on all outstanding debt, both short and long term. When the borrower defaults, output falls to \( y^{\text{def}} \), and the economy is temporarily in financial autarky; \( \theta \) is the probability that it will regain access to international credit markets each period. The value of default is then given by the following:

\[ v^d(y) = u(y^{\text{def}}) + \beta \int_{y'} \left[ \theta v^c(0, 0, y') + (1 - \theta)v^d(y') \right] f(y, y') dy'. \tag{7} \]

We are taking a simple route to model both costs of default that seem empirically relevant: exclusion from financial markets and direct costs in output. Moreover, we assume that the default value does not depend on the maturity composition of debt prior to default. This captures the idea that the maturity composition of defaulted debt is not relevant for the restructuring procedures that allow the economy to reenter the credit market.\(^{12}\)

\(^{12}\)This is consistent with empirical evidence regarding actual restructuring processes, where the maturity composition of the new debt obligations is part of the restructuring agreement (Sturzenegger and Zettelmeyer 2005).
When the borrower chooses to remain in the contract, the value is the following:

$$v^c = \max_{\{b'_S, b'_L, \ell_S, \ell_L, c\}} \left( u(c) + \beta \int_{y'} v^o(b'_S, b'_L, y') f(y, y') dy' \right)$$

(8)

subject to the budget constraint:

$$c - qS (b'_S, b'_L, y) \ell_S - qL (b'_S, b'_L, y) \ell_L = y - b_S - b_L$$

(9)

and to the laws of motion for the stock of perpetuities of short and long duration:

$$b'_S = \delta_S b_S + \ell_S$$

$$b'_L = \delta_L b_L + \ell_L.$$  

The borrower decides on optimal debt levels $b'_S$ and $b'_L$ to maximize utility. The borrower takes as given that each contract $\{b'_S, b'_L\}$ comes with specific prices $\{q^S, q^L\}$ that are contingent on today’s state $y$. The decision of whether to remain in the credit contract or default is a period-by-period decision, so that the expected value from next period forward in (8) incorporates the option to default in the future.

The default policy can be characterized by default sets and repayment sets. Let the repayment set, $R(b_S, b_L)$, be the set of output levels for which repayment is optimal when short- and long-term debt are $(b_S, b_L)$:

$$R(b_S, b_L) = \{y \in Y : v^c(b_S, b_L, y) \geq v^d(y)\},$$

(10)

and let the complement, the default set $D(b_S, b_L)$, be the set of output levels for which default is optimal for debt positions $(b_S, b_L)$:

$$D(b_S, b_L) = \{y \in Y : v^c(b_S, b_L, y) < v^d(y)\}.$$  

(11)

When the borrower does not default, optimal new debt takes the form of two decision rules mapping today’s state into tomorrow’s debt levels:

$$b'_S = \tilde{b}_S(b_S, b_L, y)$$

$$b'_L = \tilde{b}_L(b_S, b_L, y)$$

(12)

Given this characterization of debt and default decisions, we can now define the equilib-
equilibrium bond prices at which lenders are willing to offer contracts.

### 3.2 Bond Prices, Spreads, and Duration

Lenders are perfectly competitive, and value payoffs across states and time according to a pricing kernel, or stochastic discount factor, \( M(y_t, y_{t+1}) \), which we specify further below. Lenders are therefore willing to purchase a defaultable bond at a price equal to the (risk-adjusted) expected discounted value of payments received from the bond.

Each new issue of debt \( \ell_t^S > 0 \) or \( \ell_t^L > 0 \) by the borrower is a promise to pay a coupon payment every period in the future, conditional on not defaulting up to that period. If \( \ell_t^S \) or \( \ell_t^L \) is negative, then the borrower is repurchasing some of its debt. If the borrower defaults in period \( t \), we assume that lenders recover a fraction \( \varphi (b^S_t, b^L_t) \) of the outstanding value of debt. The price of a new debt issue, then, is the discounted sum of the value of the promised coupon payments, adjusted by the cumulative probability of repayment, plus the value of recovery in case of default. If the borrower’s state is \( (y_t, b^S_t, b^L_t) \), the prices \( q_t^m \) loans \( \ell_t^m \) of duration \( m = S, L \) are given by

\[
q_t^m = \sum_{n=1}^{\infty} \int_{R_{t+n-1}}^{\infty} \cdots \int_{R_{t+n-1}}^{\infty} \Phi_{t+n}^m \tilde{f} \left( y_{t+n-2}, y_{t+n-1} \right) \cdots \tilde{f} \left( y_t, y_{t+1} \right) dy_{t+n-1} \cdots dy_{t+1} \tag{13}
\]

where

\[
\Phi_{t+n}^m = \delta_m^{n-1} \int_{R_{t+n}} \tilde{f} \left( y_{t+n-1}, y_{t+n} \right) dy_{t+n} + \varphi \left( b^S_{t+n}, b^L_{t+n} \right) \int_{D_{t+n}} \tilde{f} \left( y_{t+n-1}, y_{t+n} \right) dy_{t+n} \tag{14}
\]

and \( \tilde{f} \left( y_t, y_{t+1} \right) = M \left( y_t, y_{t+1} \right) f \left( y_t, y_{t+1} \right) \).\(^{13}\) In (13)-(14), we use the shorthand \( R_t = R \left( b^S_t, b^L_t \right) \), \( D_t = D \left( b^S_t, b^L_t \right) \) for repayment and default sets. The term \( \Phi_{t+n}^m \) is the risk-adjusted expected payoff of \( m \)-maturity debt in period \( t + n \), conditional on the borrower having repaid up to period \( t + n - 1 \). It is given by a weighted sum of the coupon payment, \( \delta_m^{n-1} \), in those states in which the borrower repays, and the recovery amount, \( \varphi \left( b^S_{t+n}, b^L_{t+n} \right) \), in those states in which the borrower defaults.

Note that the price \( q_t^m \) of new debt issuances depends on current output, \( y_t \), as it influences expectations of future output realizations which determine future default decisions. The price also depends on the entire future sequence of debt levels, \( \{b^S_{t+n}, b^L_{t+n}\}_{n=1}^{\infty} \), since the outstanding debt in any period determines the decision to default, given the output shock. However, we can transform the infinite sum in (13) into a recursive expression for \( q_t^m \) by

\(^{13}\)\( \tilde{f} \) is just a normalization of the “risk-neutral” probability density, \( \frac{1}{\int_{M(y_t, y_{t+1})}^{\infty} M \left( y_t, y_{t+1} \right) f \left( y_t, y_{t+1} \right) dy_{t+1}} \).
assuming that the lender forecasts the future debt levels using the borrower’s own decision rules for debt, defined in (12), which are functions only of the debt choice next period. The sum in (13) can then be written as

\[
\int_{R(b_s', b_L', y)} f(y, y') dy' + \int_{D(b_s', b_L', y)} \varphi(b_s', b_L') f(y, y') dy' \\
+ \int_{R(b_s', b_L', y)} \delta_m \left[ \int_{R(b_s(b_s', b_L', y), b_L(b_s', b_L', y))} f(y, y') f(y', y'') dy'' \right] dy' + ...
\]

Each future debt level is replaced in sequence by the optimal decision rules \(\tilde{b}_s(b_s', b_L', y)\) and \(\tilde{b}_L(b_s', b_L', y)\). Prices for debt then satisfy the functional equations:\(^{14}\)

\[
q_s(b_s', b_L', y) = \int_{R(b_s', b_L')} \left[ 1 + \delta_S q_S \left( \tilde{b}_s(b_s', b_L', y'), \tilde{b}_L(b_s', b_L', y'), y' \right) \right] f(y', y) dy' \quad (15)
\]

\[
+ \int_{D(b_s', b_L')} \varphi(b_s', b_L') f(y, y') dy'
\]

\[
q_L(b_s', b_L', y) = \int_{R(b_s', b_L')} \left[ 1 + \delta_L q_L \left( \tilde{b}_s(b_s', b_L', y'), \tilde{b}_L(b_s', b_L', y'), y' \right) \right] f(y', y) dy' \quad (16)
\]

\[
+ \int_{D(b_s', b_L')} \varphi(b_s', b_L') f(y, y') dy'
\]

A recursive equilibrium for this economy is (i) a set of policy functions for consumption \(\bar{c}(b_s, b_L, y)\), new issuances for short-term debt \(\bar{\ell}_s(b_s, b_L, y)\) and long-term debt \(\bar{\ell}_L(b_s, b_L, y)\), perpetuity stocks for short-term debt \(\tilde{b}_s(b_s, b_L, y)\) and long-term debt \(\tilde{b}_L(b_s, b_L, y)\), repayment sets \(R(b_s, b_L)\), and default sets \(D(b_s, b_L)\), and (ii) price functions for short debt \(q_s(b_s', b_L', y)\) and long debt \(q_L(b_s', b_L', y)\), such that:

1. Taking as given the bond price functions \(q_s(b_s', b_L', y)\) and \(q_L(b_s', b_L', y)\), the policy functions \(\bar{b}_s(b_s, b_L, y), \bar{b}_L(b_s, b_L, y), \bar{\ell}_s(b_s, b_L, y), \bar{\ell}_L(b_s, b_L, y)\) and \(\bar{c}(b_s, b_L, y)\), repayment sets \(R(b_s, b_L)\), and default sets \(D(b_s, b_L)\) satisfy the borrower’s optimization problem.

2. The bond price functions \(q_s(b_s', b_L', y)\) and \(q_L(b_s', b_L', y)\) satisfy equations (15) and (16).

To compare the model’s predictions to the data, we define the yield on each bond as in the data, as the implicit constant interest rate at which the discounted value of the bond’s

\(^{14}\)We could alternatively assume that when \(\ell_s < 0\) or \(\ell_L < 0\), i.e. when the borrower is saving, the price of these savings contracts carry a different price, for example the risk-free interest rate. Results are similar with this alternative specification.
coupons equal its price. That is, given an equilibrium price \( q_m \) at any state, the yield \( r_m \) is defined from

\[
q_m = \sum_{n=1}^{\infty} e^{-nr_m} \delta_m^{n-1}.
\]

So,

\[
r_S = \log \left( \frac{1}{q_S} + \delta_S \right) \quad \text{and} \quad r_L = \log \left( \frac{1}{q_L} + \delta_L \right). \tag{17}
\]

We define the spread as the difference between the yield on a defaultable bond and the default-free rate:

\[
s_S = r_S - r^*_S \quad \text{and} \quad s_L = r_L - r^*_L.
\]

The default-free rates \( r^*_m \) are the analogues of (17) defined from the prices of default-free bonds.

As output and debt change, the probability of default varies over time, and therefore the prices of long-term and short-term debt differ, since they each put different weights on repayment probabilities in the future, as seen in (13). Spreads on short-term and long-term bonds therefore generally differ, and the relationship between the two spreads changes over time, so that the spread curve is time-varying.

Finally, we define the duration of debt issued at each date as the weighted average of the time until each coupon payment, with the weights determined by the fraction of the bond’s value on each payment date:

\[
d_m = \frac{1}{q_m} \sum_{n=1}^{\infty} ne^{-nr_m} \delta_m^{n-1}.
\]

So,

\[
d_S = \frac{1}{1 - \delta_S e^{-r_S}} \quad \text{and} \quad d_L = \frac{1}{1 - \delta_L e^{-r_L}}. \tag{18}
\]

4 Default and Optimal Maturity

In this section we illustrate the mechanisms that determine the optimal maturity composition of debt in a series of simplified example economies. We view the borrower’s choice as a portfolio allocation problem, in which the benefits and costs of short-term and long-term debt determine the relative amounts of each type issued. In the first example, we show that, in the presence of lack of commitment in future debt and default policies, short-term debt is more effective than long-term debt in transferring future resources to the present. If the borrower would try to borrow a lot of long-term debt, its price would fall to zero faster than if instead the large loan would be short-term; hence, short-term debt is beneficial for liquidity.
In the second example, we show that, in addition, short-term debt allows for more flexible default policies, which makes available resources that would not be otherwise. Finally, in the third example, we show that long-term debt allows the borrower to avoid the risk of rolling over short-term debt at prices that differ across future states due to differences in default risk; hence, long-term debt provides insurance.

We construct the simplest possible examples to illustrate the mechanisms clearly. The economy lasts for three periods. Lenders are risk-neutral, so that \( M(y, y') = 1 \). In period 0, the borrower’s income equals zero, and in periods 1 and 2 income is specified in each example. The borrower can default at any time, in which case consumption from then on is equal to \( y^{\text{def}} \). Lenders do not recover anything in case of default.

In each example, we compare the allocation with only one maturity of debt – one- or two-period bonds – against the allocation with both maturities of debt. In each economy, with both maturities available, in period 0 the borrower can issue one- and two-period bonds \( b^1_0 \) and \( b^2_0 \) given price schedules \( q^1_0(b^1_0, b^2_0) \) and \( q^2_0(b^1_0, b^2_0) \), and consumption is

\[
c_0 = q^1_0(b^1_0, b^2_0)b^1_0 + q^2_0(b^1_0, b^2_0)b^2_0.
\]

In period 1, conditional on not defaulting, new short bonds \( b^1_1 \) are issued given price schedule \( q^1_1(b^1_1) \). Consumption is equal to income plus net debt:

\[
c_1 = y_1 + q^1_1(b^1_1)b^1_1 - b^1_0.
\]

In period 2, conditional on not defaulting, the borrower pays off long- and short-term debt, and consumption equals income minus the repayment:

\[
c_2 = y_2 - (b^1_1 + b^2_0).
\]

In the cases with only one type of debt available, the budget constraints are modified accordingly.

The risk neutral lenders discount time at rate \( r \) and offer debt contracts that compensate them for the risk of default and give them zero expected profits.

4.1 Example 1: Short-Term Debt Provides Liquidity

This example illustrates that short term debt allows larger transfers to the borrower because it is a more effective instrument in dealing with the lack of commitment in savings policies.

For this example we consider the following income process. Income in period 0 is equal
Income in period 1 is equal to $y_1$. Income in period 2 equals $y_2$. Also, consumption in default, $y^{def}$, is equal to 0. To abstract from any insurance properties of debt, we assume that preferences are linear in consumption and given by

$$U = E[c_0 + \beta c_1 + \beta^2 c_2].$$

We assume that the borrower likes to front-load consumption, while lenders do not discount the future: $\beta < \frac{1}{1+r} = 1$, and we impose that consumption must be non-negative: $c_t \geq 0$ for $t = 0, 1, \text{ and } 2$.

### 4.1.1 Only Two-Period Bonds

First, consider the borrower’s problem when only two-period bonds are available in period 0, and one-period bonds are available in period 1. The solution to the borrower’s problem is the following. In period 0, the borrower borrows against all his period 2 income, at price 1, and in period 1 the borrower consumes his period 1 income, so consumption is

$$c_0 = y_2$$

$$c_1 = y_1$$

$$c_2 = 0.$$

Although the borrower does not have preferences for smoothing consumption over time, and would prefer to consume everything up front, it is not possible to consume everything in period 0, because none of the income in period 1 can be borrowed against using two-period debt. This is because such a contract would require a two-period loan with face value larger than $y_2$, so that the borrower would have to save part of the period 1 income to repay the loan in period 2. Since the borrower cannot commit to this policy in period 0, however, the optimal choice in period 1 would be not to save, and then to default in period 2 regardless of the level of income. That is, a debt contract that offered $q_0^2 l_0^2 = a + y_2$, for any $a > 0$, is not possible, because the probability of default on the loan would be equal to one, and hence the price $q_0^2$ would be zero. Effectively, the threat of punishment for default in period 2 when the two-period loan is due does not induce the borrower to repay, because the borrower discounts the future, so that reducing consumption in period 1 is worse than facing the punishment for default in period 2. At the same time, the threat of punishment for default in period 1 is irrelevant, because none of the debt is due in period 1, and the threat of punishment cannot be used to induce savings.
4.1.2 Only One-Period Bonds (or One- and Two-Period Bonds)

Now, if the borrower were able to issue one-period debt in period 0, consumption would be

\[ c_0 = y_1 + y_2 \]
\[ c_1 = 0 \]
\[ c_2 = 0. \]

Multiple possible portfolios allow this consumption pattern. The borrower could use short-term debt to borrow against all period 1 income and long-term debt to borrow against all period 2 income \((b_0^1 = y_1 \text{ with } q^1_0 = 1, b_0^2 = y_2 \text{ with } q^2_0 = 1, b_1^2 = 0)\); or, the borrower could use only short-term debt, issuing bonds in period 0 and period 1 \((b_0^1 = y_1 + y_2 \text{ with } q^1_0 = 1, b_1^1 = y_2 \text{ with } q^1_1 = 1)\). Since all consumption occurs in the first period, utility in this case is higher than in the case with long-term debt only. With one-period bonds, the threat of punishment for default is being used in both periods to induce repayment.

In this example, long-term debt is illiquid in the sense that a loan that would provide the same level of consumption in the first period does not exist, because the price of long-term debt falls to zero. This example illustrates that in the presence of lack of commitment in debt policies and default risk, short-term debt is more liquid due to more lenient bond prices, and thus it is a superior instrument to provide up-front resources.\(^{15}\)

4.2 Example 2: Short Debt Minimizes Default Costs

This example illustrates that short term debt allows the borrower to make default choices more state-contingent, which can make available more resources in the future to be allocated to consumption.

We extend Example 1 to introduce uncertainty in the endowment process. Income in periods 1 and 2 can take two values: \(y^H\) or \(y^L\) with \(y^H > y^L\). The probability of \(y^L\) is \(p\) with \(0 < p < 1\), and the realizations in periods 1 and 2 are independent. Throughout this example, we assume that parameters satisfy:

\[ \frac{(1-p) y^H - y^L}{(1-p)(y^H - y^L)} < \beta < \left( \frac{(1-p) y^H - y^L}{(1-p)(y^H - y^L)} \right)^{1/2} \]

\(\text{(19)}\)

\(^{15}\)It is easy to extend this example to an infinite horizon environment with deterministic and time varying output. A one-period bond economy can deliver higher initial consumption than a longer-term bond – two-period or perpetuity – economy. The main idea is again that the threat of punishment can be used more effectively with one-period bonds because longer-term contracts might require savings in the future which are impossible to induce with default punishments.
which ensures in each case below that the borrower is sufficiently impatient so as to borrow enough that default happens with positive probability, but also patient enough so as to value the future resources lost from defaulting.

4.2.1 Only Two-Period Bonds

With only two-period bonds available, the solution to the borrower’s problem is the following. In period 2, the borrower defaults when $y_2 = y_L$, independent whether in period 1 income is high or low. In period 0, the borrower borrows $y_H$, at price $(1 - p)$. In period 1 the borrower consumes the period 1 income. The consumption allocation is:

$$c_0 = (1 - p)y_H$$

$$c_1^i = y_i$$

$$c_2^{ij} = y_H - y_H = 0, \ c_2^{ij} = y_{def} = 0.$$ 

where $c_1^i$ denotes consumption in period 1 when income is $y_i$, and $c_2^{ij}$ denotes consumption in period 2 when $y_1 = y_i$ and $y_2 = y_j$, for $i, j = H, L$.

Without recourse to short-term debt in period 1, the borrower does not have the ability to follow different default policies depending on the state. Utility under this plan is

$$U^2 = (1 - p) y_H + \beta ((1 - p) y_H + py_L)$$

4.2.2 Only One-Period Bonds

With one-period bonds available, the solution to the borrower’s problem is to default only after two consecutive low income shocks, that is when $y_1 = y_L$ and $y_2 = y_L$. In all other states, the borrower borrows as much as possible given this default plan. In period 0, one-period debt issued is $b_0^1 = y_L + (1 - p) y_H$. In period 1, the borrower repays the one-period debt and borrows $y_L$ if $y_1 = y_H$ and $y_H$ if $y_1 = y_L$. This leads to the consumption allocation:

$$c_0 = y_L + (1 - p) y_H$$

$$c_1^H = py_H, \ c_1^L = 0$$

$$c_2^{HH} = y_H - y_L, \ c_2^{HL} = 0$$

$$c_2^{LH} = 0, \ c_2^{LL} = y_{def} = 0.$$
The borrower’s utility from this allocation is

\[ U^1 = y_L + (1 - p) y_H + \beta (1 - p) py_H + \beta^2 (1 - p)^2 (y_H - y_L) \]

The difference between utility under this allocation and utility with only two-period bonds can be written:

\[ U^1 - U^2 = (1 - \beta) y_L + \beta (1 - p) [\beta (1 - p) (y_H - y_L) - ((1 - p) y_H - y_L)] \]

The first term, \((1 - \beta) y_L\) reflects the effect discussed in Example 1: the borrower is able to transfer more resources to period 0, by increasing consumption by \(y_L\) in period 0 and reducing the resources available for the future by the same amount.

The term in square brackets reflects the tradeoff from the point of view of period 1, when income is \(y_H\), between borrowing \(y_L\) at price 1 and waiting to consume \(y_H - y_L\) in state \(y_2 = y_H\) tomorrow, or borrowing \(y_H\) at price \((1 - p)\) instead of \(y_L\), and consuming nothing tomorrow (and defaulting in the low-income state). Under the parameter restriction in (19), this difference is positive, so the borrower prefers to avoid defaulting.

In this allocation, the borrower wants to avoid borrowing too much in period 1, and would prefer the greater resources that are available by borrowing only up to the amount possible at a price of 1. The reason this is not possible when income is low in period 1 (the borrower still defaults when \(y_1 = y_2 = y_L\)) is that the borrower is constrained: consumption \(c^*_1\) in this state is already at zero, so borrowing any less would yield an infeasible consumption plan.

### 4.3 Example 3: Long-Term Debt Provides Insurance

In the first two examples, short debt dominates long debt, and long debt is redundant. For the third example, long debt will be useful for insurance. To focus on the insurance motive we assume that the borrower’s preferences are given by

\[ U = E[u(c_0) + \beta u(c_1) + \beta^2 u(c_2)] \]

with \(u(\cdot)\) strictly concave and \(\beta = 1\). We also now consider a different income process. Income in period 0 is equal to 0, income in period 1 is equal to \(y\), and income in period 2 can take two values: \(y^H\) or \(y^L\) with \(y^H > y^L\). The probability of \(y^H\) is learned in period 1 and can be either \(g\) or \(p\) with \(0 < g < 1\) and \(0 < p < 1\).
4.3.1 Only One-Period Bonds

First, consider the borrower’s choice under the assumption that only one-period bonds are available. Under the assumption that 
\[ \frac{y + \frac{\mu + \gamma}{1+\gamma} y_H}{2 + \frac{\mu + \gamma}{1+\gamma}} > y^{\text{def}} > y^L - \frac{2y^H - y}{2 + \frac{\mu + \gamma}{1+\gamma}} \]

the solution to the borrower’s problem is the following. The borrower defaults in period 2 if income is \( y^L \) and does not default in all other states. Hence, \( c_2^L (p) = c_2^L (g) = y^{\text{def}} \). Contingent on the realization of the probability \( p \) or \( g \), consumption is equalized between period 1 and the high-income state in period 2:

\[
\begin{align*}
c_1 (p) &= c_2^H (p) \\
c_1 (g) &= c_2^H (g)
\end{align*}
\]

Finally, consumption in period 0 is set to equalize expected marginal utility in period 1 to marginal utility in period 0:

\[
u' (c_0) = \frac{1}{2} (u' (c_1 (p)) + u' (c_1 (g)))
\]

Importantly, \( c_1 (p) \neq c_1 (g) \), so that consumption is not equalized across states within a period. With only short-term debt available, the borrower borrows in period 0, then borrows again in period 1. Debt issues are \( b_0^1 = c_0 \), \( b_1^1 (p) = \frac{\mu + \gamma - y}{1+p} \), and \( b_1^1 (g) = \frac{\mu + \gamma - y}{1+g} \). The price of debt issued in period 1 depends on the state realized: \( q_1^1 (p) = p \) and \( q_1^1 (g) = g \). Therefore, as long as \( p \neq g \), the price at which debt is rolled over in period 1 differs across states, and consumption differs as well.

4.3.2 One- and Two-Period Bonds

Now, if the borrower has access to both one- and two-period bonds, it is possible to equalize consumption across all states in which the borrower does not default:

\[
c_0 = c_1 (p) = c_1 (g) = c_2^H (p) = c_2^H (g) = \frac{\mu + y_H + y}{\mu + g + 2}
\]
The portfolio required involves using long-term and short-term debt in period 0, while borrowing nothing in period 1:

\[
\begin{align*}
\beta^0_0 &= \frac{2y_H - y}{(p+q) + 2} \\
\beta^1_0 &= \left(1 + \frac{p+q}{2}\right)y - \left(\frac{p+q}{2}\right)yH \\
\beta^1_1 &= 0
\end{align*}
\]

In this example the borrower faces risk because of the variation in bond prices across states in period 1 due to differences in default risk in period 2. Using long-term debt in period 0 allows the borrower to avoid the risk involved with rolling over short-term debt in period 1. The borrower benefits from this insurance with smoother consumption and higher utility.

Note that in period 0 short debt has a higher price than long debt, \(q^0_1 > q^2_0\), yet the borrower issues long-term debt. The lower discount price on long debt is the insurance premium the borrower is willing to pay for insurance against the variation in bond prices in period 1. This insurance mechanism is the same as that emphasized in Kreps (1982), Angeletos (2002) and Buera and Nicolini (2004) in their models of the optimal maturity structure of debt with incomplete markets. The difference in our model is that the source of the variation in bond prices is the government’s inability to commit to repaying, rather than the variation in the lender’s marginal rate of substitution.

### 4.4 Summary

In a standard incomplete markets model with fluctuating output and without default, a borrower would find the portfolio of long and short debt indeterminate if the risk-free rate were constant across time; the two assets would have payoffs that make them equivalent. However, in our model, the risk of default makes the two assets distinct. The first example illustrated that long-term debt is more illiquid than short-term debt due to the inability of the borrower to commit to future debt and default policies. In the second example, we showed that short-term debt also allows the borrower to make default more contingent on the state, and hence default less. However, the third example illustrated that long-term is beneficial because it provides a hedge against variations in interest rates, in effect insuring some of the default risk for the borrower.

Insurance and liquidity shape the optimal maturity structure of debt for a borrowing government. The quantitative relevance of each of these forces depends on the specifics of
preferences and the income process. Thus, in the next section we quantify these two sources by calibrating our general model to an actual emerging market economy.

5 Quantitative Analysis

5.1 Calibration

We solve the model numerically to evaluate its quantitative predictions regarding the dynamic behavior of the optimal maturity composition of debt and the spread curve in emerging markets. We calibrate an annual model to the Brazilian economy.

The utility function of the borrower is \( u(c) = \frac{c^{1-\sigma}}{1-\sigma} \). The borrower’s risk aversion coefficient is set to 2, which is a common value used in real business cycle studies. The stochastic process for output is a log-normal AR(1) process, \( \log(y_{t+1}) = \rho \log(y_t) + \varepsilon_{t+1} \) with \( E[\varepsilon^2] = \eta_o^2 \). We discretize the shocks into a seven-state Markov chain using a quadrature-based procedure (Tauchen and Hussey, 1991). We use annual series of GDP growth for 1960–2004 taken from the World Development Indicators to calibrate the volatility of output. Due to the short sample, rather than estimating the autocorrelation coefficient we choose an autocorrelation coefficient for the output process of 0.9, which is in line with standard estimates for developed countries. The decay parameters of the short and long bonds, \( \delta_S \) and \( \delta_L \), are set such that the default-free durations equal 2 and 10 years.

The function \( \varphi(b_S, b_L) \) in the pricing expressions (15)-(16) is a reduced-form representation of the fraction of debt recovered after default. We set this function to \( \varphi(b_S, b_L) = \exp\left(- (q_S^* b_S + q_L^* b_L)\right) \), with \( q_S^* \) and \( q_L^* \) equal to the default-free prices. This functional form is a convenient way to capture two features of the recovery rate: it lies between zero and one, and it declines with the quantity of debt issued. Intuitively, this is motivated by the idea that there is a fixed amount of surplus that lenders are able to extract from borrowers in the event of a default, so this amount declines as a fraction of debt as debt grows. Yue (2006) shows that, in a model in which debt recovery is determined by a bargaining process between lenders and a defaulting borrower, the endogenous recovery rate is decreasing and convex in debt, much like the function \( \varphi \). In our calibrated model, the average recovery rate is 40%, as compared to the average recovery rate of 60% reported in sovereign defaults reported by Benjamin and Wright (2009). We also choose the probability of reentering financial markets, \( \theta \), so that the average length of time in exclusion is 6 years, consistent with data presented in Benjamin and Wright (2009) on the median length of sovereign debt renegotiations.

The risk premium in our model comes from the interaction of the lenders’ pricing kernel
with default outcomes. We can rewrite equation (15) or (16) as a typical asset-pricing equation:

\[ q_m = E[M'x'_m] \]

where

\[ x'_m = \begin{cases} 
1 + \delta_m q'_m & \text{in states in which the borrower repays} \\
\varphi & \text{in states in which the borrower defaults}
\end{cases} \]

Written this way, the price of a bond is composed of the lender’s discounted expected payoff plus a risk premium term:

\[ q_m = E[M'] E[x'_m] + cov(M', x'_m) \]

To the extent that payoffs are negatively correlated with the pricing kernel, investors are compensated for this risk through a lower price \( q_m \) of debt. We specify the pricing kernel as follows:

\[ M(y_t, y_{t+1}) = \exp \left( -r - \gamma_t \varepsilon_{t+1} - \frac{1}{2} \gamma_t^2 \eta_y^2 \right) \]

\[ \gamma_t = \alpha_0 + \alpha_1 y_t \]

where \( r, \alpha_0, \) and \( \alpha_1 \) are parameters, \( \varepsilon_{t+1} \) is the shock to the borrower’s income, and \( \eta_y \) is its variance. The term \( r \) represents the risk-free interest rate; we set this at 4% annually, which equals the average annual yield of a two year U.S. bond from 1996 to 2004. The term \( \gamma_t \) controls how much lenders’ valuations are correlated with innovations \( \varepsilon_{t+1} \) to the borrower’s income level. The parameters \( \alpha_0 \) and \( \alpha_1 \) govern the size of this term and the extent to which it varies with present income, \( y_t \). (The term \( \frac{1}{2} \gamma_t^2 \eta_y^2 \) is a normalization.)

This specification is close to the form of the pricing kernel in Ang and Piazzesi (2003) and Cochrane and Piazzesi (2008) in analyzing the term structure of interest rates in the US. We define the pricing kernel as a function of only the borrower’s income because it is a parsimonious way to model risk premia that vary with the probability of default. In our model, default decisions depend on the model’s entire state, \((y, b_S, b_L)\), but our shortcut is valid for two reasons: first, default probabilities in the model are highly correlated with income. Second, in the absence of default, our model with this pricing kernel would generate a default-free interest rate that is constant and a flat default-free term structure. This is because, from equations (15) or (16), the price \( q_m^* \) of a default-free bond of duration \( m \) is
given by\textsuperscript{16}:

$$q_m^* = \int (1 + \delta_m q_m^*) M (y, y') f (y', y) dy'$$

$$= \frac{e^{-r}}{1 - \delta_m e^{-r}}$$

so the yield, defined as in equation (17), is constant and independent of the bond’s duration:\textsuperscript{17}

$$r_m^* = \log \left( \frac{1}{q_m^*} + \delta_m \right)$$

$$= r$$

This means that the size and variability of risk premia in our model reflect compensation for the risk associated with default losses and the changes in default probabilities, not the risk associated with fluctuations in the borrower’s income. The borrower’s income is a convenient state variable, because the borrower tends to default when income is low. Since income is persistent, low current income signals high default probabilities, and hence low prices, in the future. Therefore, since payoffs $x_m$ are positively correlated with the borrower’s income, the pricing kernel $M$ generates a positive risk premium if $\gamma_t \neq 0$. This risk premium varies with the state $y_t$ if $\alpha_1 \neq 0$, while its level is controlled by $\alpha_0$. If $\alpha_0$ is large, then the pricing kernel puts high weight on bad future states, and if $\alpha_1$ is large, this weight increases with the current level of the borrower’s income.

We calibrate these two parameters – $\alpha_0$ and $\alpha_1$ – as well as the output cost after default, $\lambda$, and the time preference parameter $\beta$, to match the following moments: the average 2- and 10-year spreads in Brazil, an average default probability of 4%, and a trade balance one quarter as volatile as output.

Table 4 summarizes the parameter values, and Table 5 presents some of the calibrated moments as well as other statistics from the model. The model predicts that consumption is about as variable as output, and that consumption is negatively correlated with spreads and the fraction of debt that is short term: in bad times, consumption is low, spreads are

\textsuperscript{16}This follows using

$$E [M (y, y') | y] = \exp \left( E [\log M (y, y') | y] + \frac{1}{2} \text{var} [\log M (y, y') | y] \right)$$

$$= \exp (-r)$$

since $M$ is lognormal.

\textsuperscript{17}In practice, since we discretize the state space of our model, we also have to normalize the pricing kernel so that default-free yields are constant in the discretized environment.
relatively high, and debt is mostly short term.

<table>
<thead>
<tr>
<th>Table 4: Parameters</th>
<th>Value</th>
<th>Target</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lenders’ discount rate</td>
<td>$r = 4%$</td>
<td>U.S. annual interest rate 4%</td>
</tr>
<tr>
<td>Borrower’s risk aversion</td>
<td>$\sigma = 2$</td>
<td>Standard value</td>
</tr>
<tr>
<td>Perpetuity decay factors</td>
<td>$\delta_S = 0.52$</td>
<td>Default-free durations of 2 and 10 years</td>
</tr>
<tr>
<td></td>
<td>$\delta_L = 0.936$</td>
<td></td>
</tr>
<tr>
<td>Stochastic structure</td>
<td>$\rho = 0.9, \eta = 0.022$</td>
<td>Brazil output</td>
</tr>
<tr>
<td>Probability of reentry</td>
<td>$\theta = 0.17$</td>
<td>Benjamin and Wright (2009)</td>
</tr>
<tr>
<td>Lenders’ pricing kernel</td>
<td>$\alpha_0 = 1.2$</td>
<td>Brazil average 2-year spread</td>
</tr>
<tr>
<td></td>
<td>$\alpha_1 = 15.0$</td>
<td>Brazil average 10-year spread</td>
</tr>
<tr>
<td>Output after default</td>
<td>$\lambda = 0.02$</td>
<td>Default probability 4%</td>
</tr>
<tr>
<td>Borrower’s discount factor</td>
<td>$\beta = 0.935$</td>
<td>volatility of trade balance</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Table 5: Model Statistics</th>
<th>Model statistics</th>
</tr>
</thead>
<tbody>
<tr>
<td>mean $s_S$ (percent)</td>
<td>5.63</td>
</tr>
<tr>
<td>mean $s_L$ (percent)</td>
<td>6.76</td>
</tr>
<tr>
<td>std(trade balance)/std(y)</td>
<td>0.23</td>
</tr>
<tr>
<td>mean recovery rate</td>
<td>0.41</td>
</tr>
<tr>
<td>std(c)/std(y)</td>
<td>0.99</td>
</tr>
<tr>
<td>corr(c, 2-year spread)</td>
<td>-0.33</td>
</tr>
<tr>
<td>corr(c, 10-year spread)</td>
<td>-0.27</td>
</tr>
<tr>
<td>corr(c, $\ell_S/(\ell_S + \ell_L)$)</td>
<td>-0.47</td>
</tr>
</tbody>
</table>

5.2 Results

We simulate the model, and in the following subsections we report statistics on the dynamic behavior of spreads and the maturity composition of debt from the limiting distribution of debt holdings. We first show that the model matches the data in generating time-varying differences in the pricing of short- and long-term debt, due to movements in the probability of default. The behavior of prices, in turn, reflects movements in the liquidity and insurance benefits of the two assets, and this rationalizes the maturity composition observed in the data.
5.2.1 Prices and Spreads

Default in our model happens when the economy has a low level of wealth, either due to low income or high debt. Since income and debt are persistent, this means that states with low income and high debt tend to have high spreads, as the future probabilities of default are high. We now compare spread dynamics in the model, driven by these features, to the data. The series for the data are Brazilian 2- and 10-year spreads and prices from Section 2. For this comparison, we organize the data into quantiles based on the level of the short spread. Table 6 presents average spreads for short and long debt across periods when short spreads are below their 25th and 50th percentiles and above their 50th and 75th percentiles.

<table>
<thead>
<tr>
<th>$s_S$ pct</th>
<th>DATA</th>
<th>MODEL</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$s_S$</td>
<td>$s_L$</td>
</tr>
<tr>
<td>&lt; 25</td>
<td>2.15</td>
<td>5.25</td>
</tr>
<tr>
<td>&lt; 50</td>
<td>2.68</td>
<td>5.35</td>
</tr>
<tr>
<td>≥ 50</td>
<td>8.51</td>
<td>8.93</td>
</tr>
<tr>
<td>≥ 75</td>
<td>12.25</td>
<td>10.84</td>
</tr>
<tr>
<td>Mean</td>
<td>5.60</td>
<td>7.14</td>
</tr>
</tbody>
</table>

The first two columns of Table 6 present the short and long spreads in the data, and the third and fourth columns present the model’s predictions. When default is unlikely, both spreads are low, and the spread curve is upward-sloping: when the short spread is below its 25th percentile, for example, the average short spread is 0.75%, and the average long spread is 2.95%. In contrast, when the probability of default is higher, both spreads rise, and the spread curve becomes downward-sloping: when the short spread is above the 75th percentile, the average short spread is 16.26%, and the average long spread is 14.97%. The fact that short spreads rise more than long spreads is also reflected in the difference in the volatilities of the two spread series: the standard deviation of the long spread is lower than that of the short spread. Compared to the data for Brazil, the model captures well the difference observed in the slope of the spread curve associated with periods of high and low short spreads, as well

---

As explained in Section 2, spread curves for Brazil are estimated using equation (22) in the Appendix. In the model for simplicity we are directly applying the yield to maturity formulas to compute spreads. However, we could also estimate spreads in the model using a reduced equation with only two parameters. Results under this alternative estimate are similar, although the spread curves are slightly steeper.
as the fact that the long spread is less volatile than the short spread. The overall volatility of spreads is higher than in the data.

Our model provides a decomposition of interest rate spreads into two parts: actuarially fair compensation for expected losses from default, and a risk premium. The actuarially fair price $q_{m}^{AF}$ of a bond of duration $m$ is defined by taking the bond pricing equations (15)-(16) and substituting just the probability density, $f$, for the product, $\tilde{f}$, of the pricing kernel and probability density:

$$q_{m}^{AF}(b'_S, b'_L, y) = \int_{R(b'_S, b'_L)} \left[ 1 + \delta_m q_{m}^{AF} \left( \tilde{b}_S(b'_S, b'_L, y'), \tilde{b}_L(b'_S, b'_L, y'), y' \right) \right] f(y', y) dy'$$

$$+ \int_{D(b'_S, b'_L)} \varphi(b'_S, b'_L) f(y, y') dy',$$

Then, the actuarially fair yield is given by $r_{m}^{AF} = \log \left( 1/q_{m}^{AF} + \delta_m \right)$, and the spread risk premium is defined as $r_{m} = r_{m} - r_{m}^{AF}$. The fifth and sixth columns of Table 6 show the term structure of spread risk premia in the model. Risk premia are positive, because the borrower tends to default in states with low income, and the lender’s pricing kernel $M$ is negatively correlated with income. Also, risk premia are generally small, averaging less than one percentage point. They make up a relatively large fraction of the interest rate spreads in good times, when the spreads are low, but are a much smaller fraction when spreads are high. In effect, our model says that risk premia do not need to increase a lot in bad times to account for the large increases in spreads, because the expected probability of default rises so much. However, this decomposition does not address the question of how the degree of required risk compensation affects debt and default choices, and in turn the equilibrium term structure of spreads. We examine this question in more detail in section 5.3.

Underlying the time-varying spreads is the interaction of the dynamics of income and debt with the price schedules for short and long debt. (Figure 3, in the Appendix, illustrates the equilibrium price schedules for short debt $q_{S}(b'_S, b'_L, y)$ and long debt $q_{L}(b'_S, b'_L, y)$.) However, the mapping from discount prices to spreads is not linear (eq. 1). Thus, it is informative to analyze price ratios defined as defaultable discount prices relative to default-free prices for a bond with duration $m$: $q_{m}/q_{m}^{*}$. The price ratio of each bond is the total discounted repayment probability over the lifetime of the bond, adjusted for risk and recovery. Table 7 presents statistics for these price ratios in the model and the data. The table shows that contrary to spreads, price ratios for short-term debt are always higher than for long-term debt both in the model and in the data. Moreover, price ratios are disproportionately lower
for short-term debt when spreads are high, both in the model and in the data.\textsuperscript{19}

<table>
<thead>
<tr>
<th>$s_s$ pct</th>
<th>DATA</th>
<th>$q_s/q_s^*$</th>
<th>$q_L/q_L^*$</th>
<th>MODEL</th>
<th>$q_s/q_s^*$</th>
<th>$q_L/q_L^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$&lt; 25$</td>
<td>0.96</td>
<td>0.60</td>
<td>0.99</td>
<td>0.78</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$&lt; 50$</td>
<td>0.95</td>
<td>0.59</td>
<td>0.98</td>
<td>0.75</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\geq 50$</td>
<td>0.85</td>
<td>0.43</td>
<td>0.87</td>
<td>0.59</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\geq 75$</td>
<td>0.79</td>
<td>0.35</td>
<td>0.81</td>
<td>0.50</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mean</td>
<td>0.90</td>
<td>0.51</td>
<td>0.92</td>
<td>0.67</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The distinct dynamics of price ratios and spreads can be understood as follows. Price ratios reflect cumulative repayment (and default) probabilities, whereas spreads reflect average default probabilities. Cumulative default risk for long-term debt is always larger than for short-term debt both in the data and the model. However, annualized (average) default risk can be lower on long-term debt during times when the annual default probability in the short run is larger than the annual default probability in the long run. Thus, contrary to common belief in sovereign debt markets, the interest rate spread is not a comprehensive measure of the relative cost of borrowing in different maturities of debt. In particular, in times when the probability of default is high, short-term debt may appear to be more expensive for the borrower than long-term debt, in the sense that it has a higher spread, although long-term debt is worse in the sense that it has a lower price, relative to the risk-free price. The connection between the dynamic behavior of prices and spreads in our model is borne out in the data as well.

The preceding discussion also indicates that the important feature of our model for generating the observed dynamics of prices and the spread curve is that the probability of default is mean-reverting: a period with high probability of default is followed by a period with lower probability of default, and vice versa. The effects of mean-reverting default probabilities on the spread curve are the same as those highlighted by Merton (1974) in the case of credit spreads for corporate debt. In our model the probability of default is endogenously mean-reverting as a result of the dynamics of the output process and debt accumulation. When output is high, it is also expected to be high in the near future, so the probability of default in the next period is low. The economy borrows a large amount at low interest rate spreads, so that in states where the economy is hit by a bad shock, default becomes more likely further in

\textsuperscript{19}These patterns hold for price ratios in the data for Argentina, Mexico, and Russia as well.
the future. In contrast, when the likelihood of imminent default is high, the economy avoids default in the next period only in states with high output. Conditional on not defaulting, then, output is expected to remain high, and the probability of default further in the future falls. The persistence and mean reversion of default and repayment probabilities driven by the dynamics of debt and income therefore rationalize the dynamic behavior of the spread curve observed in the data.

5.2.2 Maturity Composition

We now present the quantitative predictions for the maturity composition of debt. As discussed in Section 4, two forces in the model shape the dynamic behavior of the maturity composition. First, long-term bonds insure against future price fluctuations; we find that the insurance motive is more valuable in times of high wealth. Second, short-term bonds are more liquid and allow larger transfers of resources to the present with a smaller change in price; we find that the liquidity advantage for short debt is more valuable in times of low wealth. Given the negative correlation between wealth and spreads, these two forces lead the borrower to use long-term debt more heavily in times when spreads are low and shift toward shorter term debt when spreads are high.

<table>
<thead>
<tr>
<th>$s_S$ pct</th>
<th>MODEL</th>
<th>DATA</th>
</tr>
</thead>
<tbody>
<tr>
<td>&lt; 50</td>
<td>3.80</td>
<td>6.37</td>
</tr>
<tr>
<td>≥ 50</td>
<td>2.46</td>
<td>4.73</td>
</tr>
<tr>
<td>Overall</td>
<td>3.13</td>
<td>5.95</td>
</tr>
</tbody>
</table>

To compare issuances of long and short debt between the model and data, we compute conditional averages of the duration of new debt issuances, based on the level of the short spread. Average duration in the model is the sum of the duration (equation 18) of each new bond issuance weighted by its share in total new debt issued. Moreover, given that in the data we only compute the duration of debt issuances, and not repurchases, we do the same in the model. Table 8 reports the average duration of new debt issuances when spreads are above their median relative to when spreads are below their median in the model and in the Brazilian data. Debt duration in the model mirrors the dynamics of duration in the bond data of Brazil. In the model, average duration when spreads are low equals 3.80 years,
whereas it shortens to 2.46 when spreads are high. In Brazil, the average duration of bonds issued when spreads are high equals 6.37 years and shortens to 4.73 years when spreads are low. Although the model underpredicts the average duration of debt, the difference between debt duration when spreads are low and when spreads are high – just under one and a half years – is close to this difference in the data.

Table 9 provides more details about the maturity composition and the forces underlying its determination. The first row of Table 9 show the model’s portfolio – the fraction of the value of new debt that is short term – conditional on different levels of the short spread. When spreads are low, the borrower issues on average 34% of debt in long-term bonds, and 66% in short-term bonds. When spreads are high, the maturity composition shifts to only 11% in long-term bonds, and 89% in short-term bonds. The majority of the value of debt issuance is always in short-term debt. As discussed in Section 4, the optimal portfolio depends on the valuations of the insurance benefits of long-term debt relative to the liquidity and cost advantage of short-term debt. In the lower section of Table 9, we reports several alternative metrics to evaluate these benefits. First, to measure the cost advantage of short-term debt, we compute the ratio of the price ratios of the two debt classes: \( \frac{q^L / q_L}{q^S / q_S} \), which compares the risk-adjusted expected discounted repayment probabilities on the two types of debt. As the table shows, long-term debt is always more costly in terms of carrying lower total repayment probabilities, as \( \frac{q^L / q_L}{q^S / q_S} \) is always less than one. Increasing consumption using short-term debt is cheaper in that it implies a smaller increase in default risk. In addition, short-term debt is disproportionately cheaper in low wealth times, as the slope of price ratios is lower, 0.68 relative to 0.77. Thus, a larger share of short-term debt in low wealth times can be understood as a reaction to the more expensive long-term debt.

A more detailed measure of the liquidity benefit of short-term debt is given by calculating the increase in consumption that would be possible by marginally increasing short-term debt, relative to the increase in consumption that is possible by issuing more long-term debt. Specifically, define \( \Psi(b'_S, b'_L, b_S, b_L, y) \equiv q^S(b'_S, b'_L, y)(b'_S - \delta_S b_S) + q^L(b'_S, b'_L, y)(b'_L - \delta_L b_L) \) as the quantity of consumption that is attained with a certain debt policy \( b'_S, b'_L, \) given the state \( (b_S, b_L, y) \). We calculate the ratio of small deviations from the equilibrium debt policy for short-term debt relative to long-term debt,

\[
\frac{\Delta_S}{\Delta_L} = \frac{\Psi(b'_S + \varepsilon_S, b'_L, b_S, b_L, y) - \Psi(b'_S, b'_L, b_S, b_L, y)}{\Psi(b'_S, b'_L + \varepsilon_L, b_S, b_L, y) - \Psi(b'_S, b'_L, b_S, b_L, y)}
\]

where \( \varepsilon_S \) and \( \varepsilon_L \) are small, and are chosen so that if bond prices were always equal to the default-free prices, the ratio would be exactly equal to 1. This ratio in our model is
### Table 9: Model Maturity Composition

<table>
<thead>
<tr>
<th></th>
<th>Short Spread</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>&lt; 50 pctile</td>
<td>≥ 50 pctile</td>
<td>overall mean</td>
</tr>
<tr>
<td>$q_S \ell_S / (q_S \ell_S + q_L \ell_L)$</td>
<td>0.66</td>
<td>0.89</td>
<td>0.78</td>
</tr>
<tr>
<td>$s_S$, percent</td>
<td>1.16</td>
<td>10.09</td>
<td>5.63</td>
</tr>
<tr>
<td>$s_L - s_S$, percent</td>
<td>2.35</td>
<td>-0.08</td>
<td>1.13</td>
</tr>
<tr>
<td>$(q_L/q_S^<em>) / (q_S/q_S^</em>)$</td>
<td>0.77</td>
<td>0.68</td>
<td>0.72</td>
</tr>
<tr>
<td>$\Delta S / \Delta L$</td>
<td>1.33</td>
<td>1.82</td>
<td>1.39</td>
</tr>
<tr>
<td>std(debt)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>short</td>
<td>0.49</td>
<td>0.48</td>
<td>0.49</td>
</tr>
<tr>
<td>long</td>
<td>0.61</td>
<td>0.58</td>
<td>0.59</td>
</tr>
<tr>
<td>total</td>
<td>0.53</td>
<td>0.52</td>
<td>0.53</td>
</tr>
<tr>
<td>corr\left(debt, w'(c)/\bar{w}(c)\right)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>short</td>
<td>-0.02</td>
<td>0.11</td>
<td>0.02</td>
</tr>
<tr>
<td>long</td>
<td>-0.17</td>
<td>-0.22</td>
<td>-0.16</td>
</tr>
<tr>
<td>total</td>
<td>-0.21</td>
<td>-0.16</td>
<td>-0.16</td>
</tr>
</tbody>
</table>

always above 1, and on average it equals 1.39. Thus, short-term debt is more liquid because consumption can always be marginally increased more with short-term debt than with long-term debt. The reason is that price schedules for short-term debt are more lenient by having higher prices that decrease by less as debt increases. Looking across periods, this difference is especially large in states in which spreads are high. Thus, short-term debt is particularly useful for increasing consumption when spreads are high.

In our model, short- and long-term debt prices reflect expected repayment rates and risk premia. Thus, if the schedules of short-term debt are more lenient, this means that, adjusted for risk, lenders expect the borrower to repay in more future states. However, this does not mean that the borrower is indifferent to acquiring a certain level of resources with a small safer short-term loan, versus a large risky long-term loan. In fact, we know that if the borrower chooses to default in some future states with the long-term loan while choosing to repay in those same states with the short-term loan, he must be better off by repaying the short loan because he always has the option to default. Moreover, default risk in our model limits the maximum level of resources that the borrower can get.\(^\text{20}\) The key is that in our model these endogenous limits and price schedules are tighter for long term debt relative to short term debt. The average ratio of borrowing in each state to the short-term debt limit versus to the long-term debt limit equals 1.84. Thus, the potential increase in

\(^{20}\)Arellano (2008) shows that a one short-term asset version of our model generates an endogenous Laffer Curve for borrowing which features a debt limit.
consumption from exhausting short-term debt is 84% larger than from exhausting long-term debt. As discussed in the examples in Section 4, short-term debt can deliver larger absolute consumption levels and larger consumption with smaller loans, because of the inability of the borrower to commit to saving sufficiently to repay long-term debt. Effectively, the threat of default punishment is more effective to induce repayment of shorter-term debt because repayment of short debt does not require future savings.

The insurance benefits of long-term debt can be measured by considering the variation in value of outstanding debt. With long-term debt, the value of debt due in a period varies with the state, providing a hedge against fluctuations in the borrower’s consumption. We measure this hedging benefit in two ways: with the standard deviation of the value of each type of debt, $(1 + \delta_m q_m) b_m$ (and their sum), and the correlation of these values with the borrower’s intertemporal marginal rate of substitution (MRS). In states with low consumption (high MRS), debt provides a good hedge if its outstanding value drops, meaning the present value of the debt burden falls. The table shows that long term debt is a better hedge: its value is more volatile and more negatively correlated with the borrower’s MRS than that of short-term debt. Moreover, since the portfolio of debt includes more long-term debt in good times, the borrower achieves more hedging, measured by the correlation of total debt with the MRS, in periods with low spreads than in periods with high spreads.

In summary, through the lens of our model, the maturity structure of defaultable debt in emerging markets and its covariation with spread curves and levels can be rationalized by two factors: a hedging advantage of long-term debt for insuring against fluctuations in future default risk, and a liquidity advantage of short-term debt for providing higher resources with more lenient prices.

5.3 Counterfactuals

In section 5.2.1, on prices and spreads, we assessed the contribution of risk premia by pricing bonds with the risk-neutral pricing kernel in our model, given the borrower’s behavior. However, in our model, if the borrower were to take into account that lenders were risk neutral, debt and default decisions would change. Table 10 shows summary statistics on spreads and the maturity composition of debt when we re-solve our model under the assumption that lenders are risk-neutral.

With risk-neutral lenders, the model displays two important differences from the benchmark case: spreads are higher, and the maturity structure is more volatile. Compared to the case in which lenders require risk premia, bond price schedules are more lenient with risk-neutrality — that is, prices do not fall with the level of debt as fast — so the borrower
Table 10: Model Results with Risk Neutral Lenders

<table>
<thead>
<tr>
<th>DATA</th>
<th>MODEL, benchmark</th>
<th>MODEL, risk neutral</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s_S$</td>
<td>$s_L$</td>
<td>duration</td>
</tr>
<tr>
<td>&lt; 50</td>
<td>2.68</td>
<td>5.35</td>
</tr>
<tr>
<td>≥ 50</td>
<td>8.51</td>
<td>8.93</td>
</tr>
<tr>
<td>Mean</td>
<td>5.60</td>
<td>7.14</td>
</tr>
</tbody>
</table>

borrows more, at higher levels of spreads. In fact, the default probability is higher with risk neutral lenders, about 6.3 percent on average, compared to about 4.5 percent in our previous case. The average duration of debt is higher, and its dynamics are accentuated, under risk-neutral pricing as well. Since price schedules are more lenient, the borrower can take greater advantage of the liquidity and insurance benefits of short-term and long-term debt, and the extent to which these benefits vary over time.

### 6 Conclusion

In this paper, we have developed a dynamic model to study the maturity composition of sovereign bonds. In emerging markets data, changes in the maturity composition of debt comove with changes in the term structure of spreads: when spreads on short-term debt are low, long-term spreads are higher than short-term spreads, and the maturity of debt issued is long. When short-term spreads rise, long-term spreads rise less, and the maturity of debt shortens. Our model simultaneously reproduces the patterns observed in the term structure of spreads and bond prices, and the maturity composition of debt. Changes in the spread curve, which reflects the average default probability at different time horizons, result from the output dynamics and the endogenous dynamics of debt. Issuing long-term debt insures against future fluctuations in short-term spreads that come from changes in default risk. Short-term debt provides more liquidity because it allows the borrower to avoid the more severe commitment problem in repaying long-term debt. With these two forces, the model generates the pattern of issuances observed in the data. Long-term debt is issued mostly in times of high wealth and low spreads, when the insurance motive is the strongest. Short-term bonds are used more heavily in times when wealth is low and spreads are high, because expectations of the borrower’s future debt and default choices restrict the availability of long-term debt more heavily than of short-term debt.

Our main innovation has been to introduce multiple, long-term assets into a dynamic model with endogenous default. We view the resulting framework as useful for addressing a variety of other questions for which it is important to analyze a trade-off in maturity choice.
with defaultable debt. Natural applications are the maturity structure of consumer and corporate debt. The literature on consumer bankruptcy thus far has focused on modeling very short-term unsecured credit (Chatterjee, Corbae, Nakajima, and Rios-Rull (2007) and Livshits, MacGee, and Tertilt (2007)). However, it would be interesting to analyze both long-term and short-term defaultable loans, such as mortgages and credit card debts. In addition, the mechanisms in our model are likely to be relevant in corporate debt given the similarity between our facts on emerging market spread curves and the cross section of corporate debt spread curves. Default risk has been shown to have important implications for firm dynamics (Cooley and Quadrini (2001) and Arellano, Bai, and Zhang (2007)). The model of this paper can be used to further understand how the maturity choice can influence the entry, exit, and growth of firms. Overall, our paper provides a tractable framework to study defaultable debt of multiple maturities appropriate for these questions, and has highlighted the relevant economic trade-offs important for understanding debt maturity choice in the presence of default.
References


Appendix

Data Description

All the sovereign bond data are from Bloomberg. For the four countries we examine, we use all bonds with prices quoted at some point between March 1996 and May 2004, with the following exceptions. We exclude all bonds with floating-rate coupon payments, and at every date, we exclude bonds that are less than three months to maturity, following Gurkaynak, Sack, and Wright (2006). For each country, we estimate spreads starting from the first week for which at least four bond prices are available every week through the end of the sample. We use data from 110 bonds for Argentina, 71 for Brazil, 63 for Mexico, and 25 for Russia. To estimate default-free yield curves, we use data on U.S. and European government bond yields. The U.S. data are from the Federal Reserve Board, and the European data are from the European Central Bank.\footnote{The U.S. data are the Treasury constant maturities yields, available at http://www.federalreserve.gov/releases/h15/data.htm. The European data are Euro area benchmark government bond yields, which is an average of European national government bond yields available at http://sdw.ecb.europa.eu.} For constructing the quarterly maturity and duration statistics, we also include bonds issued during the sample period that did not have prices quoted, and use the estimated spread curve to construct their prices according to equation (21).

Spread Curve Estimation


A coupon bond is priced as a collection of zero-coupon bonds, each with maturity given by a coupon payment date, and face value given by the cash flow on that payment date. The price at date \( t \) of a bond issued by country \( i \), paying an annual coupon rate \( c \) at dates \( n_1, n_2, \ldots, n_J \) years into the future, is

\[
p_i(t, \{n_j\}) = \sum_{j=1}^{J} \exp(-n_j r_i^t(n_j)) c + \exp(-n_J r_i^t(n_J))
\]

with the face value of the bond paid on the last coupon date.

Spreads are defined as \( s_i^t(n) = r_i^t(n) - r_i^t(n) \), where \( r_i^t(n) \) is a default-free yield curve.
We define spreads as a parametric function of maturity following Nelson and Siegel (1987)

\[ s_i(n; \beta_i^t) = \beta_{1t}^i + \beta_{2t}^i \left( \frac{1 - e^{-\lambda n}}{\lambda n} \right) + \beta_{3t}^i \left( \frac{1 - e^{-\lambda n}}{\lambda n} - e^{-\lambda n} \right) \tag{22} \]

for each country \( i \), where \( \beta_i^t = (\beta_{1t}^i, \beta_{2t}^i, \beta_{3t}^i) \) and \( \lambda \) are parameters. For default-free bonds, we define

\[ r_i^\$ (n; \beta_i) = \beta_{1t}^\$ + \beta_{2t}^\$ \left( \frac{1 - e^{-\lambda n}}{\lambda n} \right) + \beta_{3t}^\$ \left( \frac{1 - e^{-\lambda n}}{\lambda n} - e^{-\lambda n} \right) \tag{23} \]

and

\[ r_i^\€ (n; \beta_i) = \beta_{1t}^\€ + \beta_{2t}^\€ \left( \frac{1 - e^{-\lambda n}}{\lambda n} \right) + \beta_{3t}^\€ \left( \frac{1 - e^{-\lambda n}}{\lambda n} - e^{-\lambda n} \right) \tag{24} \]

for US (\$) and Euro (\€) bonds.

As described by Nelson and Siegel (1987) and Diebold and Li (2006), the three components of this curve correspond to a “long-term,” or “level” factor (the constant), a “short-term,” or “slope” factor (the term multiplying \( \beta_2 \)) and a “medium-term,” or “curvature” factor (the term multiplying \( \beta_3 \)). Linear combinations of these factors can capture a broad range of shapes for the spread curve.

We first estimate the parameters \( \beta_i^\$ \) and \( \beta_i^\€ \) by OLS, using U.S. and Euro area bond yields. Throughout, we follow Diebold and Li (2006) by setting the parameter \( \lambda = 0.714 \), so that the term multiplying \( \beta_3 \) in all countries’ spread curves is maximized when \( n = 2\frac{1}{2} \) years.

Then, given a set of parameters \( \beta_i^t \), we use equation (21) to price each of country \( i \)'s bonds at date \( t \) using the risk-free yield given by (23) or (24) and the spread given by (22):

\[ p_t^i(c, \{n_j\}; \beta_i^t) = \sum_{j=1}^{J} \exp(-n_j (s_t^i(n_j; \beta_i^t) + r_{i*}^t(n_j)))c + \exp(-n_j (s_t^i(n_j; \beta_i^t) + r_{i*}^t(n_j))) \]

where \( r_{i*}^t \) refers to \( r_i^\$ \) if the bond is denominated in U.S. dollars, or \( r_{i*}^t = r_i^\€ \) if the bond is denominated in a European currency.

We estimate the parameters \( \beta_i^t \) by nonlinear least squares to minimize the sum of squared deviations of the predicted prices, \( p_t^i(c, \{n_j\}; \beta_i^t) \) from their actual values. That is, our estimated parameters solve

\[ \min_{\beta_i^t} \sum (p_t^i(c, \{n_j\}; \beta_i^t) - p_t^i(c, \{n_j\}))^2, \]

where the summation is taken over all bonds issued by country \( i \) with prices available at date \( t \). As discussed in Svensson (1994), minimizing yield to maturity errors rather than price
errors gives a better fit for short-term yields to maturity, because short-term bond prices are less sensitive to their yields to maturity than long-term bond prices.

The following features present in the data require modification of the basic bond pricing equation (21):

1. Between coupon periods, the quoted price of a bond does not include accrued interest, so we subtract from the bond price the portion of the next coupon’s value that is attributed to accrued interest.

2. For bonds with principal payments guaranteed by U.S. Treasury securities, we discount the payment of principal by the risk-free yield only, without the country spread.

3. For bonds with coupon payments that increase or decrease over time with certainty (“step-up” and “step-down” bonds, respectively), we modify the sequence of payments in equation (21) accordingly.

Table 11 below displays root mean squared errors (RMSE), as a percentage of a bond’s price. Errors are moderate, averaging three to seven percent of actual bond prices across the four countries. During periods when spreads are high, errors tend to be larger; for example, the RMSE of Russian bonds is 11 percent in periods when the 2-year spread is above its 90th percentile.

<table>
<thead>
<tr>
<th></th>
<th>average RMSE</th>
<th>percentile of 2-year spread</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( % of price)</td>
<td>&lt; 10th</td>
</tr>
<tr>
<td>Brazil</td>
<td>6.0731</td>
<td>6.1681</td>
</tr>
<tr>
<td>Mexico</td>
<td>2.6303</td>
<td>2.2326</td>
</tr>
<tr>
<td>Russia</td>
<td>7.2242</td>
<td>4.2919</td>
</tr>
</tbody>
</table>
Further Statistics on Spread Curves

Tables 12 reports further spread curves and spread volatilities for all countries.

Table 12: Average Spreads and Volatility

<table>
<thead>
<tr>
<th>Maturity (years)</th>
<th>Overall (%)</th>
<th>Std. Dev</th>
<th>When 2-year spread is above/below nth percentile</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>&lt; 10th</td>
</tr>
<tr>
<td>Argentina</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>4.77</td>
<td>6.26</td>
<td>1.10</td>
</tr>
<tr>
<td>5</td>
<td>5.27</td>
<td>3.83</td>
<td>1.97</td>
</tr>
<tr>
<td>10</td>
<td>6.22</td>
<td>3.36</td>
<td>3.19</td>
</tr>
<tr>
<td>Brazil</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>5.60</td>
<td>5.02</td>
<td>1.69</td>
</tr>
<tr>
<td>5</td>
<td>6.71</td>
<td>4.41</td>
<td>3.58</td>
</tr>
<tr>
<td>10</td>
<td>7.14</td>
<td>2.93</td>
<td>4.79</td>
</tr>
<tr>
<td>15</td>
<td>7.28</td>
<td>2.41</td>
<td>5.24</td>
</tr>
<tr>
<td>Mexico</td>
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</tr>
<tr>
<td>2</td>
<td>1.84</td>
<td>1.36</td>
<td>0.42</td>
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<td>2.46</td>
<td>0.97</td>
<td>1.57</td>
</tr>
<tr>
<td>10</td>
<td>3.45</td>
<td>0.95</td>
<td>2.40</td>
</tr>
<tr>
<td>15</td>
<td>3.87</td>
<td>0.98</td>
<td>2.72</td>
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<tr>
<td>Russia</td>
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</tr>
<tr>
<td>2</td>
<td>4.44</td>
<td>2.65</td>
<td>1.70</td>
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<tr>
<td>5</td>
<td>5.37</td>
<td>2.59</td>
<td>2.86</td>
</tr>
<tr>
<td>10</td>
<td>5.10</td>
<td>2.25</td>
<td>2.86</td>
</tr>
<tr>
<td>15</td>
<td>4.92</td>
<td>2.21</td>
<td>2.80</td>
</tr>
</tbody>
</table>
Model’s Debt Price Schedules

Figure 3: Price schedules for short- and long-term debt when income is at its mean.