PARETO OPTIMALITY AND COMPETITIVE EQUILIBRIUM IN INFINITE HORIZON ECONOMIES*

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The paper presents a general model of a decentralized economy evolving over an infinite time horizon. Alternative notions of price systems, competitive equilibria, efficiency and optimality are introduced. The main results characterize conditions under which the two fundamental theorems of welfare economics are valid in such a general framework.

1. Introduction

A cornerstone of classical economics is the idea that a competitive equilibrium is optimal in the Paretian sense that no alternative feasible allocation of commodities can improve the lot of one agent without worsening the conditions of some other individual. Equally important is the converse proposition that any given Pareto optimal allocation can be sustained by a competitive equilibrium. A prime achievement of welfare economics has been to establish conditions that are, roughly speaking, 'necessary and sufficient' for the validity of these conclusions in finite economies (i.e., economies in which the numbers of commodities and economic agents are finite). On the other hand, it is known that in non-finite economies, these propositions may fail even when the sufficient conditions of the finite case are met. In this paper we shall restrict our attention to a class of infinite horizon economies, typified by von Neumann growth models, in which the production possibilities are not constrained by non-producible

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factor. For these economies, it is verified that the welfare propositions mentioned above hold under conditions closely paralleling those relevant for finite economies. The relation between competitive equilibrium and efficiency or Pareto optimality for static economies has been reviewed in Koopmans (1975). References to the partial extensions of these results to some infinite horizon economies are given in Majumdar–Mitra–McFadden (1976). No attempt will be made, therefore, to review the extensive literature on intertemporal efficiency or optimality. In section 2, a general model of a decentralized economy over time is presented. In section 3, various notions of price systems, competitive equilibria, efficiency and optimality are introduced. The main results in section 4 give necessary and sufficient conditions under which a strong competitive equilibrium [see Definition (3.5)] is Pareto optimal, and a Pareto optimal allocation is a valuation equilibrium [see Definition (3.6)]. Some other logical connections among the various concepts are examined in section 3, with a number of counterexamples to possible implications.

2. A general decentralized economy

2.1. Commodity space

Consider an economy in which the numbers of commodities and economic agents within each period \( t = 0, 1, 2 \ldots \) is finite. Each economic agent (consumer or firm) is assumed to have a finite life.

Let \( \mathcal{G}_t \) denote the commodity space in period \( t \); it is a finite-dimensional real linear vector space whose vectors have a component for each commodity existing in the economy in period \( t \). Define the real linear vector space \( \mathcal{S} \) consisting of all infinite sequences (or programs) or commodity vectors, \( v = (v^0, v^1, \ldots) \) with \( v^t \in \mathcal{G}_t \) for \( t = 0, 1, 2, \ldots \). Then \( \mathcal{S} \) is the commodity space for the infinite horizon economy.

2.2. Firms

The generation of firms initially formed in period \( t \) will be numbered \( j = 1, 2, \ldots, J_t \). The typical firm \( j \) in generation \( t \) will have a finite lifetime \( (w_{jt}, \ldots) \), and will have for each period \( \tau (\tau = 0, 1, 2, \ldots) \) an input–output pair \((a_{jt}^\tau, b_{jt}^\tau)\) in \( \mathcal{G}_t \times \mathcal{G}_{t+1} \), where \((a_{jt}^\tau, b_{jt}^{\tau+1}) = (0, 0)\) for \( \tau < t \), and \( \tau \geq t + (w_{jt} - 1) \), and \( b_{jt}^0 = 0 \) for \( \tau \geq 0 \).

The typical firm \( j \) in generation \( t \) will then have, for each period \( \tau \), a net output vector \( y_{jt}^\tau = b_{jt}^\tau - a_{jt}^\tau \) in \( \mathcal{G}_t \), where \( y_{jt}^\tau = 0 \) for \( \tau < t \) and \( \tau > t + (w_{jt} - 1) \).

An input program for a typical firm is \( a_{jt} = (a_{jt}^0, a_{jt}^1, \ldots) \) in \( \mathcal{G} \), whose components are zero outside the firm's lifetime, in the sense indicated above. Similarly, an output program and a net output program for the typical firm
are \( b_j = (b_j^0, b_j^1, \ldots) \) and \( y_j = (y_j^0, y_j^1, \ldots) \), both in \( \mathcal{G} \). Let \( \mathcal{Y}_j \) denote the finite-dimensional linear subspace of \( \mathcal{G} \) spanned by the net output programs \( y_j^\tau \) in \( \mathcal{G} \).

An aggregate input bundle in period \( \tau \) is denoted by

\[
a^\tau = \sum_{\tau=0}^{\tau} \sum_{j=1}^{J_j} a_{j,\tau}^\tau, \quad \tau \geq 0. \tag{2.1}
\]

An aggregate output bundle in period \( (\tau + 1) \) is

\[
b^{\tau+1} = \sum_{\tau=0}^{\tau} \sum_{j=1}^{J_j} b_{j,\tau+1}^\tau, \quad \tau \geq 0, \quad \text{and} \quad b^0 = \sum_{j=1}^{J_j} b_{j,0}^0 = 0. \tag{2.2}
\]

An aggregate net output bundle in period \( \tau \) is

\[
y^\tau = b^\tau - a^\tau = \sum_{\tau=0}^{\tau} \sum_{j=1}^{J_j} y_{j,\tau}^\tau, \quad \tau \geq 0. \tag{2.3}
\]

Aggregate input, output, and net output programs are defined by

\[
a = \sum_{\tau=0}^{\infty} \sum_{j=1}^{J_j} a_{j,\tau} = (a^0, a^1, \ldots) \quad \text{in} \ \mathcal{G},
\]

\[
b = \sum_{\tau=0}^{\infty} \sum_{j=1}^{J_j} b_{j,\tau} = (b^0, b^1, \ldots) \quad \text{in} \ \mathcal{G}, \tag{2.4}
\]

\[
y = \sum_{\tau=0}^{\infty} \sum_{j=1}^{J_j} y_{j,\tau} = (y^0, y^1, \ldots) \quad \text{in} \ \mathcal{G}.
\]

[Clearly, \( y = (b^0 - a^0, b^1 - a^1, \ldots) \) is in \( \mathcal{G} \).]

For each firm \( j \) in generation \( t \), a technology set \( \mathcal{T}_{j,t}^\tau \), on \( R^+_t \times R^+_t \), defines the input–output pairs which the firm can obtain by production in period \( \tau \). An input–output pair \( (a_{j,\tau}^\tau, b_{j,\tau+1}^\tau) \) is technologically feasible if it belongs to \( \mathcal{T}_{j,t}^\tau \).

The following assumptions on the technology sets \( \mathcal{T}_{j,t}^\tau \) will be used:

**Assumption T**

(T.1) \( \mathcal{T}_{j,t}^\tau \) is independent of the behavior of any other agent (i.e., no externalities).

(T.2) \( (0,0) \in \mathcal{T}_{j,t}^\tau \) (inaction is possible).
(T.3) \((a, b) \in \mathcal{T}_j^r\) and \(a' \geq a, \quad 0 \leq b' \leq b\), implies \((a', b') \in \mathcal{T}_j^r\) (free disposal).

(T.4) \((a, b) \in \mathcal{T}_j^r\) and \(a = 0\) implies \(b = 0\) (impossibility of free production).

(T.5) \(\mathcal{T}_j^r\) is closed and convex (continuity and non-increasing returns).

For each firm \(j\) in generation \(t\), a production possibility set \(Y_j\) defines the net output programs \(y_{j_t}\) which the firm can achieve, i.e.,

\[
Y_j = \{y_{j_t} | y_{j_t} = b_{j_t} - a_{j_t}, \quad (a_{j_t}, b_{j_t}^{+1}) \in \mathcal{T}_j^r \quad \text{for} \quad \tau \geq 0\}.
\]

The following consequences of Assumption T for the sets \(Y_j\) may, then, be noted:

(Y.1) \(Y_j\) is independent of the behavior of any other economic agent.

(Y.2) The null net output program is in \(Y_j\).

(Y.3) \(Y_j\) has a convex, free-disposal hull.\(^1\)

(Y.4) \(Y_j\) is closed under pointwise convergence.\(^2\)

(Y.5) If \(y_{j_t} \in Y_j\), and \(y_{j_t}\) is bounded below, then \(y_{j_t}\) is bounded above.\(^3\)

Aggregate production possibilities for the economy are defined by

\[
Y = \sum_{t=0}^{\infty} \sum_{j=1}^{J_t} Y_j = \left\{y \in \mathcal{Y} \mid y = \sum_{t=0}^{\infty} \sum_{j=1}^{J_t} y_{j_t}, \quad y_{j_t} \in Y_j \right\}.
\]

The following consequence of Assumption T for the set \(Y\), may then be proved:

**Lemma 2.1** If (Y.2)-(Y.5) hold, then \(Y\) is closed under pointwise convergence.

**Proof.** Suppose a sequence of programs \((y)\) \(y\) in \(Y\) converges pointwise to

\(^1\)The free disposal hull of \(Y_j\) is the set \(\{y \in \mathcal{Y}_j \mid y \leq y' \text{ for some } y' \in Y_j\}\).

\(^2\)A sequence of programs \((y)\) \(y\) converges pointwise to a program \(y^*\) if each component of the vector \((y)\) converges to the corresponding component of \(y^*\). Formally, given \(t\) and \(\varepsilon > 0\), there exists \(n_0(t, \varepsilon)\) such that for \(n > n_0(t, \varepsilon)\), \(\|y^n - y^*\| < \varepsilon\). When the vectors are confined to a finite-dimensional subspace, pointwise convergence is equivalent to ordinary Euclidean convergence.

\(^3\)Otherwise, there is a sequence \((y)\) \(y_j \in Y_j\) such that \((y)\) \(y_j\) is bounded below, but \((y)\) \(y_j\) is unbounded above. This means by definition of the \(Y_j\) that for some \(\tau \leq \tau^*\), \((y)\) \(y_j\) is bounded above, \((y)\) \(b_{j_t}^{+1}\) is unbounded above \(\|y_j^{+1}\| > 1\), and \((y)\) \(a_{j_t}, b_{j_t}^{+1}\) \(\in \mathcal{T}_j^r\). Define \((x)\) \(x_j = (y_j^{+1}, y_j^{+1} / b_{j_t}^{+1}) \in \mathcal{T}_j^r\). Then, by (T.2) and (T.5), \((x)\) \(x_j\) \(\in \mathcal{T}_j^r\) and \((x)\) \(a_j, b_j^{+1}\) \(\rightarrow 0, b_j^{+1} = 1\). Hence, there is a convergent subsequence of \(n\) (call it \(n\) again) such that \((x)\) \(a_j \rightarrow 0\), and \((x)\) \(b_j^{+1} \rightarrow \beta\). By (T.5), \((0, \beta) \in \mathcal{T}_j^r\). But \(\|\beta\| = 1\), which violates (T.4).
\[ \tilde{y} \in \mathscr{G}. \text{ There exists a corresponding sequence } (n) y_{j_t} \in Y_{j_t}, \text{ such that } (n) y = \sum_{t=0}^{\infty} \sum_{j=1}^{F_j} (n) y_{j_t}. \]

Suppose each \((n) y_{j_t}\) sequence is bounded. Then, it has a subsequence which converges pointwise to some \(\tilde{y}_{j_t}\) in \(Y_{j_t}\) by (Y.4). The Cantor diagonal process can then be used to extract a subsequence converging pointwise to \(\sum_{t=0}^{\infty} \sum_{j=1}^{F_j} \tilde{y}_{j_t}\) in \(Y\). But \(\sum_{t=0}^{\infty} \sum_{j=1}^{F_j} \tilde{y}_{j_t} = \tilde{y}\) by the supposition that \((n) y\) converges pointwise to \(\tilde{y}\). Hence, \(\tilde{y}\) is in \(Y\).

Alternatively, suppose for a subsequence (denote it by \(n\) again) \((n) y_{j_t}\) is unbounded. If \((n) y_{j_t}\) is bounded below, \((n) y_{j_t}\) is bounded above by (Y.5). So, under the supposition that \((n) y_{j_t}\) is unbounded, there is a first period \(\tau\) in which \((n) y_{j_t}\) is unbounded below. Since \(\tau\) is the first period this happens, \((n) y_{\tau-1}\) is bounded below, and hence \((n) y_{\tau}\) is bounded above. Thus \((n) y_{\tau} \to -\infty\) as \(n \to \infty\) (for some component), contradicting the hypothesis that \((n) y\) converges pointwise to \(\tilde{y}\). Q.E.D.

A net output plan for this economy is a complete description of the net output program of each firm, and may be represented formally as a sequence of vectors of these programs:

\[ s = (y, y_1, \ldots, y_{j_0}, y_{j_1}, \ldots, y_j, y_{j_2}, \ldots), \]

where \(y = \sum_{t=0}^{\infty} \sum_{j=1}^{F_j} y_{j_t}\) is the aggregate net output program.

A net output plan is possible if each \(y_{j_t}\) is in the corresponding production possibility set \(Y_{j_t}\).

A possible net output plan \(s\) is said to be efficient if no alternative possible net output plan \(s'\) yields an aggregate net output program which is at least as large in every component and larger in at least one component. That is, if \(y\) and \(y'\) are the net output programs associated with \(s\) and \(s'\), respectively, and \(y' \geq y\), then \(y' = y\).

An aggregate net output program will be called possible (efficient) if it can be associated with some net output plan which is possible (efficient).

2.3. Consumers

The generation of consumer units initially formed in period \(t\) will be numbered \(k = 1, 2, \ldots, K_t\). The typical consumer unit \(k\) in generation \(t\) will consume in each period \(\tau\) of its finite lifetime (\(\gamma_k\) periods) a non-negative commodity bundle \(c_{k_t}\) in \(\mathscr{G}\). We define a finite-dimensional linear subspace \(\mathscr{G}_{k_t}\) of \(\mathscr{G}\) spanned by the consumption programs \(c_{k_t} = (c_{k_t}^0, c_{k_t}^1, \ldots)\) in \(\mathscr{G}\), whose components are zero outside the consumer's lifetime \([i.e., c_{k_t}^0 = 0 \text{ for } \tau < t, \text{ and } \tau > t + (\alpha_k - 1)]\). In order to avoid complications of questionable economic interest, assume that there is a positive (finite) number \(\alpha\) such that \(\alpha \geq \alpha_k\) for all \(k, t\).
The typical consumer unit will have a desired set $D_{kt}$, a subset of $\mathcal{C}_{kt}$, consisting of the non-negative consumption programs on which it can subsist. On the desired set $D_{kt}$, the consumer unit will have a preference preordering (i.e., a complete, transitive, reflexive binary relation) $\succeq_{kt}$. For $y, y'$ in $D_{kt}$, we use the notation $y \succ_{kt} y'$ (resp. $y \sim_{kt} y'$) if $y \succeq_{kt} y'$ and not $y' \succeq_{kt} y$ (resp. $y\succeq_{kt} y'$ and $y' \succeq_{kt} y$). We shall employ several or all of the following conditions on preferences.

**Assumption P.** The desired set $D_{kt}$ and the preference preordering $\succeq_{kt}$ have some or all of the following properties for $k = 1, 2, \ldots, K_t$, and $t = 0, 1, 2, \ldots$

(P.1) $D_{kt}$ and $\succeq_{kt}$ are independent of the consumption programs of other consumer units, and the net output programs of firms (no externalities).

(P.2) $D_{kt}$ is convex and monotone above (i.e., $c_{kt} \in D_{kt}$, $c_{kt}' \in \mathcal{C}_{kt}$, and $c_{kt}^\prime \geq c_{kt}$ imply $c_{kt}' \in D_{kt}$), and the set of non-negative consumption vectors in $\mathcal{C}_{kt}$, which are not in $D_{kt}$ is closed under pointwise convergence. (This set may be empty.)

(P.3) At any $c_{kt} \in D_{kt}$, the upper contour set $U_{kt}(c_{kt}) = \{ c_{kt}' \in D_{kt} | c_{kt}' \succeq_{kt} c_{kt} \}$ is closed under pointwise convergence, relative to $D_{kt}$ (continuity of preferences).

(P.4) At any $c_{kt} \in D_{kt}$, the upper contour set $U_{kt}(c_{kt})$ is convex (i.e., if $c\prime, c\prime\prime \in D_{kt}$ satisfy $c\prime \succeq_{kt} c_{kt}$ and $c\prime\prime \succeq_{kt} c_{kt}$, then $\theta c\prime + (1 - \theta)c\prime\prime \succeq_{kt} c_{kt}$ for $0 < \theta < 1$).

(P.5) If $c_{kt}' \geq c_{kt}$ in $D_{kt}$, then $c_{kt}' \in U(c_{kt})$ (monotonicity of preferences).

(P.6) At any $c_{kt} \in D_{kt}$, there exists $c_{kt}' \in D_{kt}$, which is strictly preferred, i.e., $c_{kt} \not\in U_{kt}(c_{kt}')$ (non-saturation).

(P.7) At any $c_{kt} \in D_{kt}$, there exists a sequence $(a_n)c_{kt} \in D_{kt}$ converging pointwise to $c_{kt}$ such that each $(a_n)c_{kt}$ is strictly preferred to $c_{kt}$ (local non-saturation).

An aggregate consumption bundle in period $t$ can be defined by $c^t$ (in $\mathcal{G}_t$),

$$c^t = \sum_{t = 0}^{\tau} \sum_{k = 1}^{K_t} c_{kt} = \sum_{t = \tau - 1}^{\tau} \sum_{k = 1}^{K_t} c_{kt}.$$  \hspace{1cm} (2.7)

An aggregate consumption program is then given by

*Several comments may be useful in clarifying these properties. If $D_{kt}$ consists of all consumption programs which are positive in one subset of commodities and non-negative in the remaining commodities over the lifetime of the consumer unit, then (P.2) is satisfied. If preferences are representable by continuous utility functions, then (P.3) holds. If there is a some commodity which is essential to subsistence, divisible, and always desired, and $D_{kt}$ is as described above, then preferences satisfy (P.6) and (P.7).*
\[ c = \sum_{t=0}^{\infty} \sum_{k=1}^{K_t} c_{kt} = (c_0, c_1, \ldots) \] in \( G \). \hspace{1cm} (2.8)

A distribution plan in this economy is a complete description of the consumption program of each consumer unit, and may be represented formally as a sequence of vectors \( d = (c, c_{10}, \ldots, c_{00}, c_{11}, \ldots, c_{k1}, c_{12}, \ldots) \), where

\[ c = \sum_{t=0}^{\infty} \sum_{k=1}^{K_t} c_{kt} \] \hspace{1cm} (2.9)

is the aggregate consumption program.

A distribution plan \( d \) is desirable if each \( c_{kt} \) is in the corresponding desired set \( D_{kt} \). We let \( D \) denote the set of desirable distribution plans. A distribution plan \( d \) in \( D \) is said to be Pareto preferable to a plan \( d' \) in \( D \), if \( c_{kt} \succeq_{kt} c'_{kt} \) for all consumer units, and \( c_{kt} \succ_{kt} c'_{kt} \) for at least one consumer unit.

### 2.4. Feasible allocations and Pareto optimality

The economy is assumed to have a vector of non-produced resources, a non-negative bundle in \( G \), which initially becomes available in period \( t \). This bundle is denoted by \( e^0 \), and the resource supply program is denoted by \( e \) \( = (e^0, e^1, e^2, \ldots) \). When \( y \) is a possible aggregate net output program, \( c = y + e \) will be termed a possible supply program.

A possible net output plan \( s \), a desirable distribution plan \( d \), and a resource supply program \( e \) define a feasible allocation \( h = (s, d, e) \) if the material balance condition \( c = y + e \) is met by the aggregate consumption program and net output program determined in \( d \) and \( s \), respectively. We let \( H \) denote the set of feasible allocations in this economy.

A feasible allocation \( h \) is Pareto optimal if there is no Pareto preferable feasible allocation \( \bar{h} \). It is short-run Pareto optimal if there is no Pareto preferable feasible allocation \( \bar{h} \), such that every consumer unit living on or after some period \( L \) gets the same allocation in \( \bar{h} \) and in \( h \).

### 3. Concepts of price system, equilibrium, non-decomposability, and reachability

#### 3.1. Prices and equilibrium

A price system on a subspace \( G \) of the commodity space \( G \) is a linear function \( P \) which is not identically zero on \( G \) and is non-negative, i.e., \( P(c) \geq 0 \) for any non-negative \( c \) in \( G \). If \( G \) is a topological space, and \( P \) is
continuous on $\mathcal{G}_1$, we call $P$ a valuation function [see Debreu (1954)]. When $P$ is representable as an infinite sequence $p = (p^t)$ with $P(c) = \sum_{t=0}^{\infty} p^t c^t$ for all $c$ in $\mathcal{G}_1$, $p$ is termed a price sequence. In this case the notation $p \cdot c = \sum_{t=0}^{\infty} p^t c^t$ is adopted.

Consider the subspace $\mathcal{G}_{1f}$ of $\mathcal{G}_1$ consisting of all sequences $c$ in $\mathcal{G}_1$ which have a finite number of non-zero components. Then, for any price-system $P$ on $\mathcal{G}_1$, there exists a (possibly zero) sequence $p = (p^t)$ which is a unique representation of $P$ on $\mathcal{G}_{1f}$. If $\mathcal{G}_1$ consists only of programs with a finite number of non-zero components, any price system $P$ on $\mathcal{G}_1$ will have a unique representation as a non-zero price sequence $p = (p^t)$.

We shall, now, consider a number of possible concepts of a competitive equilibrium in our infinite horizon economy.

A feasible allocation $\overline{h}$, and a non-zero price sequence $p = (p^t)$ define a competitive equilibrium $(\overline{h}, p)$ if:

(i) For all $k = 1, 2, \ldots, K$, and for all $t = 0, 1, \ldots$

\[ p \cdot c_{kt} \leq p \cdot \overline{c}_{kt}, \quad c_{kt} \in D_{kt} \] implies \[ \overline{c}_{kt} \preceq_{kt} c_{kt} \] \quad (3.1)

(ii) For all $j = 1, 2, \ldots, J$, and for all $t = 0, 1, \ldots$

\[ p^{t+1} B_{jt}^{t+1} - p^t \overline{a}_{jt} \geq p^{t+1} b_j - p^t a \quad \text{for} \quad (a, b) \in \mathcal{F}_{jt}, \quad \tau \geq 0. \] \quad (3.2)

Eq. (3.2) is equivalent to the condition that for all $j = 1, 2, \ldots, J$, and for all $t = 0, 1, \ldots$

\[ p \cdot \overline{y}_{jt} \leq p \cdot y_{jt} \quad \text{for all} \quad y_{jt} \in Y_{jt}. \] \quad (3.3)

To prove this, note that on the subspace $\mathcal{G}_{1H}$ of $\mathcal{G}_1$ consisting of all programs which are zero after any period $H$, the function $P$ has a unique continuous representation $(P_{(i)t}, \ldots, P_{(i)H})$ [Dunford (1958, p. 245)]. This is also true for $\mathcal{G}_{1H-1}$, yielding prices $(P_{(i)t-1}, \ldots, P_{(i)H-1})$. But $\mathcal{G}_{1H-1}$ is itself a subspace of $\mathcal{G}_{1H}$, implying $P_{(i)t-1} = P_{(i)H}$ for $t = 0, \ldots, H-1$. Induction on $H$ completes the proof.

Conversely, if $p \cdot y_{jt} \leq p \cdot \overline{y}_{jt}$, $y_{jt} \in Y_{jt}$ and (3.2) is violated for some $\tau^*$ and $(a, b) \in \mathcal{F}_{jt}$, then
A competitive equilibrium \((\bar{h}, p)\) is a Malinvaud equilibrium if, for every feasible allocation \(\bar{h}\), and time period \(L\) such that all consumer units living on or after \(L\) receive the same consumption program under either \(\bar{h}\) or \(\bar{h}\), it follows that

\[
\sum_{t=0}^{\infty} p^t (\bar{y}^t - \bar{y}^t) \leq 0. 
\]  

(3.4)

Condition (3.4) for a Malinvaud equilibrium can be interpreted as requiring that among the set of all feasible aggregate net output programs which differ from \(\bar{y}\) in at most a finite number of components, present value is maximized at \(\bar{y}\). From material balance, feasible allocations \(\bar{h}\) and \(\bar{h}\) satisfy

\[
\sum_{t=0}^{L} p^t (\bar{y}^t - \bar{y}^t) = \sum_{t=0}^{L} p^t (\bar{c}^t - \bar{c}^t). 
\]  

(3.5)

Hence, an equivalent interpretation of the Malinvaud equilibrium is that among the set of all feasible aggregate consumption programs which differ from \(\bar{c}\) in a finite number of periods, present value is maximized at \(\bar{c}\).

A competitive equilibrium \((\bar{h}, p)\) is a strong competitive equilibrium if, for every feasible allocation \(\bar{h}\), the following condition holds:

\[
\lim_{l \to \infty} \inf \sum_{t=0}^{L} \sum_{k=1}^{K_t} p^t (\bar{c}_{kt} - \bar{c}_{kt}) \leq 0. 
\]  

(3.6)

Condition (3.6) can be interpreted as requiring that the incremental ‘present value’ of consumption associated with a shift from \(\bar{h}\) to another feasible allocation \(\bar{h}\) (which may differ from \(\bar{h}\) in infinitely many periods) not be positive.

Note that it is not necessary in this definition that ‘present value’ be a well-defined number. A valuation equilibrium imposes the same economic conditions, and requires in (3.7) that ‘present value’ be well-defined. Thus, a strong competitive equilibrium \((\bar{h}, p)\) defines a valuation equilibrium if the

consider \(y_p \in Y_p\) given by

\[
(y_p^t - \bar{a}_p^t, \bar{b}_p^t - \bar{a}_p^t, \ldots, \bar{b}_p^{t+1} - \bar{a}_p^{t+1}, \ldots).
\]

Then

\[
p \cdot y_p^t - p \cdot \bar{y}_p^t = [p^{t+1} - (-a)] + p^{t+1} \cdot (-a) = \bar{y}_p^t [p^{t+1} - (-a)]
\]

a contradiction.
non-zero price-sequence \( p = (p') \) is a continuous linear function on a topological subspace of \( \mathcal{B} \) which contains the set of all possible non-negative net supply programs \( c = y + e \), and the inequality

\[
p' \cdot y \leq p' \cdot \tilde{y}
\]

holds for this set of net supply programs.

A competitive equilibrium \((\tilde{h}, p)\) has the insignificant future property if

\[
\lim_{\tau \to \infty} p' a' = 0 \quad \text{and} \quad \lim_{\tau \to \infty} p' e' = 0.
\]

(3.7)

(3.8)

Thus, (3.7) imposes the requirement that the values of aggregate consumption and inputs in period \( \tau \) go to zero as \( \tau \) goes to infinity.

We shall now discuss how the various concepts of competitive equilibria are inter-related. The interested reader might compare (3.2), (3.4), (3.6), and (3.7) to the definitions of efficiency prices of different ‘types’ discussed by Peleg and Yaari (1970). Clearly, one set of implications is immediate from the sequence of definitions: valuation equilibrium implies strong competitive equilibrium implies competitive equilibrium, and Malinvaud equilibrium implies competitive equilibrium. Some less apparent implications will now be derived, and in such derivations (which are only sketched) some accounting identities will be useful.

Let \( A_T \) be the value of aggregate consumption from period 1 through period \( T \) at prices \( p = (p') \). Recalling that \( c' = e' + y' \),

\[
A_T \equiv \sum_{\tau = 0}^{T} p' c' \\
= \sum_{\tau = 0}^{T} p'(e' + y') \\
= \sum_{\tau = 0}^{T} p'e' + p^0 b^0 - p^T a^T + \sum_{\tau = 0}^{T-1} [p^{\tau+1} b^{\tau+1} - p' a'].
\]

(3.9)

Using (3.9), we can show that strong competitive equilibrium implies Malinvaud equilibrium. Suppose there is a feasible allocation \( \tilde{h} \) and a period \( L \) such that every consumer unit living on or after \( L \) receives the same consumption program as in \( \tilde{h} \). Consider the identity

\[
\sum_{\tau = 0}^{L} \sum_{k=1}^{K_{\tau}} p' \cdot c_{kl} = \sum_{\tau = 0}^{L} p' \cdot c' = \sum_{\tau = 0}^{L} \sum_{k=1}^{K_{\tau}} \sum_{\tau = L+1}^{\infty} p' \cdot c_{kl}'.
\]

(3.10)

For \( \tilde{h} \) and \( h \), the right-hand side of (3.10) is the same, implying
\[
\sum_{t=0}^{L} p'(\bar{c}^t - \bar{c}^t) = \sum_{t=0}^{L} \sum_{k=1}^{K_t} p \cdot (\bar{c}_{kt} - \bar{c}_{kt}) = \lim_{l \to \infty} \inf \sum_{t=0}^{l} \sum_{k=1}^{K_t} p(\bar{c}_{kt} - \bar{c}_{kt}). \tag{3.11}
\]

Then (3.5) and (3.9) imply that (3.4) holds.

Next, we shall show that a valuation equilibrium \((\bar{h}, \bar{p})\) has the insignificant future property. In order to verify (3.8) note first that

\[
p'\bar{y} = \sum_{t=0}^{\infty} p'(c^t + e^t) \geq \sum_{t=0}^{\infty} p'c^t. \tag{3.12}
\]

By definition of a valuation equilibrium, \(p'\bar{y}\) is finite. From (3.12) it follows directly that

\[
\lim_{t \to \omega} p'c^t = 0. \tag{3.13}
\]

If, now, \(\lim_{t \to \infty} \sup p'a^t \neq 0\), there must be some \(\delta > 0\) and a subsequence (retain the same notation for the subsequence) such that \(p'a^t \geq \delta\) for all \(t\). Choose some \(T^*\) such that \(\sum_{t=T^*}^{\infty} p'tc^t < \delta/2\) and define \(\bar{c} = (\bar{c}^t)\) as \(\bar{c}^t = c^t\) for all \(t \leq T^*\); \(\bar{c}^{T^*+1} = b^{T^*+1} + e^{T^*+1}\); and \(\bar{c}^t = 0\) for all \(t > T^*+1\). Then

\[
p'\bar{c} \geq \sum_{t=0}^{T^*} p'tc^t + p{T^*+1}a^{T^*+1} \geq \sum_{t=0}^{\infty} p'tc^t + \delta/2. \tag{3.14}
\]

But (3.14) contradicts (3.7), implying

\[
\lim_{t \to \infty} p'a^t = 0. \tag{3.15}
\]

The next interesting implication is that a competitive equilibrium with the insignificant future property (3.8) is a strong competitive equilibrium (3.6).

To see this, use (3.10) to get

\[
\sum_{t=0}^{l} \sum_{k=1}^{K_t} p(\bar{c}_{kt} - \bar{c}_{kt}) = \sum_{t=0}^{l+\alpha} p'(\bar{c}^t - \bar{c}^t) - \sum_{t=l+\alpha}^{l+\alpha} p'(\bar{c}^t - \bar{c}^t) \nonumber
\]

\[
+ \sum_{t=0}^{l} \sum_{k=1}^{K_t} \sum_{\tau=l+1}^{\infty} p(\bar{c}_{kt} - \bar{c}_{kt}). \tag{3.16}
\]

Now use (3.9) to get the following bound on the first term on the right side of (3.16):
\[ \sum_{t=0}^{l+s} p^t(\epsilon^t - \bar{c}^t) \leq \sum_{t=0}^{l+s} d^{l+s} \cdot p^t. \] (3.17)

The inequality (3.18) is now derived by using (3.17) and dropping a term from the right side of (3.16) that is negative,

\[
\sum_{t=0}^{l} \sum_{k=1}^{k_t} p^t(\bar{c}^t - \bar{\epsilon}^t) + \sum_{t=0}^{l+s} p^t \bar{c}^t - \sum_{t=1}^{l+s} p^t \bar{c}^t + \sum_{t=0}^{l+s} p^t \bar{\epsilon}^t - \sum_{t=1}^{l+s} p^t \bar{\epsilon}^t \\
\leq \sum_{t=1}^{l+s} d^{l+s} p^t + \sum_{t=0}^{l+s} p^t \bar{c}^t. \] (3.18)

Now use the insignificant future property to get (3.6) from (3.18).

Two examples will now be used to illustrate the distinction between the various notions of competitive equilibria. In both these examples we consider an economy with a single commodity and no net production. There is an 'old' consumer in period 0, disappearing at the end of that period. A single consumer unit is born in period \( t \geq 0 \), disappearing at the end of period \( t + 1 \). The utility function of all the consumers is the same, namely the sum of consumptions in two periods (the utility of the old consumer in period 0 is just his consumption in period 0).

Example 3.1. Let \( \epsilon^t = 2 \) for all \( t \geq 0 \). We examine the following allocations:

Allocation (A): In each period \( t \geq 0 \), the available supply (two units of the commodity) is divided equally between the two consumers.

Allocation (B): The available supply is distributed entirely to the 'old' consumer in each period \( t \geq 0 \).

Allocation (C): Assuming that the good can be stored free without any depreciation, the 'old' consumer in period 0 gets 1 unit; all other consumers get 2 units in their 'old' age.

A comparison of the allocation (A) with (B) shows that (A) is not Pareto optimal. At price \( p = (1, 1, 1, \ldots) \), the allocation (A) is a Malinvaud equilibrium [since the absence of production allows (3.4) to be satisfied automatically]. Thus, a Malinvaud equilibrium need not be Pareto optimal. The allocation (A), is not, however, a strong competitive equilibrium, as can be concluded from a comparison with (B). At \( p = (1, 1, \ldots) \) the allocation (C) is a competitive equilibrium, but not a Malinvaud equilibrium, as can be concluded from a comparison of (A) and (C). Finally, the allocation (B) is an
example of a strong competitive equilibrium which is not a valuation equilibrium.\footnote{Let \( \hat{\mathbf{h}} \) denote allocation \( \mathbf{B} \) and let \( \check{\mathbf{h}} \) denote an alternative feasible allocation in which a proportion \( \theta_t \) of the aggregate endowment in period \( t \) is given to the \( t \)th consumer unit. Then \( \sum_{t=0}^{T} p \cdot (\hat{\mathbf{e}}_t - \check{\mathbf{e}}_t) = -\theta_n \) and (3.6) is satisfied. Since \( \sum_{t=0}^{T} p \cdot \check{\mathbf{e}} = H_p \cdot \check{\mathbf{e}} \) does not exist, and \( p \) is not a valuation function.}

\textbf{Example 3.2.} Let \( e^0 = 1, \ v^t = 1/t \) for \( t \geq 1 \). The available supply is allotted to the ‘old’ consumer in each period. At the price system \( p = (1, 1, \ldots) \), this allocation is a competitive equilibrium. It is also true that \( p^t v^t = 1/t \) for \( t \geq 1 \), so that the insignificant future property holds. However, \( \sum_{t=0}^{T} p^t v^t = 1 + \sum_{t=1}^{T} 1/t \) and this goes to infinity with \( T \). Thus, we have an example of a competitive equilibrium with the insignificant future property that is not a valuation equilibrium.

For a number of simpler models, including the one we considered in Majumdar–Mitra–McFadden (1976), it has been shown that a competitive equilibrium that is also efficient is necessarily long-run Pareto optimal, or that short-run Pareto optimality together with (long-run) efficiency implies long-run Pareto optimality [see Cass and Yaari (1967, p. 249) and Bose (1974) in this connection]. Indeed for ‘interior’ programs in models in which the technology has appropriate differentiability properties, some remarkably strong implications can be shown to hold, as our earlier exercise seems to indicate. For treating other aspects of intertemporal welfare economics, it might be useful to start with such a simple model. The next example shows that in general efficiency and short-run Pareto optimality need not imply Pareto optimality.

\textbf{Example 3.3.} Again we consider an economy in which a single consumer is born in each period, living for two periods. Similarly, in each time period a ‘generation of firms’ (consisting of a single firm) is born, living for two periods. The technology set \( \mathcal{F} \) for all firms is the same and is given by

\begin{equation}
\mathcal{F} = \{(a, b) \geq 0 : A z \leq a, B z \geq b \text{ for some } z \geq 0\},
\end{equation}

where

\[ A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ 0 & 1 \end{pmatrix}. \]

Clearly the second activity has the second commodity as a non-depreciating, durable capital good which generates a net production of the first commodity.

The preferences for consumers are given by utility functions (for \( t \geq 0 \))
\[ U_{1t}(c_{1t}^{i}, c_{1t}^{i+1}) = \frac{1}{2} c_t^i + \frac{1}{2} c_{t+1}^i + \frac{1}{2} c_t^{i+1} + \frac{1}{2} c_{t+1}^{i+1} + 2 c_t^{i+1}, \quad (3.20) \]

where \( i \) denotes consumption of commodity \( i = (1, 2) \).

The resource supply is given by \( e^0 = (0, 1), e^t = (0, 0) \) for \( t \geq 1 \). Consider the allocation given by \( \bar{a} = (0, 1), t \geq 0; \; \bar{c}^0 = (1, 0) \) for \( t \geq 0; \; \bar{b}_{t+1}^{i+1} = \frac{1}{4}, t \geq 0 \).

One can verify that the allocation is feasible and efficient. Furthermore, it is short-run Pareto optimal.

However, it is not long-run Pareto optimal. Consider the allocation given by

\[
\bar{a}^0 = (0, 1), \quad \bar{a} = \left( 0, 1 - \sum_{t=0}^{i-1} \frac{1}{2^t} \right), \quad t \geq 1,
\]

\[
\bar{b}^t = \left[ \frac{1}{4}, 1 \right], \quad \bar{b}^t = \left[ \frac{1}{4} \left( 1 - \sum_{t=0}^{i-2} \frac{1}{2^t} \right), \left( 1 - \sum_{t=0}^{i-2} \frac{1}{2^t} \right) \right], \quad t \geq 2
\]

\[
\bar{c}^0 = (0, 0), \quad \bar{c}^t = \left[ \frac{1}{4}, \frac{1}{2} \right], \quad \bar{c}^t = \left[ \frac{1}{4} \left( 1 - \sum_{t=0}^{i-2} \frac{1}{2^t} \right), \frac{1}{2} \right], \quad t \geq 2,
\]

\[
\bar{c}_{1t}^0 = (0, 0), \quad \bar{c}_{1t}^{i+1} = \bar{c}_{1t}^{i+1}, \quad t \geq 0.
\]

Then, it can be checked that this allocation is feasible, and has

\[
U_{1t}(\bar{c}_{1t}^{i}, \bar{c}_{1t}^{i+1}) > U_{1t}^0(\bar{c}_{1t}^{i}, \bar{c}_{1t}^{i+1}) \quad \text{for all} \quad t \geq 0.
\]

3.2. Non-decomposability and reachability

A feasible allocation \( \bar{h} = (\bar{s}, \bar{d}, e) \) is non-decomposable if, for each partition of the consumer units into two non-empty subsets \( I_1 \) and \( I_2 \), there exists a feasible allocation \( \bar{h} = (\bar{s}, \bar{d}, e) \), with the same net supply program, which is Pareto preferable to the allocation \( \bar{h} \) for the consumer units in \( I_1 \).

This condition implies, in particular, that no consumer is satiated and that between any two groups \( I_1 \) and \( I_2 \), there exists at least one commodity desired by some unit in \( I_1 \) which is held by some unit in \( I_2 \). Non-decomposability also excludes the ‘extreme’ distributions in which some unit is at subsistence or some other unit has the ‘largest’ feasible consumption program. A sufficient condition for non-decomposability is that there exists a commodity which is essential and always desirable to every unit, and that at least one unit in each cohort survives for more than one period.

Consider a possible aggregate net output program \( \bar{y} \) and a resource supply program \( e \), such that \( \bar{c} = \bar{y} + e \) is non-negative. Then, \( (\bar{y}, e) \) is reachable if for
any positive scalar $\mu$ there exists a possible net output program $\tilde{y}$ such that $\tilde{c} = \tilde{y} + e$ is non-negative, and there is $L(\mu)$, such that $\tilde{c}^t = \mu \tilde{c}^t$ for $t > L(\mu)$.

Reachability implies, roughly, that starting from an arbitrarily small proportion of the original resource program, one can, by pure accumulation, reach, after a sufficiently long term, the path of an originally possible net supply program. In an economy in which all commodities can be produced, resources do not limit the long-run growth rate, and consumption requires the diversion of productive commodities, feasible allocations will generally be reachable.\(^8\)

4. Equilibrium, Pareto optimality, and efficiency

4.1. Equilibrium is optimal

We shall now prove that a strong competitive equilibrium is Pareto optimal.

**Theorem 4.1.** Under (T.1), (P.1), and (P.7), a strong competitive equilibrium $(\bar{h}, p)$ is Pareto optimal.

**Proof.** Suppose not. Then, there is a feasible allocation $\tilde{h}$, which is Pareto preferable to $\bar{h}$. Then $\tilde{c}_{kt} \geq k \bar{c}_{kt}$ for each consumer unit. We claim that this implies that $p \cdot \tilde{c}_{kt} \geq p \cdot \bar{c}_{kt}$ for all $k, t$. If not, then $p \cdot \tilde{c}_{kt} < p \cdot \bar{c}_{kt}$ for some $k, t$. By (P.7), there exists a $\tilde{c}_{kt}'$ in a sequence converging to $\tilde{c}_{kt}$, such that $p \cdot \tilde{c}_{kt}' \leq p \cdot \tilde{c}_{kt}$ and $\tilde{c}_{kt}' > k \bar{c}_{kt}$. Since $\tilde{c}_{kt} \geq k \tilde{c}_{kt}$, so $\tilde{c}_{kt} > k \tilde{c}_{kt}$, contradicting (3.1), and establishing our claim.

We also know that $\tilde{c}_{kt} > k \tilde{c}_{kt}$ for some $k, t$. We claim this implies that $p \cdot \tilde{c}_{kt} > p \cdot \bar{c}_{kt}$ for this $k, t$. If not, then $p \cdot \tilde{c}_{kt} \leq p \cdot \bar{c}_{kt}$, implying $\tilde{c}_{kt} \geq k \tilde{c}_{kt}$ by (3.1), a contradiction, which again establishes our claim.

Hence, from the above two results, we know that there is $\theta > 0$ such that

$$\sum_{t=0}^{l} \sum_{k=1}^{K} p \cdot (\tilde{c}_{kt} - \bar{c}_{kt}) \geq \theta > 0$$

for all $l$ sufficiently large, contradicting (3.5) for a strong competitive equilibrium. Hence $(\bar{h}, p)$ is Pareto optimal.

4.2. Reachable optimal allocations are equilibria

A partial converse of Theorem 4.1 can be proved when a Pareto-optimal allocation is reachable. The resulting price system will define a valuation equilibrium, implying the existence of a finite present-value associated with each aggregate consumption program.

\(^8\)Several examples of economies with this property are given in McFadden (1967).
Theorem 4.2. Under (T.1)–(T.5), and (P.1)–(P.6), if a feasible allocation \( \bar{h} \) is efficient,\(^9\) reachble, non-decomposable, and Pareto optimal, then there exists a non-zero price sequence \( p \) such that \((\bar{h}, p)\) is a valuation equilibrium.

Proof. Let \( G \) in \( \mathcal{A} \) denote the free disposal hull of the set of possible aggregate net supply programs, i.e., \( G \) is the set of all \( c \leq y + e \), where \( y \) is a possible aggregate net output program. Let \( G_0 \) denote the subset of non-negative net supply programs in \( G \) and let \( \mathcal{X} \) denote the real linear vector space spanned by \( G_0 \). Define \( F = G \cap \mathcal{X} \). All programs in \( G_0 \) are contained in \( F \), and all efficient programs in \( F \) are possible aggregate net supply programs. We call \( F \) the set of admissible aggregate net supply programs.

The set \( G \) is convex, and contains its free-disposal hull, and \( G_0 \) is pointwise closed and bounded by Assumption T (and hence, Lemma 1). Hence \( G \) and \( \mathcal{X} \) satisfy Assumption (A) of the appendix. Hence, by Lemmas A.1–A.6 of the appendix, there exists a norm \( \delta \) on \( \mathcal{X} \), such that \( \mathcal{X} \) is a Banach space, and \( G_0 = \{ c \in \mathcal{X} | c \geq 0 \ \text{and} \ \delta(c) \leq 1 \} \).

Define, for each consumer unit \( k \) in cohort \( t \), the finite-dimensional subspace \( \mathcal{X}_{kt} = \mathcal{C}_{kt} \cap \mathcal{X} \). Since \( \bar{h} = (\bar{s}, \bar{d}, e) \) is Pareto optimal, and non-decomposable, there must exist for any consumer unit \( kt \) an alternative feasible allocation which is Pareto preferred by the consumer units other than \( kt \), and hence is strictly inferior for \( kt \). Hence by (P.3), the zero bundle is not in \( U_k(\bar{c}_{kt}) \). This, along with (P.2)–(P.6), imply that \( \bar{a} \) is 'non-extreme', 'non-decomposable', and that preferences satisfy, at \( \bar{a} \), Assumption (B) of the appendix.

Since \( \bar{a} \) is reachable, given any \( \mu > 0 \), there is a possible net output plan \( \bar{s} \) such that \( \bar{c} = \bar{y} + e \) is non-negative, and \( \bar{c} \) and \( \mu \bar{c} \) are identical after a finite number of periods, \( L(\mu) \). Hence, one can take \( \bar{c}_{kt} = \mu \bar{c}_{kt} \) for \( t \geq L(\mu) \), and so,

\[
1 \geq \delta(\bar{c}) \geq \delta\left( \sum_{t=L(\mu)}^{\infty} \sum_{k=1}^{K_t} \mu \bar{c}_{kt} \right) = \mu \delta\left( \sum_{t=L(\mu)}^{\infty} \sum_{k=1}^{K_t} \bar{c}_{kt} \right).
\]

Thus, letting \( \mu \to \infty \),

\[
\lim_{\mu \to \infty} \delta\left( \sum_{t=L(\mu)}^{\infty} \sum_{k=1}^{K_t} \bar{c}_{kt} \right) = 0.
\]

The hypotheses of Theorem A.8 of the appendix are then met; so, there exists a non-negative continuous linear functional \( P \) on \( \mathcal{X} \), such that:

(i) \( P(c) \leq P(\bar{c}) = 1 \) for all \( c \in F \), and
(ii) for each \( kt, \ c \in D_{kt} \) and \( P(c) \leq P(\bar{c}_{kt}) \) implies \( \bar{c}_{kt} \geq c, \).

\(^9\)If (P.7) holds, then Pareto Optimality implies efficiency. Hence, if (P.7) is assumed, we do not require the assumption that the allocation \( \bar{h} \) is efficient.
$P$ has a unique representation as a price sequence $p = (p^0, p^1, \ldots)$ on the subspace $\mathcal{X}_f$ of $\mathcal{X}$ consisting of programs with a finite number of non-zero components. Since $D_\mu$ is finite-dimensional, (P.6) implies that $p$ is a non-zero price sequence, satisfying (3.1).

For any $y_{jt} \in Y_{jt}$,

$$P(\tilde{y} + (y_{jt} - \tilde{y}_{jt})) \leq P(\tilde{y}),$$

or

$$P(y_{jt} - \tilde{y}_{jt}) = p \cdot (y_{jt} - \tilde{y}_{jt}) \leq 0,$$

so (3.3), and, hence, (3.2) are satisfied.

Since $\lim_{t_0 \to \infty} \delta(\sum_{t = t_0}^{t_0} \sum_{k=1}^{K_t} \tilde{c}_{kt}) = 0$, continuity of the linear functional $P$ implies that

$$P(\tilde{c}) = \lim_{t_0 \to \infty} \sum_{t=0}^{t_0} \sum_{k=1}^{K_t} p \cdot \tilde{c}_{kt}. \quad \text{(10)}$$

For $c \in G_0$, non-negativity implies

$$p \cdot c = \lim_{t_0 \to \infty} \sum_{t=0}^{t_0} p^l c^l \leq P(\tilde{c}). \quad \text{(11)}$$

and $p$ is bounded on $G_0$. Since $G_0$ spans $\mathcal{X}$, $p$ is a continuous linear

$$P(\tilde{c}) = P\left(\sum_{t=0}^{t_0} \sum_{k=1}^{K_t} \tilde{c}_{kt}\right) = P\left(\sum_{t=t_0}^{t_0} \sum_{k=1}^{K_t} \tilde{c}_{kt}\right) = P\left(\sum_{t=t_0}^{t_0} \sum_{k=1}^{K_t} \tilde{c}_{kt}\right) + P\left(\sum_{t=t_0}^{t_0} \sum_{k=1}^{K_t} \tilde{c}_{kt}\right) = \sum_{t=0}^{t_0} p \cdot \tilde{c}_{kt} + P\left(\sum_{t=t_0+1}^{t_0} \sum_{k=1}^{K_t} \tilde{c}_{kt}\right).$$

Taking limits on both sides as $t_0 \to \infty$,

$$P(\tilde{c}) = \lim_{t_0 \to \infty} \sum_{t=0}^{t_0} \sum_{k=1}^{K_t} p \cdot \tilde{c}_{kt} + \lim_{t_0 \to \infty} P\left(\sum_{t=t_0+1}^{t_0} \sum_{k=1}^{K_t} \tilde{c}_{kt}\right) = \lim_{t_0 \to \infty} \sum_{t=0}^{t_0} \sum_{k=1}^{K_t} p \cdot \tilde{c}_{kt} = \lim_{t_0 \to \infty} \sum_{t=0}^{t_0} p^l c^l = 0.$$

Since $c \in G_0$, $P(\tilde{c}) \geq P(c)$, write $c = (c^0, c^1, \ldots) = (c^0, c^1, \ldots, c^{t_0} 0, 0, \ldots) + (0, 0, \ldots, c^{t_0+1}, c^{t_0+2}, \ldots) = c^* + c^\dagger$. Then

$$P(c) = P(c^*) + P(c^\dagger) \geq P(c^*) = \sum_{t=0}^{t_0} p^l c^l = \frac{t_0}{0} p^l c^l \leq P(\tilde{c}) \text{ for each } t_0.$$ 

Non-negativity implies that $\lim_{t_0 \to \infty} \sum_{t=0}^{t_0} p^l c^l \leq P(\tilde{c})$ exists. But $\lim_{t_0 \to \infty} \sum_{t=0}^{t_0} p^l c^l = p \cdot c$. Hence

$$p \cdot c = \lim_{t_0 \to \infty} \sum_{t=0}^{t_0} p^l c^l \leq P(\tilde{c}).$$
functional on $\mathcal{X}$,\footnote{p is bounded on $G_0$, hence continuous on $G_0$. Since $G_0$ spans $\mathcal{X}$, given $c' \in \mathcal{X}$, write $c' = \sum_{i=0}^n \alpha_i c_i$ where $c_i \in G_0$. Then \[ P(c') = \sum_{i=0}^n \alpha_i P(c_i) = \sum_{i=0}^n \alpha_i (p \cdot c_i) = \sum_{i=0}^n p \cdot (\alpha_i c_i) = p \cdot c'. \] Thus $p$ is defined on $\mathcal{X}$, and continuous on $G_0$. Hence $p$ is bounded on $\mathcal{X}$. Hence $p$ is continuous on $\mathcal{X}$.} and $p \cdot c \leq p \cdot \bar{c} = 1$ for $c \in G_0$. Hence, $p$ is a valuation function on $\mathcal{X}$.

Finally, note that for any feasible allocation, $\bar{h}$, one has $\sum_{t=0}^\infty \sum_{k=1}^{K_t} \bar{c}_{kt}$ in $G_0$, implying $\lim_{\epsilon \to 0} \sum_{t=0}^\infty \sum_{k=1}^{K_t} p \cdot \bar{c}_{kt} = p \cdot \bar{c} \leq 1$. Hence, $(\bar{h}, p)$ satisfies (3.5) for a strong competitive equilibrium, and is, therefore, a valuation equilibrium.

Appendix

The set of admissible solutions in many problems in economics and control theory can be characterized as a subset of the non-negative orthant of a real linear vector space.\footnote{A typical example for economics is the problem of optimizing a social objective function over the set of all programs of non-negative consumption of commodities which are feasible in a growing economy over an infinite horizon.} When the admissible set is convex and 'pointwise' bounded, and all solutions smaller than a given admissible solution are also admissible, we establish the existence of a norm on the space spanned by the admissible set such that the admissible set coincides with the non-negative programs in the unit sphere. This result is closely related to the theorem of Kolmogoroff (1934) that a topological linear space is homeomorphic to a normed linear space if and only if there exists a bounded convex neighborhood of the origin. The norm topology introduced here is homeomorphic to the core topology of a linear space introduced by Klee (1951).

For a countable non-empty set $T$, consider the linear space $\mathcal{Y}$ of all real-valued functions on $T$, and let $\Omega = \{ y \in \mathcal{Y} | y(t) \geq 0 \ \text{for all} \ t \in T \}$ denote the non-negative orthant of $\mathcal{Y}$. A partial ordering $\leq$ on $\mathcal{Y}$ is defined by $y \leq y'$ if and only if $y' - y \in \Omega$ for $y, y' \in \mathcal{Y}$.

A sequence $\{ y_n \}$ in $\mathcal{Y}$ converges pointwise to $y_0$ in $\mathcal{Y}$ [notation: $y_n \to y_0$] if, given $\epsilon > 0$ and $t \in T$, there exists $n(\epsilon, t)$ such that $|y_n(t) - y_0(t)| < \epsilon$ for $n > n(\epsilon, t)$.\footnote{The space $\mathcal{Y}$ is the Cartesian product over $T$ of the space of real numbers, and pointwise convergence is simply convergence in the product topology of $\mathcal{Y}$ when the real line is given its natural topology. Pointwise closure (resp. boundedness) corresponds to closure (resp. boundedness) in the product topology. If sequences are replaced by generalized sequences, all the proofs of this paper hold when $T$ is an arbitrary non-empty set, not necessarily countable.} A set $Y$ in $\mathcal{Y}$ is pointwise-closed (p-closed) if $\{ y_n \} \subseteq Y, y_n \to y_0$ imply $y_0 \in Y$. The set $Y$ is pointwise bounded (p-bounded) if $\{ y(t) | y \in Y \}$ is a
bounded set for each $t \in T$. The set $Y$ is monotone below relative to a set $Y'$ containing $Y$ if $y \in Y, y' \in Y', y' \leq y$ imply $y' \in Y$.

The $(p$-closed) convex hull of a set $Y$ is the intersection of all $(p$-closed) convex sets in $\mathcal{Y}$ containing $Y$.

Suppose a non-empty set $Y_0$ is given which is contained in the non-negative orthant $\mathcal{O}$. Let $Y_1$ denote the convex hull of $Y_0$, and let $\mathcal{X}$ denote the real linear space spanned by $Y_1$. Let $Y_2$ denote the intersection of all sets, monotone below relative to $\mathcal{X}$, containing $Y_1$.

**Lemma 1.** The origin of $\mathcal{X}$ is an internal point of $Y_2$ in $\mathcal{X}$.

**Proof.** Since $Y_2$ is convex, any point $\hat{y} \in \mathcal{X}, \hat{y} \neq 0$, can be written as

$$\hat{y} \leq \alpha_1 y_1 + \alpha_2 y_2, \quad y_j \in Y_1, \quad \alpha_j \geq 0.$$ 

Then, for $0 \leq \theta \leq 1/\alpha_1 + \alpha_2,$

$$\theta \hat{y} \leq \theta \alpha_1 y_1 + \theta \alpha_2 y_2 \leq \frac{\alpha_1}{\alpha_1 + \alpha_2} y_1 + \frac{\alpha_2}{\alpha_1 + \alpha_2} y_2 \in Y_2,$$

and $\theta \hat{y} \in Y_2$ by monotonicity. Q.E.D.

Define the support function of $Y_2, \phi(y) = \inf \{\mu | \mu \text{ a positive scalar, } 1/\mu y \in Y_2\}$. The function $\phi$ exists and is non-negative, positive linear homogeneous, and convex [cf. Dunford and Schwartz (1958)]. Define $\delta(y) = \phi(y) + \phi(-y)$. Then, $\delta(y)$ satisfies

(i) $\delta(y) \geq 0,$

(ii) $\delta(y + y') \leq \delta(y) + \delta(y'),$

(iii) $\delta(\alpha y) = |\alpha| \delta(y),$

for $y, y' \in \mathcal{X}$ and any scalar $\alpha$, and is a pseudo-norm on $\mathcal{X}$ [cf. Kelley (1955, p. 18)].\textsuperscript{15} Note that $y \in Y_0$ implies $\delta(y) \leq 1$, and that $\delta(y) < 1$ implies $y \in Y_2$.

**Lemma 2.** If $Y_0$ is $p$-bounded, then $\delta(y)$ is a norm on $\mathcal{X}$; i.e., $y \neq 0$ implies $\delta(y) > 0$.

**Proof.** Clearly, $Y_0$ $p$-bounded implies $Y_1$ $p$-bounded. If $\hat{y} \neq 0$, then $\hat{y}(t_0) \neq 0$ for some $t_0 \in T$. We can assume (by changing the sign of $\hat{y}$ if necessary) that $\hat{y}(t_0) > 0$. By $p$-boundedness, there then exists a positive scalar $\mu$ such that $\hat{y}(t_0) > \mu \sup \{y(t_0) | y \in Y_1\}$, implying $(1/\mu)\hat{y} \notin Y_1$, and $\delta(\hat{y}) \geq \phi(\hat{y}) \geq \mu > 0$. Q.E.D.

\textsuperscript{15}Condition (ii) follows from the convexity of $\phi$; condition (iii) follows from the homogeneity of $\phi$ and the sign-symmetry of $\delta$. 
Corollary. The norm $\delta$ satisfies the bound $|\hat{y}(t)| \leq \delta(\hat{y}) \sigma(t)$ for any $\hat{y} \in \mathcal{X}$, $t \in T$, where $\sigma(t) = \sup \{y(t) \mid y \in Y_t\}$.

A sequence $\{y_n\}$ in $\mathcal{X}$ is a $\delta$-Cauchy sequence if $\lim_{m,n \to \infty} \delta(y_n - y_m) = 0$. By the bound in the Corollary to Lemma 2, if $\{y_n\}$ is a $\delta$-Cauchy sequence in $\mathcal{X}$, then the sequence $\{y_n(t)\}$ is a Cauchy sequence in the real line for each $t \in T$, and there exists a pointwise limit $y_0 \in \mathcal{Y}$ of $\{y_n\}$; i.e., $y_n \to_t y_0$.

Let $\mathcal{F}$ denote the space of all pointwise limits of $\delta$-Cauchy sequences in $\mathcal{X}$, and define on $\mathcal{F}$ the function

$$\delta(y) = \delta(y) \text{ if } y \in \mathcal{X},$$

$$= \inf \{ \mu \mid \mu = \lim_{n \to -\infty} \delta(y_n), \{y_n\} \text{ a } \delta\text{-Cauchy sequence in } \mathcal{X} \text{ with } y_n \to_t y \}$$

if $y \in \mathcal{F}$, $y \not\in \mathcal{X}$.

Lemma 3. If $Y_0$ is $p$-bounded, then $\delta$ is a norm on $\mathcal{F}$, and $\mathcal{F}$ is a complete vector space (hence, a Banach space).

Proof. A direct verification can be made following Köthe (1969, p. 126, §14.3.1).

Corollary. $\mathcal{X}$ is a dense subspace of $\mathcal{F}$ in the $\delta$ norm.

The results in the lemmas and corollaries above remain valid if $Y_1$ is re-defined as the $p$-closed convex hull of $Y_0$.

Lemma 4. If $Y_1$ is $p$-closed, $p$-bounded, and monotone below relative to $\Omega$, then $\mathcal{X}$ with the $\delta$-norm is complete, i.e., $\mathcal{X} = \mathcal{F}$.

Proof. Suppose $\hat{y} \in \mathcal{F}$. Then, there exists a $\delta$-Cauchy sequence $\{y_n\}$ in $\mathcal{X}$ such that $y_n \to_t \hat{y}$. Since $\delta(y_n)$ is bounded, we can assume (by re-scaling) that $\delta(y_n) < 1$. Then, $\phi(y_n) < 1$ and $\phi(-y_n) < 1$, implying $y_n \in Y_2$ and $-y_n \in Y_2$. Then, by definition, there exist $y'_n, y''_n \in Y_1$ such that $y_n \leq y'_n$ and $-y_n \leq y''_n$. Since $Y_1$ is $p$-bounded, there exists a subsequence (retain notation) such that $y'_n \to_t y'_0$ and $y''_n \to_t y''_0$ with $y'_0, y''_0 \in Y_1$ by $p$-closure. Then, $\hat{y} \leq y'_0$ and $-\hat{y} \leq y''_0$ by the $p$-closure of the partial ordering $\leq$. Define $\hat{y}'$ and $\hat{y}''$ by $\hat{y}'(t) = \max \{0, \hat{y}(t)\}$ and $\hat{y}''(t) = \min \{0, \hat{y}(t)\}$. Then, $0 \leq \hat{y}' \leq y'_0$ and $0 \leq -\hat{y}'' \leq y''_0$, implying by the monotonicity in $\Omega$ of $Y_1$ that $\hat{y}', -\hat{y}'' \in Y_1$. Hence, $\hat{y} = \hat{y}' + \hat{y}'' \in \mathcal{X}$. Q.E.D.

Corollary. $Y_1 = \{y \in \mathcal{X} \mid 0 \leq y \text{ and } \delta(y) \leq 1\}$.

Let $\mathcal{X}^*$ denote the space of all linear functional on $\mathcal{X}$ which are
continuous in the $\delta$-norm. Similarly, let $\tilde{\mathcal{F}}^*$ be the space of all $\delta$-continuous functionals on $\tilde{\mathcal{F}}$.

Lemma 5. If $Y_0$ is $p$-bounded, then for each $x \in \mathcal{X}^*$, there exists $\tilde{x} \in \tilde{\mathcal{F}}^*$, such that $x(y) = \tilde{x}(y)$ for all $y \in \mathcal{F}$.


Let $\Gamma = \mathcal{X} \cap \Omega$ denote the non-negative orthant of $\mathcal{X}$, and let $\Gamma^* = \{x \in \tilde{\mathcal{F}}^* | x(y) \geq 0 \text{ for all } y \in \Gamma\}$ denote the non-negative orthant of $\tilde{\mathcal{F}}^*$.

Lemma 6. If $Y_0$ is $p$-bounded, and a convex subset $Y_3$ of $\Gamma$ has $\delta(y) \geq 1 \text{ for all } y \in Y_3$, then there exists $P \in \Gamma^*$ such that $P(y) \leq 1 \text{ for all } y \in Y_2$ and $P(y) \geq 1$ for all $y \in Y_3$.

Proof. The sets $Y_3$ and $Y_4 = \{y \in Y_2 | y \leq y', \delta(y') \leq 1\}$ are disjoint since $y \in Y_4 \cap \Gamma$ implies $\delta(y) < 1$. $Y_4$ has a non-empty interior. Hence, there exists a non-zero functional $P \in \Gamma^*$ and some scalar $\alpha$ such that $P(y) \leq \alpha$ for $y \in Y_4$, and $P(y) \geq \alpha$ for $y \in Y_3$. $Y_4$ contains $-\Gamma$, implying $-\lambda P(y) \leq \alpha$ for $y \in \Gamma$, all positive scalars $\lambda$, implying in turn that $P \in \Gamma^*$. By $\delta$-continuity, $P(y) \leq \alpha$ for $y$ satisfying $\delta(y) \leq 1$, and hence $P(y) \leq \alpha$ for $y \in Y_2$. Now, $Y_4 \subseteq Y_2$, $\mathcal{X}$ spanned by $Y_4$ and $P \neq 0$, imply $P(\hat{y}) > 0$ for some $\hat{y} \in Y_4$. Hence, $\alpha > 0$. Normalizing $\alpha = 1$ completes the proof. Q.E.D.

Consider a given subset $Y$ of $\mathcal{Y}$ and a non-empty, finite or infinite, set of integers $K = \{k | k = 0, 1, 2, \ldots\}$. Suppose a linear subspace $\mathcal{Y}_k \subseteq \mathcal{Y}$ is defined for each $k \in K$, and define $Y^k = \{y \in \mathcal{Y}_k | y \in Y\}$. Let $\mathcal{S}$ denote the linear space of all vectors $s = (y^0, y^1, \ldots)$ with $y^k \in \mathcal{Y}_k$, $k \in K$, and define $S_0 = \{s \in \mathcal{S} | y^k \in Y^k, k \in K; y = \sum_{k \in K} y_k \in Y\}$.

We make the following assumption:

Assumption A

$Y$ is convex and monotone below relative to $\mathcal{Y}$, and the set $Y_0 = Y \cap \Omega$ of non-negative points in $Y$ is non-empty, $p$-bounded, and $p$-closed.

Let $\mathcal{X}$ denote the real linear space spanned by $Y_0$, and let $\Gamma = \Omega \cap \mathcal{X}$ denote its non-negative orthant. Define $\mathcal{X}_k = \mathcal{X} \cap \mathcal{Y}_k$ and $\Gamma_k = \Omega \cap \mathcal{X}_k$ for $K$. Since $Y_0 = \mathcal{Y} \cap \Omega \subseteq Y_0$, we have $Y_0 \subseteq \Gamma_k$. Let $Y_k = Y \cap \mathcal{X}_k$. Then, Lemmas 1-6 apply to $\mathcal{X}$ and $Y_k$.\footnote{In the statement of Lemma 6, replace $Y_2$ by $Y_2$, and in its proof, replace $Y_4$ by $Y_4$.} We shall again denote by $Y_2$ the intersection of all sets in $\mathcal{X}$, monotone below with respect to $\mathcal{X}$, containing $Y_0$.\footnote{In the statement of Lemma 6, replace $Y_2$ by $Y_2$, and in its proof, replace $Y_4$ by $Y_4$.}
Suppose, for each $k \in K$, a partial preordering $\succeq_k$ (reflexive, transitive relation) of a subset $D_k$ of $\Gamma_k$ is given. Define $y \succeq_k y'$ (resp. $y \sim_k y'$) if and only if $y \succeq_k y'$ and not $y' \succeq_k y$ (resp. both $y \succeq_k y'$ and $y' \succeq_k y$); $y, y' \in D_k$. Let $D_k$ denote the subset of points, $s = (y^0, y^1, \ldots) \in S$ such that $y^k \in D_k$ for $k \in K$. For any non-empty subset $K_1$ of $K$, define a partial preordering $\succeq_k$ on $D$ by $s \succeq_k s'$ if and only if $y^k \succeq_k y'$ for every $k \in K_1$; $s, s' \in D$. Define $s >_k s'$ (resp. $s \sim_k s'$) if and only if $s \succeq_k s'$ and not $s' \succeq_k s$ (resp. both $s \succeq_k s'$ and $s' \succeq_k s$); $s, s' \in D$. Define $S_1 = S_0 \cap D$.

A point $\bar{s} \in S_1$ is maximal in $S_1$ relative to a non-empty subset $K_1$ of $K$ if any $s' \in S_1$ with $s' \succeq_k \bar{s}$ has $s' \sim_k \bar{s}$. A point $\bar{s} \in S_1$ is non-decomposable if, for every partition of $K$ into two proper subsets $K_1, K_2$ there exists $s' \in S_1$ with $y(s') = y(\bar{s})$ such that $s' >_k \bar{s}$. For $\bar{s} \in S_1$, define the set $U_k(\bar{y}^k) = \{y \in D_k | y \succeq_k \bar{y}^k\}$. The point $\bar{s} \in S_1$ is non-extreme if the origin is not contained in the $p$-closure of $U_k(\bar{y}^k), k \in K$.

We shall consider the following assumption on the relations $\succeq_k$ for a given non-extreme $\bar{s} \in S_1$:

**Assumption B**

$U_k(\bar{y}^k)$ is convex, and $y \in U_k(\bar{y}^k)$, $y \succeq_k y' \in \Gamma_k$ implies $y' \in U_k(\bar{y}^k)$. If $y' >_k \bar{y}$, $y' \succeq_k \bar{y}^k$, then $\theta y' + (1-\theta) y >_k \bar{y}^k$ for $0 < \theta < 1$, and $\theta y <_k \bar{y}^k$ for some $\theta' < 1$. At any $\bar{s} \in S_1$, there exists $k' \in K$ and $y \in \Gamma_k$, such that $\bar{y}^k + \bar{y} >_{k'} \bar{y}^k$.

Define the set

$U(\bar{s}) = \{y(s) \in \Gamma_k | s >_k \bar{s}\}$ for $\bar{s} \in S_1$.

If $\bar{s}$ is maximal in $S_1$ relative to $K$, then $U(\bar{s})$ and $Y_0$ are disjoint. If, further, (B) holds, then $U(\bar{s})$ is convex, and Lemma 6 establishes the existence of a non-zero continuous linear functional (in the $\delta$-norm) separating $U(\bar{s})$ and $Y_\delta$. A stronger separation theorem will now be established.

**Theorem 8.** If (A) and (B) hold at a point $\bar{s} \in S_1$ which is maximal relative to $K$, non-decomposable, and non-negative, and if $\lim_{\delta \to \infty} \delta (\sum_{k \in K} \bar{y}^k) = 0$, then there exists $P \in \Gamma^\ast$ such that $P(y) \leq 1$ for all $y \in Y_\delta$, $P(y) > 1$ for all $y \in U(\bar{s})$, and, for each $k \in K$, $P(y^k) \leq P(\bar{y}^k)$ and $y^k \succeq_k y^k$ imply $y^k \sim_k y^k$.

**Proof.** For the set $Y_\delta$, define the support function

$\phi_\delta(y) = \inf\{\mu \in \mu \text{ a positive scalar}, (1/\mu) y \in Y_\delta\}$.

Let $\phi_\gamma(y)$ denote the support function of $Y_\gamma$. Since $Y_\gamma \subseteq Y_\delta$, it follows that $\phi_\delta(y) \leq \phi_\gamma(y)$, and since $Y_2 \cap \Gamma = Y_2 \cap \Gamma, \phi_\gamma(y) = \phi_\gamma(y)$ for $y \in \Gamma$. 


Consider the function

$$f_k(y) = \inf \{ z > 0 \mid (1/z) y \notin U_k(y^k) \}, \quad k \in K, \quad y \in \Gamma.$$ 

We first show that $f_k(y)$ is well-defined and bounded above on sets bounded in the $\delta$-norm. Suppose one could find a sequence $\{y_n\}$ in $\Gamma$ with $\delta(y_n) \leq 1$ such that $(1/n)y_n \in U_k(y^k)$. Then, $(1/n)y_n \rightarrow 0$, contradicting the hypothesis that $\delta$ is non-extreme. Hence, $f_k(y)$ is defined and bounded on $y \in \Gamma$ with $\delta(y) \leq 1$. By Assumption (B), $f_k(y)$ is concave and positive linear homogeneous on $\Gamma$ and satisfies $f_k(y) \geq 1$ for $y \in U(y^k)$ and $f_k(y^k) = 1$.

Define

$$\mu_k = -1 + \sup \{ f_k(y + y^k) \mid y \in \Gamma, \quad \delta(y) \leq 1 \}.$$ 

Since $y^k \in Y_0$ has $f_k(2y^k) = 2$, we have $\mu_k \geq 1$. Since $f_k(y)$ is bounded on $Y_0$, $\mu_k$ is finite.

Let $v = (v_0, v_1, \ldots)$ denote a sequence of real numbers with a component for each $k \in K$, and define a real Banach space

$$V = \left\{ v \left| \sum_{k \in K} \left| v_k \right| \mu_k = \zeta(v) < +\infty \right. \right\}.$$ 

Then, $V$ is homeomorphic to $l_1$, and the space of continuous linear functionals on $V$ is

$$V^* = \left\{ z = (z_0, z_1, \ldots) \left| \sup_{k \in K} \left| z_k \right| < +\infty \right. \right\}.$$ 

Let $R$ denote the real line, and form the linear space $W = V \times R \times \mathcal{F}$ with norm

$$\| w \| = \| (v, r, y) \| = \sum_{k \in K} |v_k|/\mu_k + |r| + \delta(y).$$

Define the set

$$E = \left\{ (v, r, y) \in W \left| v_k \leq f_k(y_1^k) - f_k(y^k), \quad \gamma \leq 1 - \phi_\delta(y_2), \right. \right\} \quad \gamma = \sum_{k \in K} y_k^k - y_2; \quad y_1 \in \Gamma; \quad y_2, \quad \sum_{k \in K} y_k^k \in \mathcal{F}.$$ 

Since each function $f_k$ is concave and $\phi_\delta$ is convex, the set $E$ is convex. The point $(\theta, 0, 0)$ is in $E$ (i.e., take $y_1 = y^k_1$, $y_2 = y$). We will now show that $(\theta, 0, 0)$
is a non-interior point of $E$. Suppose there exists $(v, r, y) \in E$ with $v \geq 0$, $r > 0$, and $y \geq 0$. Then, there exists $\tilde{s} \in S$, such that $f_k(\tilde{y}^k) \geq f_k(\tilde{y}^k)$ for all $k$, implying $\tilde{y}^k \sim_k \tilde{y}^k$ by the maximality of $\tilde{s}$, and such that $\phi_{\tilde{s}}(y(\tilde{s})) = 1 - \varepsilon < 1$. By Assumption (B), there exists $k' \in K$ and $y \in \Gamma_{k'}$ such that $\theta y^k + \tilde{y}^k \succ_k \theta y + (1 - \varepsilon)(\tilde{y}^k \sim_k \tilde{y}^k)$ for $0 < \varepsilon < 1$. Choose $\theta = \varepsilon/(1 + \delta(\tilde{y}))$. Then, $\phi_{\tilde{s}}(\theta y^k) < e$, and $s'$ defined with $y^k = \tilde{y}^k$, $k \neq k'$, and $y^k = y^k + \theta \tilde{y}$ has $s' \succ_k s$, contradicting the maximality of $\tilde{s}$. Hence, $(v, r, y) \in E$ and $r > 0$, $y \geq 0$ implies that some component of $v$ is negative, and $(0, 0, 0)$ is non-interior to $E$.

We next show that $\tilde{w} = (0, -3, 0)$ is an interior point of $E$. Consider any point $w = (v, r, y)$ satisfying $\|w - \tilde{w}\| \leq \frac{1}{2}$, and define $\theta_k = (\|v_k\| + 2^{-k+2})/\mu_k$. Then $\sum_{k \in K} \theta_k \leq \frac{1}{2}$ and $\leq 1$. Let $\tilde{y}^k$ denote a point in $Y_0^k$ which satisfies $f_k(\tilde{y}^k) = \max\{\mu_k - 2^{-k+2}, f_k(\tilde{y}^k)\}$, and define $y^{\ast k} = \theta_k \tilde{y}^k + \tilde{y}^k$. By Lemma 5, $\mathscr{X}$ is a compact and the unit sphere is closed in $\delta$-norm, implying that $y^* = \sum_{k \in K} y^{\ast k}$ has $\phi_{\tilde{s}}(y^*) \leq 2$. Define

$$v^*_k = f_k(y^{\ast k}) - f_k(\tilde{y}^k) \geq \theta_k (\mu_k - 2^{-k+2}),$$

by concavity. Then, the point $(v^*, 1 - \phi_{\tilde{s}}(y^* - y), y)$ is in $E$. But

$$1 - \phi_{\tilde{s}}(y^* - y) \geq 1 - \phi_{\tilde{s}}(y^*) - \delta(y) \geq -2,$$

and

$$v^*_k \geq \theta_k (\mu_k - 2^{-k+2}) \geq v_k,$$

imply

$$(v^*, 1 - \phi_{\tilde{s}}(y^* - y), y) \in (v, r, y) = w,$$

and $w \in E$.

Since $(0, 0, 0)$ is a boundary point of a convex set with a non-empty interior, there exists a non-zero linear functional $(z, b, -P) \in V^* \times R \times \mathcal{X}^*$ such that $z(v) + br - P(y) \leq 0$ for all $(v, r, y) \in E$ [cf. Dunford and Schwartz (1957, pp. 447–449)]. From the construction of $E$, $z_k \geq 0$ for $k \in K$, $b \geq 0$, and $P \in \Gamma^*$. Take $y^*_k = \tilde{y}^k$, $k \in K$ and $y_2 \in \mathcal{X}$. Then one has $b(1 - \phi_{\tilde{s}}(y^*_2)) - P(\tilde{y} - y_2) \leq 0$. Suppose $b = 0$. Then $P(y_2) \leq P(\tilde{y})$ for all $y_2 \in \mathcal{X}$ implies $P = 0$. But then, taking $y^*_k = (1 + 2^{-k})\tilde{y}^k$ and $y_2 = \tilde{y}$, one has $f_k(y^*_k) - f_k(\tilde{y}^k) = 2^{k+2} - v_k$, implying $v \in V$ and $z(v) = \sum_{k \in K} 2^{-k} z_k \leq 0$. But this contradicts the previous conclusion that $(z, b, -P)$ is non-zero and $z_k \geq 0$, and hence the supposition that $b = 0$ is false. Normalize $b = 1$. Setting $y^*_1 = \tilde{y}^k$ and $y_2 = 0$ and $2\tilde{y}$ establishes that $b = P(\tilde{y}) = 1$.

Next consider $y^*_k$ equal to $0$ or $2\tilde{y}^k$ for some $k' \in K$, $y^*_k = \tilde{y}^k$ for $k \in K$, $k \neq k'$, and $y_2 = \tilde{y}$. Then, $-z_k + P(y^*_k) \leq 0$ and $+z_k - P(y^*_k) \leq 0$, implying $z_k = P(\tilde{y}^k)$. Now,
\[
\lim_{k_0 \to \infty} \delta \left( \sum_{k \leq k_0} \sum_{k \in K} \tilde{y}^k \right) = 0
\]

implies
\[
P(\tilde{y}) = \lim_{k_0 \to \infty} P \left( \sum_{k \leq k_0} \tilde{y}^k \right) = \lim_{k_0 \to \infty} \sum_{k \leq k_0} z_k = 1,
\]

by the continuity of the linear functional. Hence, at least one \(z_k\) is positive.

Suppose not all \(z_k\) are positive. Partition \(K\) into sets \(K_1, K_2\) such that \(z_k > 0\) for \(k \in K_1\); \(z_k = 0\) for \(k \in K_2\). By non-decomposability there exists \(\tilde{s} \in S_1\) with \(\tilde{s} \succ_{K_1} \tilde{s}\). Hence, for some \(k' \in K_2\), \(\tilde{y}_k^k >_{k'} \tilde{y}_k^{k'}\). By Assumption (B), \(\theta' \tilde{y}_k^k >_{k'} \tilde{y}_k^{k'}\) for some \(\theta' < 1\), implying \(f_k(\tilde{y}_k^k) > 1\). Define \(v_k = 2^{-k}[f_k(\tilde{y}_k^k) - f_k(\tilde{y}_k^{k'})]\). Then, \((u, 0, 0) \in E\) implies \(\sum_{k \in K_1} z_kv_k \leq 0\), contradicting the previous results that \(z_k > 0\), \(v_k \geq 0\) for \(k \in K_1\), and \(v_k > 0\). Hence, \(z_k > 0\) for \(k \in K\).

Finally, consider \(y_2 = \tilde{y}, \ y_1^k = \tilde{y}_k^k\) for \(k \in K\), \(k \neq k'\). Then
\[
z_k[f_k(\tilde{y}_k^k) - f_k(\tilde{y}_k^{k'})] - P(\tilde{y}_1^k - \tilde{y}_k^{k'}) \leq 0,
\]
or
\[
z_k f_k(\tilde{y}_k^k) \leq P(\tilde{y}_k^k) \quad \text{and} \quad z_k f_k(\tilde{y}_k^{k'}) = P(\tilde{y}_k^{k'}) \quad \text{for} \quad k \in K, \ y_k \in \Gamma_k.
\]

Hence, \(P(\tilde{y}_k^k) \leq P(\tilde{y}_k^{k'})\) implies \(f_k(\tilde{y}_k^k) > f_k(\tilde{y}_k^{k'})\). But \(\tilde{y}_k^k >_{k'} \tilde{y}_k^{k'}\) would imply \(f_k(\tilde{y}_k^k) > f_k(\tilde{y}_k^{k'})\) by Assumption (B). Hence, \(P(\tilde{y}_k^k) \leq P(\tilde{y}_k^{k'})\) implies \(\tilde{y}_k^k \succ_{k'} \tilde{y}_k^{k'}\).

Take \(\tilde{y} \in U(\tilde{s})\), and let \(\tilde{s}\) be an associated point in \(S_1\). Then, \(f_k(\tilde{y}_k^{k'}) \geq f_k(\tilde{y}_k^{k'})\), implying \(P(\tilde{y}_k^k) \geq P(\tilde{y}_k^{k'})\) for \(k \in K\). Further \(f_k(\tilde{y}_k^{k'}) > f_k(\tilde{y}_k^{k'})\) for some \(k' \in K\), implying \(P(\tilde{y}_k^k) > P(\tilde{y}_k^{k'})\). Now, \(P(\tilde{y}) \geq \lim_{k_0 \to \infty} P(\sum_{k \leq k_0} \tilde{y}_k^k)\) by the non-negativity of the \(\tilde{y}_k^k\) and \(P(\tilde{y}) = \lim_{k_0 \to \infty} P(\sum_{k \leq k_0} \tilde{y}_k^k)\) was previously established. Hence, \(P(\tilde{y}) > P(\tilde{y})\) for all \(\tilde{y} \in U(\tilde{s})\).

The condition \(\phi_5(y_2) \geq P(y_2)\) is obtained by setting \(y_2 = \tilde{y}_k^k\) for \(k \in K\). Then \(\phi_5(y_2) \leq 1\) for \(y_2 \in Y_5\) and \(1 = P(\tilde{y})\) implies \(P(y) \leq P(\tilde{y})\) for \(y \in Y_5\). This completes the proof of the theorem. Q.E.D.

References


