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SOME USES OF THE EXPENDITURE FUNCTION IN PUBLIC FINANCE

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1. Introduction

Increasingly, uses are being found for dual approaches to the consumer and the firm.[1] In consumer analysis one can use the expenditure function, which relates the minimal level of income necessary to achieve a given utility level to the vector of commodity prices. In teaching public finance, we have found three problems which are nicely approached in this way – the deadweight burden of taxation, optimal commodity taxes, and criteria for indivisible public investments financed by lump sum taxation. This paper presents these three uses after briefly reviewing the properties of this function. The appendix contains more details on the mathematics of the expenditure function.

2. Expenditure function[2]

The conventional treatment of consumer choice is to consider the maximization of utility subject to a budget constraint:

$$\begin{aligned}&\text{Maximize } U(x)\\&\text{subject to } q\cdot x \leqq I,\end{aligned} \tag{1}$$

where x represents the vector of net demands (with commodities supplied, like labor, therefore measured negatively), q and I being consumer prices and income. If we consider the dual to this problem we would express it as[3]

$$\begin{aligned}&\text{Minimize } q\cdot x\\&\text{subject to } U(x) \geqq u.\end{aligned} \tag{2}$$

*We wish to thank the National Science Foundation for financial assistance.

[1]See e.g. E. Diewert.

[2]For more details on the expenditure function see the appendix. For earlier presentations see S. Karlin (1959) and L. McKenzie (1957).

[3]We assume local nonsatiability, which converts the inequality constraint into equality. In addition we assume U to be strictly quasi-concave.

Let us denote by $E(q;u)$ the level of income needed at this minimization. By definition, we have

$$E(q;u) = \sum q_i x_i^* = \sum q_i x_i(q;u), \tag{3}$$

where x_i^* represents the optimal level of demand to solve the minimization in eq. (2), and depends functionally on prices and the utility constraint.

The principal properties of the expenditure function are summarized in the appendix; these properties include concavity and homogeneity of degree one in prices. What is extremely useful for later analysis is that the partial derivative of the expenditure function with respect to the ith price is precisely the optimal demand for the ith good. Hence, this price derivative yields the Hicksian or compensated demand curve.[4] A simple argument for this conclusion can be given. Note first that if $x^* = x^*(q;u)$ is the optimal demand vector in eq. (3), then $U(x^*) \geqq u$. Hence, the minimization in eq. (2) implies that for any positive price vector q',

$$q' \cdot x^* - E(q';u) \geqq 0. \tag{4}$$

The expression (4) attains a minimum of zero at $q' = q$, and hence must satisfy the first order condition that its partial derivatives with respect to q', evaluated at q, equal zero; i.e.,

$$x_i^* - \partial E(q;u)/\partial q_i = 0. \tag{5}$$

Thus, the partial derivatives of the expenditure function with respect to prices are indeed the optimal quantities.[5] Of course the derivatives of the demand curves are just the second derivatives of the expenditure function and we can obtain the familiar properties of the Slutsky matrix from the fact that it is the matrix of second derivatives of E,

$$E_i = x_i(q, u),$$
$$E_{ij} = \frac{\partial x_i(q, u)}{\partial q_j}. \tag{6}$$

3. Deadweight burden of taxation

3.1. *Definition*

There are several different definitions of deadweight burden that seem natural to pursue.[6] The question being considered is measuring the loss to the economy

[4]See J.R. Hicks (1939) or P. Samuelson (1947).

[5]This argument is due to W.M. Gorman. We have assumed implicitly that E is differentiable in prices; the first proposition of the appendix verifies that this is always the case.

[6]For a discussion of different measurements see H. Mohring (1971). We shall consider a single consumer representing either a one consumer economy or a many consumer economy which redistributes to maximize an individualistic social welfare function, see P. Samuelson (1956).

from the use of distorting rather than nondistorting taxes. For our purposes we shall define the deadweight burden, or loss, as the excess of the income we must give a consumer to restore him to his pretax indifference curve over the tax revenue collected from him. Let us denote consumer prices by q and producer prices by p, so that taxes t are the difference between them. While it is not clear that this is the most intuitive notion, for consistency we measure the tax revenue for this definition as the level collected at the consumer equilibrium after the consumer has been restored to his original indifference curve. Let us define the compensated tax revenue function, T,

$$T(q, p, u) \equiv \sum(q_i - p_i)E_i(q, u) = \sum t_i E_i(q, u). \tag{7}$$

It is also necessary to select the utility level for this measure. Naturally it is the level of utility achieved by the consumer in the absence of taxation. If there were no lump sum income at the pretax equilibrium, the pretax utility and prices would satisfy

$$E(p, u) = 0. \tag{8}$$

We assume that there are fixed producer prices (and so constant returns to scale) and no other sources of lump sum income, so that eq. (8) holds. (Any

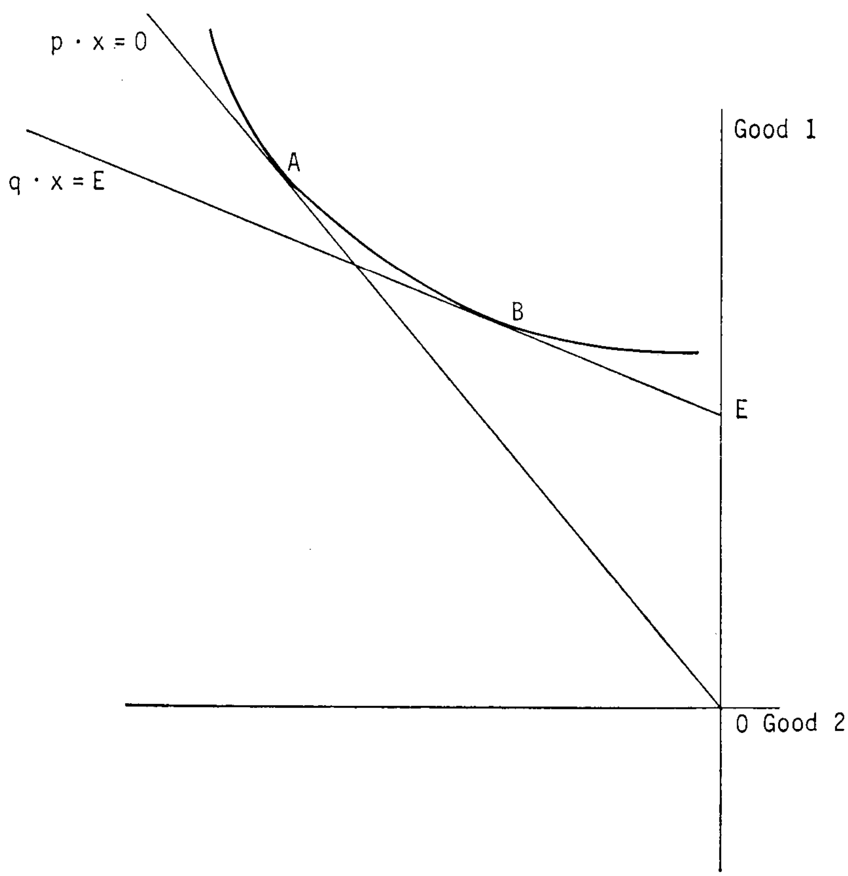

Fig. 1

fixed endowment has been subtracted out to obtain utility in terms of net trades. Thus taxes are on transactions, not consumption.) We can now define the loss function, L, as the difference between income needed at consumer prices q and tax revenue collected,

$$L(q,p,u) = E(q,u) - T(q,p,u). \tag{9}$$

For the two commodity case we can show the loss measurement in a diagram assuming that good one is numeraire. The diagram is in the fourth quadrant representing a demand for good one (a consumer good) and a supply of good two (e.g. labor). Without taxes the consumers budget line passes through the origin (since he has no lump sum income) and leads to the choice of A as the utility maximizing point. With taxation of good two, giving rise to prices q, we must shift the budget line upwards by the amount of the expenditure function to permit the same utility level to be achieved. This is shown in fig. 1 where B represents the compensated post tax consumer equilibrium.

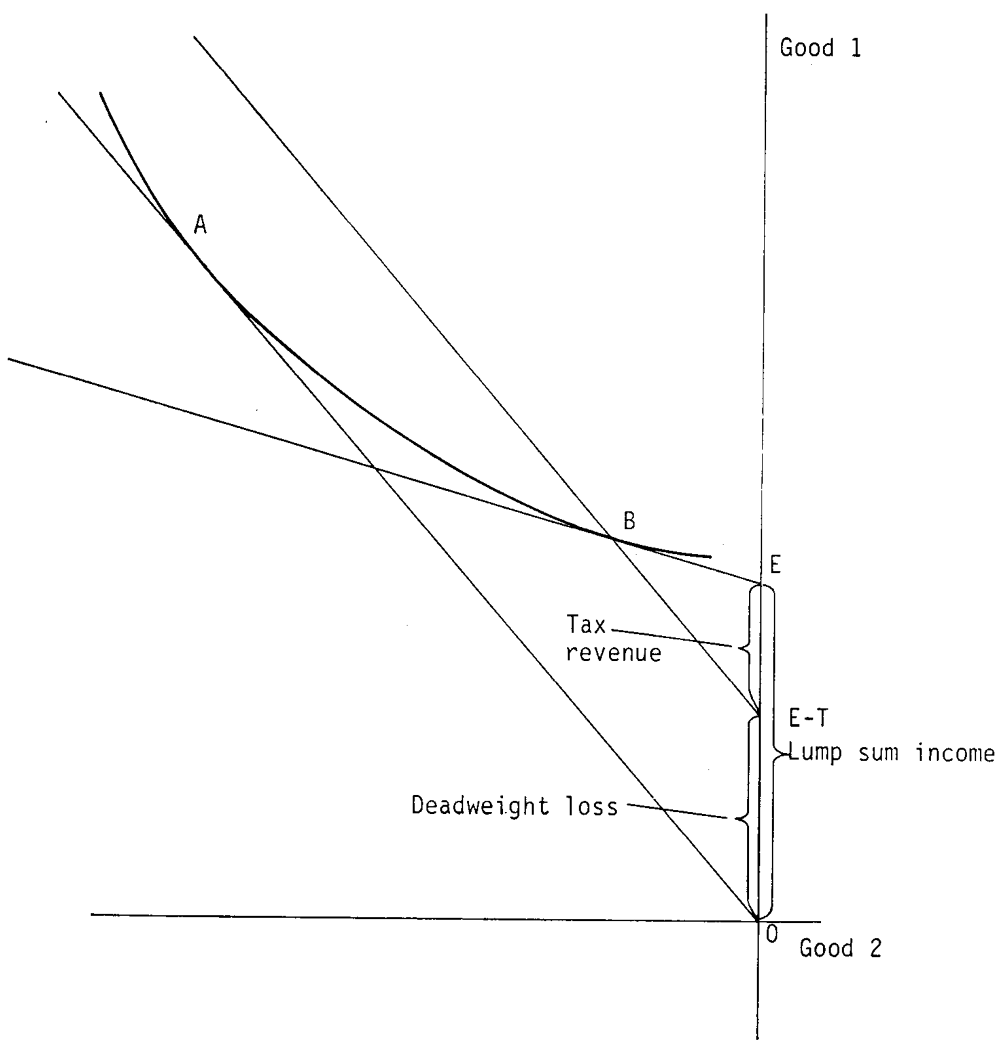

Fig. 2

To determine the division of OE between tax revenue and loss, we draw in the budget line through B which is parallel to the pretax budget line. With good 1 as numeraire, this line has the equation[7] $p \cdot x = E - T$. Thus the loss is the distance from the origin to the intercept of the line through the post tax equilibrium with slope determined by pretax prices.

3.2. Marginal deadweight burden

Given a concept of deadweight burden it is natural to ask how it varies with the tax structure. To answer this question we can differentiate the definition of the loss with respect to a consumer price, holding producer prices constant

$$\frac{\partial L(q, p, u)}{\partial q_k} = E_k - \frac{\partial T}{\partial q_k} = -\sum (q_i - p_i) E_{i_k}(q, u) = -\sum t_i E_{i_k}. \tag{10}$$

It is natural also to relate the marginal loss to the marginal tax revenue

$$\frac{\partial L / \partial q_k}{\partial T / \partial q_k} = \frac{-\sum t_i E_{i_k}}{E_k + \sum t_i E_{i_k}}. \tag{11}$$

(This expression will return below when we consider the optimal tax, which requires constancy in this ratio for all taxed goods.) Given the definition of marginal deadweight burden we can, naturally, integrate back to obtain the total loss, integrating from p to q. By suitable choice of the approximation to this exact expression we can then obtain the deadweight loss measure of A. Harberger (1964), interpreting his analysis as using derivatives of compensated demands. To perform the integration let us integrate from p_i to q_i for each commodity in succession (i.e., move out parallel to the axes successively). When integrating with respect to the ith tax the $i-1$ prices that come earlier are at their q levels while the $n-i$ prices that come later are at the p levels:

$$L(p, q, u) = -\sum_{i=1}^{n} \int_{p_i}^{q_i} \{ \sum_{j=1}^{i-1} (q_j - p_j)\, E_{ji}(q_1, q_2, \ldots, q_{i-1}, s, p_{i+1}, \ldots, p_n) + (s - p_i) E_{ii}(q_1, q_2, \ldots, q_{i-1}, s, p_{i+1}, \ldots, p_n) \}\, \mathrm{d}s. \tag{12}$$

Let us approximate[8] these integrals by choosing one price vector $\hat{q}$ and evaluating all the derivatives of E at this single price vector rather along the path from p to q. With this approximation we have

[7]At the point B we have $q \cdot x = E$ and $p \cdot x = E - T$. Thus $tx = T$.

[8]While in each integral we can use some intermediate value of E_{ji} to preserve the value of the integral, we cannot generally use the value obtained at the same point in all the integrals we are evaluating.

$$L(p,q,u) = -\sum_{i=1}^{n}\int_{p_i}^{q_i}\{\sum_{j=1}^{i-1}(q_j-p_j)E_{ji}+(s-p_i)E_{ii}\}\,\mathrm{d}s$$
$$= -\sum_{i=1}^{n}\{\sum_{j=1}^{i-1}(q_i-p_i)(q_j-p_j)E_{ji}+\tfrac{1}{2}(q_i-p_i)^2E_{ii}\} \tag{13}$$
$$= -\tfrac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}t_it_jE_{ij},$$

where the last two steps follow first by integrating and then, recognizing that $E_{ij} = E_{ji}$, by realizing that each cross term appears once so that we get $\frac{1}{2}$ the number of cross terms in the full sum in the final expression. This is precisely the expression obtained by Harberger (1964).

3.3. Example

Let us consider the Cobb–Douglas two-good case:

$$u = x_1^{\alpha}(x_2+A)^{1-\alpha};\quad x_1 > 0,\quad 0<\alpha<1,\quad -A<x_2<0. \tag{14}$$

This gives Marshallian demands

$$x_1 = \alpha q_1^{-1}(q_2A+I), \tag{15}$$
$$x_2 = -A+(1-\alpha)q_2^{-1}(q_2A+I),$$

and substituting from demands in the utility function, the indirect utility function

$$u = \alpha^{\alpha}q_1^{-\alpha}(1-\alpha)^{1-\alpha}q_2^{\alpha-1}(q_2A+I). \tag{16}$$

Inverting, i.e., solving for I, we have the expenditure function

$$E = -q_2A+u\alpha^{-\alpha}(1-\alpha)^{\alpha-1}q_1^{\alpha}q_2^{1-\alpha} = -q_2A+q_1^{\alpha}q_2^{1-\alpha}B, \tag{17}$$

which serves as the definition of B. Differentiating E we have

$$x_1 = \alpha Bq_1^{\alpha-1}q_2^{1-\alpha},\qquad x_2 = -A+(1-\alpha)Bq_1^{\alpha}q_2^{-\alpha}. \tag{18}$$

If we choose B such that $E(p,u) = 0$, then

$$B = Ap_1^{-\alpha}p_2^{\alpha}. \tag{19}$$

Now consider a tax on good one

$$L(q,p,u) = -q_2A+q_1^{\alpha}q_2^{1-\alpha}p_1^{-\alpha}p_2^{\alpha}A-\alpha t_1Ap_1^{-\alpha}p_2^{\alpha}q_1^{\alpha-1}q_2^{1-\alpha} \tag{20}$$
$$= p_2A\left[-1+\left(\frac{p_1+t_1}{p_1}\right)^{\alpha}-\alpha\frac{t_1}{p_1}\left(\frac{p_1+t_1}{p_1}\right)^{\alpha-1}\right].$$

Let $\tau = t_1/p_1$.

Turning to marginal loss, we can differentiate eq. (20)

$$\begin{aligned}\frac{\partial L}{\partial t_1} &= p_2 A\left[\frac{\alpha}{p_1}\left(\frac{p_1+t_1}{p_1}\right)^{\alpha-1} - \frac{\alpha}{p_1}\left(\frac{p_1+t_1}{p_1}\right)^{\alpha-1}\right. \\ &\quad \left. + \frac{\alpha(1-\alpha)}{p_1}\frac{t_1}{p_1}\left(\frac{p_1+t_1}{p_1}\right)^{\alpha-2}\right] \\ &= p_2 A \frac{\alpha}{p_1}(1-\alpha)\frac{t_1}{p_1}\left(\frac{p_1+t_1}{p_1}\right)^{\alpha-2} \\ &= p_1^{-1} p_2 A\, \alpha(1-\alpha)\tau(1+\tau)^{\alpha-2}. \end{aligned} \tag{21}$$

3.4. Varying prices

To consider the case of varying prices, let us return to the diagrammatic treatment. In an economy with fixed producer prices, the production frontier is a straight line. Thus we can interpret the diagram as describing the actual technology rather than merely the budget constraint of the consumer. With this interpretation, the provision of loss from outside the economy to restore the consumer to his original utility level must be in real resources. With a linear technology it does not matter which commodity is used for compensation. With a nonlinear technology it matters which good is numeraire, i.e. is the good used for compensation. Let us assume that the consumer good (good one) is used for this purpose. Then the production frontier shifts up by the amount of compensation. We can now construct the analogues to figs. 1 and 2, showing the pre-tax equilibrium A and the post-tax equilibrium B.

In fig. 3 we have equilibrium without any tax at A where an indifference curve is tangent to the production frontier. The line through A tangent to both curves represents both the budget line of the consumer and the maximal isoprofit line which can be reached by the firm. Thus the height of its intercept represents the level of profits measured in units of good one. In fig. 4 we have the equilibrium after-tax at B. The production function has been shifted vertically by the amount of compensation. The line tangent to the production frontier at B represents the maximal isoprofit line (measured in producer prices) which can be reached by the firm. Thus the height of its intercept above the origin for the shifted production frontier, 0′, is the level of profits measured in units of good one. The line tangent to the indifference curve through B is the budget line of the consumer. Thus the height of its intercept is the level of lump sum income which the consumer has. This equals the value of the expenditure function at these prices.

To set this up algebraically, let us introduce the profit function,[9] $\pi(p)$, giving the level of maximal profits available to a price taking firm when facing prices p.

[9]For details see D. McFadden.

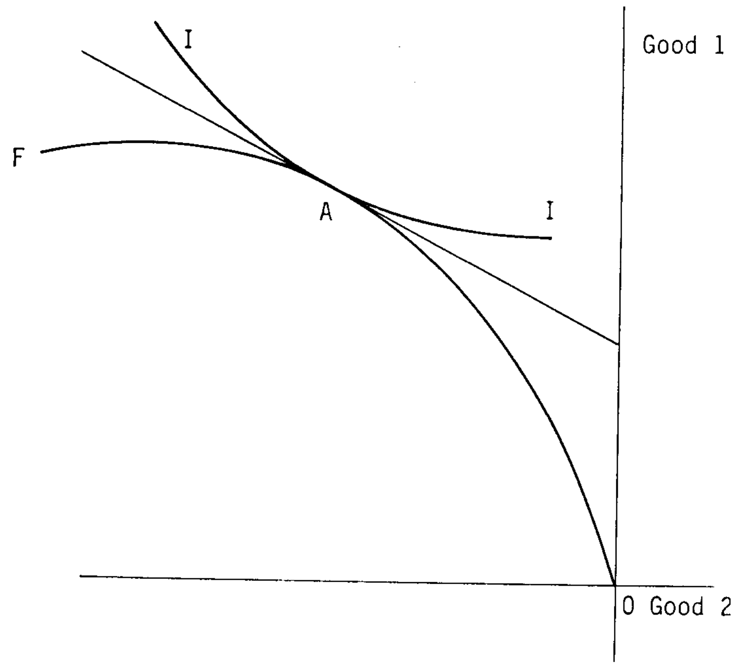
I
Good 1
F
A
I
0 Good 2

Fig. 3

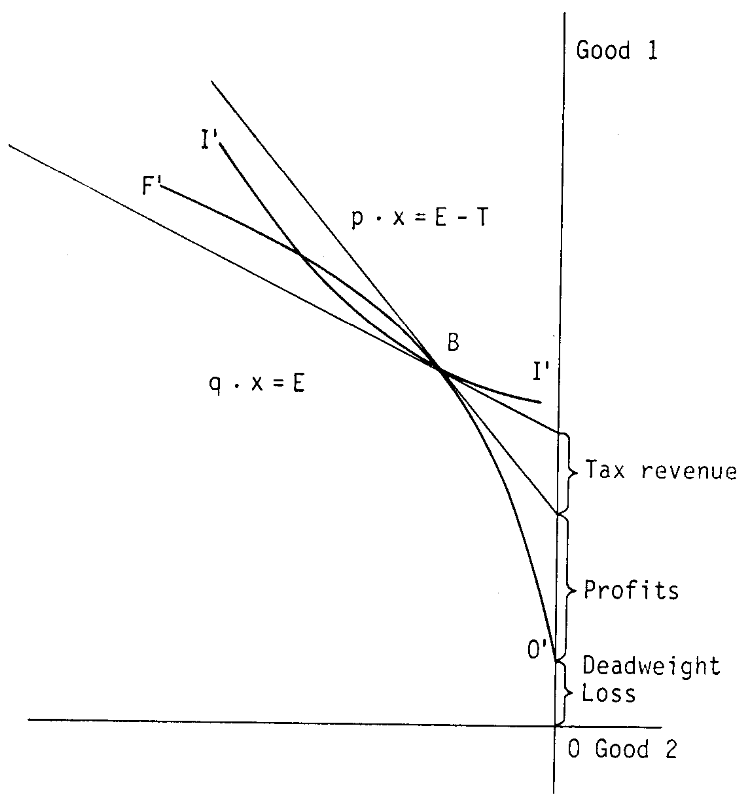
Good 1
I'
F'
p · x = E - T
B
I'
q · x = E
Tax revenue
Profits
0'
Deadweight
Loss
0 Good 2

Fig. 4

The loss is now the value of the expenditure function less the sum of tax revenue and production profit. (As before, the utility level u is the no tax equilibrium level, but now results in an expenditure function level at producer prices equal to profits at the no tax equilibrium.)

$$L(p, q, u) = E(q, u) - T(q, p, u) - \pi(p). \tag{22}$$

Before determining the marginal loss we must relate producer prices at equilibrium to the tax structure which is chosen. To obtain this relationship we must set up the market clearance equations. As with the expenditure function, the derivatives of the profit function are the quantities supplied by the firm. For goods 2 through n, market clearance requires equality of compensated demand and supply,

$$E_i(q, u) = \pi_i(p); \qquad i = 2, 3, \ldots, n. \tag{23}$$

With good one, however, market clearance includes the supply of compensation coming from outside the economy,

$$E_1(q, u) = \pi_1(p) + L. \tag{24}$$

It is a good exercise in the algebra (particularly the homogeneity) of expenditure and profit functions to show that the two expressions for loss, eq. (22) and eq. (24), are in fact equivalent. We can now write the marginal loss by differentiating eq. (22), substituting $p+t$ for q and determining p as a function of t by the $n-1$ equations (23) and the choice of numeraire, $p_1 = 1$:

$$L(t, u) = E[p(t)+t, u] - \sum t_i E_i[p(t)+t, u] - \pi[p(t)]. \tag{25}$$

Differentiating this expression we have

$$\begin{aligned} \frac{\partial L}{\partial t_k} &= -\sum_i t_i E_{ik} + \sum_i (E_i - \pi_i) \frac{\partial p_i}{\partial t_k} - \sum_i t_i \sum_j E_{ij} \frac{\partial p_i}{\partial t_k} \qquad (26) \\ &= -\sum t_i \left(E_{ik} + \sum_j E_{ij} \frac{\partial p_j}{\partial t_k} \right), \end{aligned}$$

with the last step making use of market clearance and the choice of numeraire. Rather than getting lost in the maze of the determination of the equilibrium price vector, let us relate the loss derivative to the change in quantities in the full equilibrium. Since the equilibrium quantities are just the expenditure function derivatives evaluated at the equilibrium prices, we have

$$\frac{\partial x_i}{\partial t_k} = E_{ik} + \sum_j E_{ij} \frac{\partial p_j}{\partial t_k}. \tag{27}$$

Thus we can express the loss as the change in tax revenue arising from the alteration of compensated equilibrium quantities in response to the relative price changes

$$\frac{\partial L}{\partial t_k} = -\sum t_i \frac{\partial x_i}{\partial t_k}. \tag{28}$$

This equation also appears in Harberger (1964).

4. Taxes and welfare

4.1. Optimal commodity taxes

Having set up a loss function, it is natural to minimize it subject to the constraint of raising a given amount of revenue. (We again assume that producer prices are fixed.)

$$\begin{aligned} &\text{Minimize } L(q, p, u) \\ &\text{subject to } T(q, p, u) = \text{constant}. \end{aligned} \tag{29}$$

For this solution to reflect a possible equilibrium in the economy, we would have to choose the appropriate utility level u. If the government were keeping the revenue for expenditures and if there were no lump sum income in the economy, the appropriate level of u would coincide with a zero level for the expenditure function at the gross of tax prices q for this would coincide with the equilibrium utility level. If $E(q, u)$ is zero, then $\sum q_i E_i(q, u)$ is also zero. Thus the revenue needs of the economy cannot be satisfied by distortion free proportional taxation on all commodities, since no revenue would be gained. Thus the calculus solution will be the second best solution not the first best.[10] With a Lagrange multiplier, λ, the first order conditions are

$$\frac{\partial L}{\partial q_k} = \lambda \frac{\partial T}{\partial q_k}; \qquad k = 1, 2, \ldots, n. \tag{30}$$

This expression has already been calculated above, eq. (11), giving the first order conditions

$$\frac{-\sum t_i E_{ik}}{E_k + \sum t_i E_{ik}} = \lambda; \qquad k = 1, 2, \ldots, n. \tag{31}$$

This coincides with the conditions in the optimal tax literature. See, e.g., W. Baumol and D. Bradford (1970) or P. Diamond and J. Mirrlees (1971).

4.2. Movement from proportional taxation

In addition to considering the determination of the full optimum, it is useful

[10]Since the equilibrium quantities are homogeneous of degree zero in consumer prices, an alternative approach would be to normalize, e.g. $q_1 = p_1$. However the value of λ depends on the choice of numeraire, see A. Atkinson and N. Stern.

to consider piecemeal welfare economics, evaluating the change in loss resulting from a change in taxes starting from an arbitrary position and continuing to raise the same revenue. One example of such analysis was the proof by W.C. Corlett and D.C. Hague (1953) that a movement away from proportional consumer good taxation (or equivalently away from an income tax) increased utility when it increased the tax on the good with the lower compensated cross elasticity with the wage (in a two consumer good, one type of labor model). To consider this proposition in a loss setting, we wish to consider the change in loss from an increase in q_1 adjusted by a decrease in q_2 which keeps T constant, assuming $q_3 = p_3$. Taking this derivative [and using eq. (10)] we have

$$\begin{aligned}\left(\frac{\mathrm{d}L}{\mathrm{d}q_1}\right)_T &= \frac{\partial L}{\partial q_1}+\left(\frac{\mathrm{d}q_2}{\mathrm{d}q_1}\right)_T\frac{\partial L}{\partial q_2} \\ &= \frac{\partial L}{\partial q_1}-\frac{\partial T/\partial q_1}{\partial T/\partial q_2}\frac{\partial L}{\partial q_2} \\ &= \left(\frac{\partial T}{\partial q_2}\right)^{-1}\left((\textstyle\sum t_iE_{i2})(E_1+\sum t_iE_{i1})-(\sum t_iE_{i1})(E_2+\sum t_iE_{i2})\right) \\ &= \left(\frac{\partial T}{\partial q_2}\right)^{-1}(E_1\textstyle\sum t_iE_{i2}-E_2\sum t_iE_{i1}).\end{aligned} \tag{32}$$

Assuming that tax revenue increases with a tax on the second good, that there are three goods, and that we are evaluating from a position of proportional taxation

$$t_1 = \tau q_1, \quad t_2 = \tau q_2, \quad t_3 = 0.$$

The change in loss has the same sign as the second term in parentheses, which we can evaluate:

$$\begin{aligned}E_1\textstyle\sum t_iE_{i2}-E_2\sum t_iE_{i1} &= \tau(E_1q_1E_{12}+E_1q_2E_{22}-E_2q_1E_{11} \\ &\quad -E_2q_2E_{21}) \\ &= \tau(E_2q_3E_{31}-E_1q_3E_{32}).\end{aligned} \tag{33}$$

The last step follows from the homogeneity of degree zero of E_i in prices (recalling the symmetry of the Slutsky matrix), i.e.

$$\sum_j E_{ij}q_j = 0. \tag{34}$$

To complete the Corlett–Hague analysis we merely convert the demand derivatives into elasticities

$$\left(\frac{\mathrm{d}L}{\mathrm{d}q_1}\right)_T = \left(\frac{\partial T}{\partial q_2}\right)^{-1}\tau E_1E_2\left(\frac{q_3E_{13}}{E_1}-\frac{q_3E_{23}}{E_2}\right). \tag{35}$$

Thus the loss goes down with an increase in the tax on the good with the lower compensated elasticity with the wage.[11]

5. Investment criteria

5.1. Area to the left of the demand curve

A standard procedure in considering lumpy investments is to compare the cost of the investment with the area under the compensated demand curve. We can use the expenditure function to obtain a simple justification for this familiar procedure. We shall obtain an expression, in terms of the expenditure function, which measures exactly whether the investment increases utility. By differentiation, this can be expressed in terms of the area to the left of the demand curve. Then, using integration by parts, we shall convert the expression into one reflecting the area below the demand curves. The section concludes with an example.

For the next subsection it is convenient to use a numeraire, so we introduce it at the start.

Suppose the first commodity is numeraire, and let q and x denote prices and quantities, respectively, of the remaining commodities. Also, let z denote a quantity of the numeraire commodity; and I, income. Assume that without the project, the economy has an equilibrium with prices q^A and quantities z^A, x^A. Associated with this equilibrium is an income $I^A = z^A + q^A \cdot x^A$ and utility level $u^A = U(z^A, x^A)$. Being the equilibrium level, this utility level also satisfies

$$E(1, q^A, u^A) = I^A. \tag{36}$$

Denote equilibrium values after construction of the project by q^B, z^B, x^B, I^B, u^B. Tautologically, the project is worth undertaking if and only if $u^B \geqq u^A$. The cost of the project, net of receipts on the project and of the change in profits on other projects arising because of this construction, is $I^A - I^B$. This is to be compared to an evaluation of benefits in numeraire units.

Since the expenditure function is monotone increasing in utility, one has for any price vector q that $u^B \geqq u^A$ if and only if $E(1, q, u^B) \geqq E(1, q, u^A)$. Hence the project is worth undertaking if and only if the following expression, which we term surplus, S, is non-negative:[12]

$$\begin{aligned} S &= E(1, q^B, u^B) - E(1, q^B, u^A) \\ &= E(1, q^B, u^B) - E(1, q^A, u^A) + E(1, q^A, u^A) - E(1, q^B, u^A) \\ &= E(1, q^A, u^A) - E(1, q^B, u^A) - (I^A - I^B). \end{aligned} \tag{37}$$

[11]A similar proposition holds for the position at the optimum defined by eq. (34). At the optimum in a three good model the good with the lower compensated elasticity with the wage is taxed more heavily. See Meade (1955) or Diamond and Mirrlees (1971).

[12]The selection of q^B as the price vector at which the expenditure comparison is made is not fundamental. Choice of q^A or another alternative leads to an analogous expression for surplus S. The magnitude of S is not invariant with respect to this choice of q, but its sign is invariant and gives an unambiguous criterion for the worth of the project.

Thus, the project is worthwhile when the change in the expenditure function (evaluated at the original utility level) induced by the change in prices from constructing the project exceeds the cost of the project. Recalling that $x = X(q, u) \equiv E_q(1, q, u)$ is the compensated demand for the x commodities, eq. (41) can be written

$$S = \int_{q^B}^{q^A} X(q, u^A) \cdot \mathrm{d}q - (I^A - I^B). \tag{38}$$

When there is a single commodity in addition to the numeraire, or if other prices do not change, the integral in this expression is the area shown in fig. 5 assuming that good n is being produced by the project. More generally with many commodities, the integral is a line integral evaluated on any path of

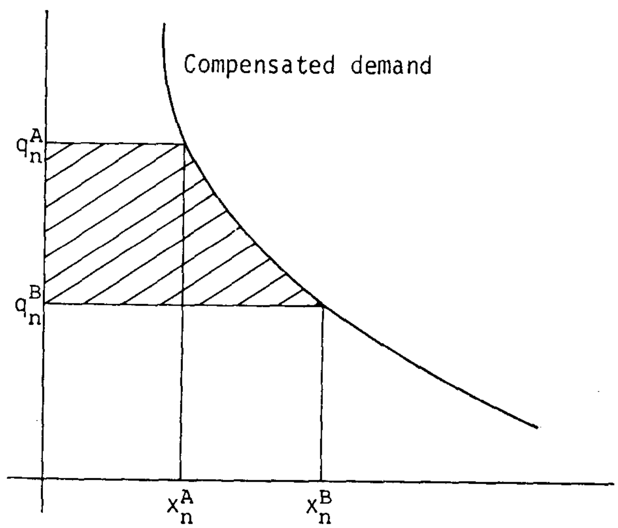

Fig. 5. Horizontal axis: quantity; vertical axis: price.

prices leading from q^B to q^A. In particular, one can choose the path in which components of q are changed successively from their values in q^B to their values in q^A, and the integral equals the sum over non-numeraire commodities of areas such as in fig. 5 with the appropriate intermediate price endpoints. Surplus evaluated in this manner is independent of the path of integration; for the specific path above, independent of the order in which the areas in the sum are calculated.

5.2. Area below the demand curve

A more familiar form of the investment criterion (37) can be obtained by using integration by parts. For this we need to introduce the inverse of the demand curves. When the utility function is continuously differentiable and strictly quasi-concave in (z, x), the expenditure function $E(1, q, u)$ is continuously differentiable and strictly concave in q, and the system of demand equations

$x = E_q(1, q, u)$ has a unique inverse $q = Q(x, u)$ giving the prices at which x will be chosen. If we let $z = Z(x, u)$ denote an indifference curve, satisfying $u \equiv U[Z(x, u), x]$, then it follows from the first-order conditions for utility maximization and differentiation of this identity that $-Z_x(x, u) = Q(x, u)$.

It is a straightforward calculus exercise to verify the following identity for these functions, which in the case of a single non-numeraire commodity reduces to the ordinary formula for integration-by-parts:

$$\int_{q^B}^{q^A} X(q, u^A)\cdot \mathrm{d}q = q^A\cdot x^A - q^B\cdot x^B - \int_{q^B}^{q^A} Q(x, u^A)\cdot \mathrm{d}x. \tag{39}$$

Substituting this expression in eq. (38) yields

$$S = \int_{x^A}^{x^B} Q(x, u^A)\cdot \mathrm{d}x - (I^A - q^A\cdot x^A - I^B + q^B\cdot x^B)$$

$$= \int_{x^A}^{x^B} Q(x, u^A)\, \mathrm{d}x - (z^A - z^B), \tag{40}$$

$$= z^B - Z(x^B, u^A). \tag{41}$$

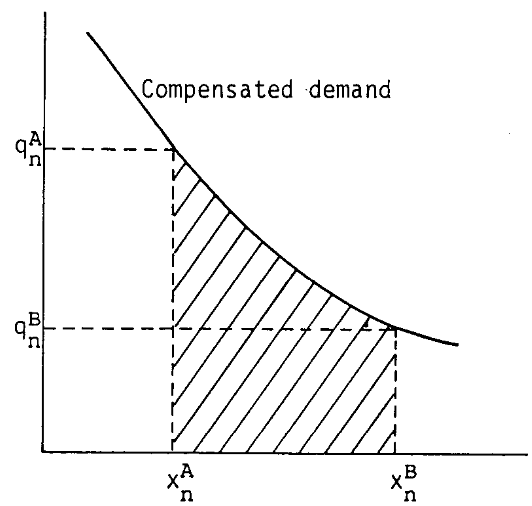

Fig. 6. Horizontal axis: quantity; vertical axis: price.

From the last line of the equality (41) we have the obvious condition that the project is worth undertaking if the quantity of numeraire good in equilibrium with the project exceeds the level necessary for achieving the utility level without the project, given the quantities of other goods produced in equilibrium with the project. From eq. (40) we can state the investment criterion as a comparison of the area under the compensated inverse demand curve with the cost of the project measured in terms of net numeraire units foregone. In the case of a single non-numeraire commodity or a project small enough to leave all other non-numeraire quantities constant, the integral in eq. (41) gives the area shown in fig. 6. With labor as numeraire, $z^A - z^B$ then equals the change in product cost between the two equilibria. In the multiple-commodity case, the integral in eq. (41) is again a line integral, independent of path, to be evaluated on any path from x^A to x^B. A further division of project cost into fixed and marginal costs would allow eq. (40) to be written as the area between the inverse demand function and the marginal cost curve for any path from x^A to x^B, less the fixed cost of the project in numeraire units.

5.3. Example

Consider an economy with a single consumer with utility function

$$u = x^{\frac{1}{2}}(24+z)^{\frac{1}{2}}. \tag{42}$$

Assume there are two production processes for converting good two (z) to good one (x)

$$x+z = 0; \qquad z \leqq 0, \tag{43}$$

and

$$x+a(z+4) = 0; \qquad z \leqq -4, \qquad a > 1. \tag{44}$$

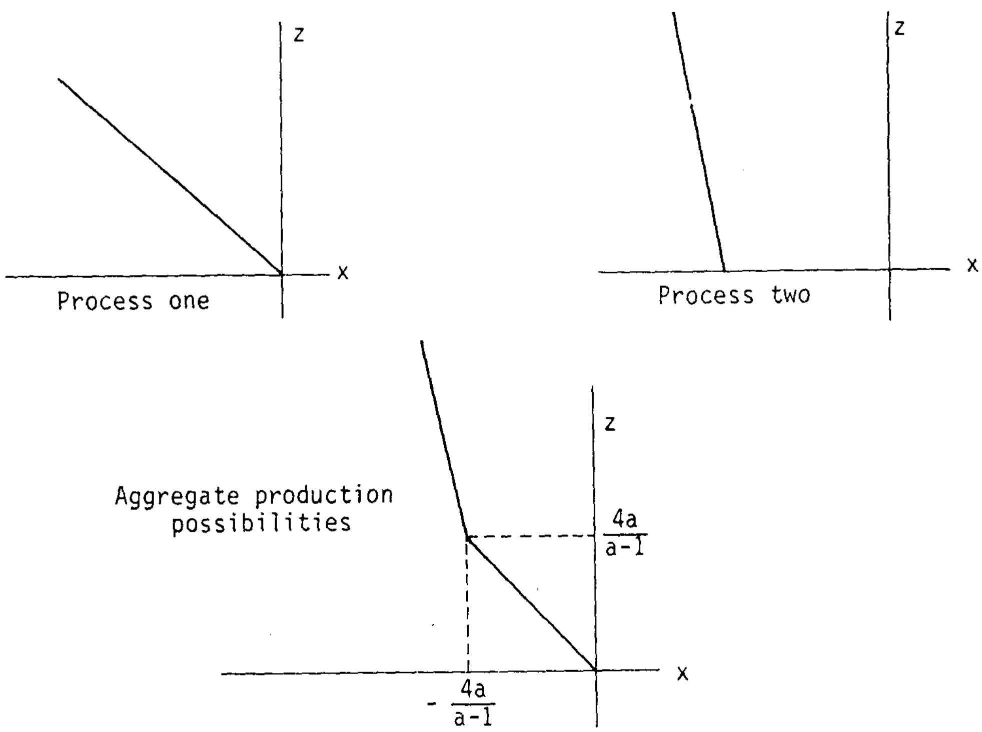

Fig. 7

From the example above, we have the expenditure function [see eq. (17)]

$$E(q_1, q_2, u) = 2uq_1^{\frac{1}{2}}q_2^{\frac{1}{2}} - 24q_2. \tag{45}$$

If just process one is available the optimum occurs at

$$x = 12, \quad z = -12, \quad u = 12, \tag{46}$$

and

$$E(1, 1, 12) = 0. \tag{47}$$

With good two as numeraire assuming undertaking of the project to provide process two, the equilibrium price satisfies $q_1 = a^{-1}$, and the profit is

$$\pi = q_1 x + q_2 z = a^{-\frac{1}{2}}(-az - 4a) + z = -4. \tag{48}$$

Evaluating the expenditure function at these prices

$$E(a^{-1}, 1, 12) = 24(a^{-\frac{1}{2}} - 1). \tag{49}$$

Thus

$$E(a^{-1}, 1, 12)(\lesseqgtr) - 4, \text{ as } a^{\frac{1}{2}}(\lesseqgtr)1.2. \tag{50}$$

Thus process two should be adopted for $a^{\frac{1}{2}} \geqq 1.2$.

Let us check this by calculating utility if just process 2 is available. Because the utility function is Cobb–Douglas we know that we want $x > 0$. Substituting we can express the maximization as

$$\max_x (x)^{\frac{1}{2}}(24 - 4 - a^{-1}x)^{\frac{1}{2}}. \tag{51}$$

For this the FOC is

$$\tfrac{1}{2}x^{-1} - \tfrac{1}{2}a^{-1}(24 - 4 - a^{-1}x)^{-1} = 0, \tag{52}$$

$$x = 10a, \tag{53}$$

$$u = (10a)^{\frac{1}{2}}(10)^{\frac{1}{2}} = 10a^{\frac{1}{2}}.$$

Thus utility with just process two exceeds that with just process one if $a^{\frac{1}{2}} > 1.2$.

Appendix

Properties of the expenditure function

The expenditure function, defined in eq. (2) by

$$E(q;u) = \text{Min}\, q \cdot x \text{ subject to } U(x) \geqq u, \tag{A1}$$

provides an indirect representation of preferences which is a useful starting point for many applications of demand theory. The following propositions summarize its properties.

Proposition 1. If the utility function U is continuous, strictly quasi-concave, and locally non-satiated, then the expenditure function E is

(1) homogeneous of degree one, concave, and continuously differentiable in positive prices;

(2) strictly increasing in u; and

(3) continuous jointly in prices and u.

The partial derivative of E with respect to the ith price equals the compensated demand function for the ith commodity.

Proofs of Proposition 1 can be found in McFadden (forthcoming), Rockafellar (1970), and Shephard (1970). The demonstration that the expenditure function is homogeneous of degree one and concave in prices is simple and illustrates the spirit of the analysis, and hence is reproduced here: The homogeneity property is a consequence of the fact that a proportional change in prices leaves the opportunity set, hence the optimal commodity vector, unchanged. To show concavity, suppose q and q' are two price vectors, and $q^* = \theta q+(1-\theta)q'$ for some $0 < \theta < 1$. Let x^* be the expenditure-minimizing commodity vector at the price vector q^*. Since $U(x^*) \geqq u$, cost minimization implies $q\cdot x^* \geqq E(q, u)$ and $q'\cdot x^* \geqq E(q';u)$. Hence,

$$E(\theta q+(1-\theta)q', u) = q^*\cdot x^* = \theta q\cdot x^*+(1-\theta)q'\cdot x^*$$

$$\geqq \theta E(q, u)+(1-\theta)E(q', u),$$

the defining inequality for concavity.

An elementary derivation of the derivative property of the expenditure function which also establishes the existence of the derivative is given in McFadden (forthcoming). Arguments drawing more heavily on mathematical properties have been given by Rockafellar (1970) and Shephard (1970). The monotonicity and joint continuity properties (2) and (3) are established in McFadden (forthcoming).

Proposition 2. If a function E satisfies properties (1)–(3) in Proposition 1, then there exists a continuous, strictly quasi-concave, locally non-satiated utility function U such that E equals the expenditure function derived from U. The function U satisfies

$$U(x) = \text{Max}\{u|q'x \geqq E(q';u) \text{ for all positive } q'\}. \tag{A2}$$

Proposition 2 is called the Shephard–Uzawa duality theorem, and is proved in the three references above. The significance of this proposition lies in the fact that the previously established properties of expenditure functions completely characterize such functions. In applications, this implies that any function E satisfying (1)–(3) must necessarily be derivable from some utility function U. Consequently, the applied analyst can proceed to work with this 'expenditure' function without explicitly displaying the underlying utility function.

The *bordered Hessian* of a utility function U is the matrix

$$\begin{bmatrix} 0 & \partial U/\partial x_1 & \dots & \partial U/\partial x_n \\ \partial U/\partial x_1 & \partial^2 U/\partial x_1^2 & \dots & \partial^2 U/\partial x_1 \partial x_n \\ \vdots & \vdots & \ddots & \vdots \\ \partial U/\partial x_n & \partial^2 U/\partial x_n \partial x_1 & \dots & \partial^2 U/\partial x_n^2 \end{bmatrix}. \tag{A3}$$

The *Hessian* of an expenditure function E is the matrix

$$\begin{bmatrix} \partial^2 E/\partial q_1^2 & \cdots & \partial^2 E/\partial q_1 \partial q_n \\ \vdots & \ddots & \vdots \\ \partial^2 E/\partial q_n \partial q_1 & \cdots & \partial^2 E/\partial q_n^2 \end{bmatrix}. \tag{A4}$$

Since the first partial derivatives of E with respect to prices are the compensated demand functions, the Hessian matrix of E is the Slutsky or substitution matrix of cross-price effects.

Proposition 3. If U is continuous, strictly quasi-concave, and locally non-satiated, then the Hessian matrix of E exists, and is symmetric and negative semidefinite, for almost all positive price vectors.[13] If U is, further, continuously differentiable, then E is strictly quasi-concave in prices. If U is, further, twice continuously differentiable with a non-singular bordered Hessian, then E is twice continuously differentiable and the Hessian matrix of E is of rank $n-1$ for all positive prices.

This proposition is proved in McFadden (forthcoming) and proved in a slightly different context by Samuelson (1947). It provides the basis for neoclassical comparative statics analysis, with the non-singularity of the bordered Hessian matrix of U guaranteeing the existence of implicitly defined derivatives of compensated demand.

As indicated in the text, the expenditure function $I = E(q;u)$ can be inverted to give the indirect utility function $u = E^{-1}(q;I) \equiv V(q/I)$. The identity $I \equiv E[q, V(q/I)]$ and the derivative property of the expenditure function can be used to derive the classical relationship between market demand functions and the derivatives of the indirect utility function (the Roy identity):

$$x_i(q, I) = -(\partial V/\partial q_i)/(\partial V/\partial I).$$

[13]This conclusion requires *only* that a utility-maximizing vector exist for each positive income and price vector. In particular, concavity and continuity of U are *not* required for this result.

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