STRUCTURAL MODELS CONTAINING A DUMMY ENDOGENOUS VARIABLE

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y endogenous vector,
w covariates, unobservables, parameters
Structural model: $y = H(y, w)$.

A model is defined to be coherent if, for each $w \in \Omega$ there exists a corresponding unique value for y that satisfies the model.

Denote the unique y that corresponds to each w by the reduced form equation $y = G(w)$, which must satisfy $G(w) = H[G(w), w]$.

Example $y = (y_1, y_2)$, $w = (\alpha, e_1, e_2)$

$$\begin{aligned} y_1 &= I(y_2 + e_1 \geq 0) \\ y_2 &= \alpha y_1 + e_2 \end{aligned}$$

Then

$$y_1 = I(\alpha y_1 + e_1 + e_2 \geq 0)$$

So $y_1 = 0$, $y_2 = e_2$ if

$$0 = I(e_1 + e_2 \geq 0), \quad \text{so } e_1 + e_2 < 0$$

and $y_1 = 1$, $y_2 = a + e_2$ if

$$1 = I(\alpha + e_1 + e_2 \geq 0), \quad \text{so } \alpha + e_1 + e_2 \geq 0$$

Both $y_1 = 0$ and $y_1 = 1$ are solutions if $-a \leq e_1 + e_2 < 0$.

Neither $y_1 = 0$ nor $y_1 = 1$ will satisfy this model if $0 \leq e_1 + e_2 < -a$.

This model is incoherent unless $e_1 + e_2$ is constrained not to lie between zero and $-a$.

Heckman (1978)

Gourieroux, Laffont, and Monfort (1980)

Blundell and Smith (1994)

Dagenais (1997)

Bresnahan and Reiss (1991)

Tamer (2003)

Aradillas-Lopez (2005)

Let $y = (y_1, y_2)$,
y_1 is a dummy endogenous
Provide necessary and sufficient conditions for coherence of

$$\begin{aligned} &(1) \qquad y_1 = H_1(y_1, y_2, w) \\ &(2) \qquad y_2 = H_2(y_1, y_2, w) \end{aligned}$$

for arbitrary functions H_1 and H_2, where H_1 can only equal zero or one.

Examples:
discrete endogenous regressor models,
regime shift models,
treatment response models,
sample selection models,
joint continuous-discrete demands,
simultaneous discrete choice models

$$
\begin{aligned}
&(1) \qquad y_1 = H_1(y_1, y_2, w) \\
&(2) \qquad y_2 = H_2(y_1, y_2, w)
\end{aligned}
$$

Theorem 1: Assume $y_1 \in \{0, 1\}$. The system of equations (1) and (2) is coherent iff for some g

$$
\begin{aligned}
&(3) \qquad H_1[1, g(1, w), w] = H_1[0, g(0, w), w] \\
&(4) \qquad y_2 = g(y_1, w)
\end{aligned}
$$

To prove, solve (2) to get (4), substitute (4) into (1), and show incoherent whenever $H_1[y_1, g(y_1, w), w]$ is not the same for both values of y_1. Required nondependence of this expression on y_1 shows severity of coherence with a dummy endogenous variable.

Corollary 1: The general endogenous selection model, in which y_1 indexes whether y_2 is observed,

$$
\begin{aligned}
y_1 &= R(y_2, w) \\
y_2 &= r(w)y_1
\end{aligned}
$$

is coherent iff R is independent of y_2.

Proof: By (3) coherency requires $R[r(w), w] = R(0, w)$, so $R(y_2, w) = R(0, w)$.

No cohorent selection model can be endogenous, where endogeneity is defined as having the selection criterion y_1 depend on the observed outcome y_2.

Coherence is possible using some other notion of endogeneity, such as having y_1 depend on the latent outcome $r(w)$.

Replace (1) and (2) with

$$(5) \qquad y_1 = I[h(y_1, y_2, w) + e_1 \geq 0]$$

$$(2) \qquad y_2 = H_2(y_1, y_2, w)$$

for some function h, where $e_1 \in w$.
Define $s_y(w) = h[y, g(y, w), w]$.

Theorem 2. The system (5) and (2) is coherent iff $y_2 = g(y_1, w)$ and either $s_0(w) = s_1(w)$, or $e_1 \notin$ interval $[-s_0(w), -s_1(w)]$.

$s_0(w) = s_1(w)$ holds iff for some f

$$(6) \; y_1 = I[f[y_2 + [g(0, w) - g(1, w)]y_1, w] + e_1 \geq$$

$$y_2 = g(y_1, w)$$

$s_0(w) = s_1(w)$ holds iff for some ϕ and some binary $d(w)$

$$(7) \qquad y_1 = I[\phi[(1 - d(w))y_2, w] + e_1 \geq 0]$$

$$(8) \qquad y_2 = g[d(w)y_1, w]$$

Applying Theorem 1, coherency requires $I[s_0(w) + e_1 \geq 0] = I[s_1(w) + e_1 \geq 0]$, which holds if $s_0(w) = s_1(w)$ (triangular), or by limiting e_1.

Theorem 2 shows that, with a binary choice equation, must either restrict error support (Dagenais 1997), or make system triangular.

Two representation of triangular: Generalize Blundell and Smith (1994):

(6) $y_1 = I[f[y_2 + [g(0, w) - g(1, w)]y_1, w] + e_1 \geq$

(4) $y_2 = g(y_1, w)$

or generalize Heckman (1978):

$$(7) \qquad y_1 = I[\phi[(1 - d(w))y_2, w] + e_1 \geq 0]$$

$$(8) \qquad y_2 = g[d(w)y_1, w]$$

EXAMPLES:

Nonparam Dummy Endogenous Regressor

$$
\begin{aligned}
y_1 &= G_1(y_2, x, e_1) \\
y_2 &= G_2(y_1, x) + e_2
\end{aligned}
$$

For y_1 discrete, Das (2001) estimates G_2, implicitly assuming coherency.

G_2 can be conditional average outcome of endogenous treatment y_1.

By Theorem 1, coherency requires $G_1[G_2(y_1, x) + e_2, x, e_1]$ independent of y_1.

By (6), a coherent model is

$$
\begin{aligned}
y_1 &= G_1[y_2 + [G_2(0, x) - G_2(1, x)]y_1, x, e_1) \\
y_2 &= G_2(y_1, x) + e_2
\end{aligned}
$$

permits Das estimator for G_2.

Another coherent is

$$
\begin{aligned}
y_1 &= G_1[(1 - d)y_2, x, e_1) \\
y_2 &= G_2(dy_1, x) + e_2
\end{aligned}
$$

Linear Dummy Endogenous Regressor

$$
\begin{aligned}
y_1 &= I[x'\beta_1 + y_2\alpha_1 + e_1 \geq 0] \\
y_2 &= x'\beta_2 + y_1\alpha_2 + e_2
\end{aligned}
$$

Heckman (1978): Coherent if $\alpha_1 = 0$ or $\alpha_2 = 0$, triangular systems.

Blundell and Smith (1994) is

$$
\begin{aligned}
y_1 &= I[x'\beta_3 + y_2\alpha_1 + y_1\gamma_1 + e_3 \geq 0] \\
y_2 &= x'\beta_2 + y_1\alpha_2 + e_2
\end{aligned}
$$

coherent if $\gamma_1 = -\alpha_1\alpha_2$. This is (4) and (6) with f and g are linear.

Can let ϕ and g in (7) and (8) be linear.
Then get coherent

$$
\begin{aligned}
y_1 &= I[x'\beta_1 + (1-d)y_2\alpha_1 + e_1 \geq 0] \\
y_2 &= x'\beta_2 + d y_1\alpha_2 + e_2
\end{aligned}
$$

Endogenous Regime Switching

$$\begin{aligned} y_1 &= I[x'\beta_1 + y_2\alpha_1 + e_1 \geq 0] \\ y_2 &= x'\beta_2 + e_2 + (x'\beta_3 + e_3)y_1 \end{aligned}$$

not coherent except under
severe restrictions such as $\alpha_1 = 0$.

Theorem 2 suggests two
coherent alternatives:

$$\begin{aligned} y_1 &= I[(x'\beta_1 + y_2\alpha_1 + y_1 x'\beta_4 + e_1 \geq 0] \\ y_2 &= x'\beta_2 + e_2 + (x'\beta_3 + e_3)y_1 \end{aligned}$$

is coherent if $\beta_4 = -\alpha_1\beta_3$,
and

$$\begin{aligned} y_1 &= I[x'\beta_1 + (1-d)y_2\alpha_1 + e_1 \geq 0] \\ y_2 &= x'\beta_2 + e_2 + (x'\beta_3 + e_3)dy_1 \end{aligned}$$

is coherent.

Simultaneous Binary Choices

$$\begin{aligned} y_1 &= I[h_1(y_1, y_2, w) + e_1 \geq 0] \\ y_2 &= I[h_2(y_1, y_2, w) + e_2 \geq 0] \end{aligned}$$

Are interrelated choices substitutes or complements?

Dagenais (1997) coherence by imposing linearity and restricting the support of (e_1, e_2).

By Theorem 2, coherent is

$$\begin{aligned} y_1 &= I[f_1[y_2 - r(w)y_1, w] + e_1 \geq 0] \\ y_2 &= I[f_2(y_1, w) + e_2 \geq 0] \end{aligned}$$

where

$$r(w) = I[f_2(1, w)+e_2 \geq 0]-I[f_2(0, w)+e_2 \geq 0]$$

is coherent for any f_1, f_2.

Another coherent is

$$\begin{aligned} y_1 &= I[\phi_1[(1 - d)y_2, w] + e_1 \geq 0] \\ y_2 &= I[\phi_2[dy_1, w] + e_2 \geq 0] \end{aligned}$$

for any ϕ_1, ϕ_2, and dummy d.

An example is

$$y_1 = I[x'\beta_1 + (1-d)y_2\alpha_1 + e_1 \geq 0] \tag{9}$$

$$y_2 = I[x'\beta_2 + dy_1\alpha_2 + e_2 \geq 0] \tag{10}$$

d may be included in x.

An example d is let $d = 1$ if decide y_1 first.

Signs of α_1 and α_2 indicate substitute or complement. Can have opposite signs, e.g., if $\alpha_1 > 0$ and $\alpha_2 < 0$, then individuals having $d = 1$, view the choices as substitutes.

Let $P_{y_1,y_2} = prob(y_1, y_2)$.

$$P_{11}(\theta \mid x) = \int_{-x'\beta_2-d\alpha_2}^{\infty}\left(\int_{-x'\beta_1-(1-d)\alpha_1}^{\infty} f(e_1, e_2)de_1\right)$$

$$P_{01}(\theta \mid x) = \int_{-x'\beta_2}^{\infty}\left(\int_{-\infty}^{-x'\beta_1-(1-d)\alpha_1} f(e_1, e_2)de_1\right)de$$

$$P_{10}(\theta \mid x) = \int_{-\infty}^{-x'\beta_2-d\alpha_2}\left(\int_{-x'\beta_1}^{\infty} f(e_1, e_2)de_1\right)de_2$$

$$P_{00}(\theta \mid x) = \int_{-\infty}^{-x'\beta_2}\left(\int_{-\infty}^{-x'\beta_1} f(e_1, e_2)de_1\right)de_2$$

With n iid draws, log likelihood is

$$\sum_{i=1}^{n} y_{1i}y_{2i} \ln P_{11}(\theta \mid x_i)+(1-y_{1i})y_{2i} \ln P_{01}(\theta \mid x_i)+$$

$$y_{1i}(1-y_{2i}) \ln P_{10}(\theta \mid x_i)+(1-y_{1i})(1-y_{2i}) \ln P_{00}(\theta \mid x_i$$

Behavioral Models

Previous models ad hoc, though with d sequential decision making.

Resolve incoherency by more fully modeling behavior, e.g., modeling choice among multiple equilibria.

Example: two simultaneous binary decisions. Consider naive model

$$y_1 = I[x'\beta_1 + y_2\alpha_1 + e_1 \geq 0] \tag{11}$$

$$y_2 = I[x'\beta_2 + y_1\alpha_2 + e_2 \geq 0] \tag{12}$$

Resolve incoherency by McFadden random utility modeling.

$$
\begin{aligned}
(11) \qquad y_1 &= I[x'\beta_1 + y_2\alpha_1 + e_1 \geq 0] \\
(12) \qquad y_2 &= I[x'\beta_2 + y_1\alpha_2 + e_2 \geq 0]
\end{aligned}
$$

Let U_j = utility from choice y_j. If

$$
\begin{aligned}
U_1 &= (x'\beta_1 + y_2\alpha_1 + e_1)y_1 \\
U_2 &= (x'\beta_2 + y_1\alpha_2 + e_2)y_2
\end{aligned}
$$

Then $y_1 = \arg\max U_1$ gives (11) and $y_2 = \arg\max U_2$ gives (12).

To resolve the incoherency, let $(y_1, y_2) = \arg\max U_1 + U_2$.

Let $V(y_1, y_2)$ be $U_1 + U_2$ given y_1, y_2 and let $a = a_1 + a_2$. Then

$$
\begin{aligned}
V(0,0) &= 0 \\
V(1,0) &= x'\beta_1 + e_1 \\
V(0,1) &= x'\beta_2 + e_2 \\
V(1,1) &= x'(\beta_1 + \beta_2) + a + e_1 + e_2
\end{aligned}
$$

$(y_1, y_2) = \arg\max V(y_1, y_2)$.

$$
\begin{aligned}
V(0,0) &= 0 \\
V(1,0) &= x'\beta_1 + e_1 \\
V(0,1) &= x'\beta_2 + e_2 \\
V(1,1) &= x'(\beta_1 + \beta_2) + a + e_1 + e_2
\end{aligned}
$$

$(y_1, y_2) = \arg\max V(y_1, y_2)$.

Is a special case of ordinary multinomial choice, general is $V(1,1) = x'\beta_3 + e_3$.

Is coherent if e_1, e_2 continuous.

$a > 0$ increases the $prob(y_1 = y_2 = 1)$, choices are complements if a is positive, substitutes if a is negative.

Let $P_{y_1,y_2} = prob(y_1, y_2)$.

$$P_{11}(\theta \mid x) = \int_{-x'\beta_2-a}^{\infty} \left(\int_{\max[-x'\beta_1-a,-x'(\beta_1+\beta_2)-a-e_2]}^{\infty} f(e_1, e_2) de_1 \right) de_2$$
$$P_{01}(\theta \mid x) = \int_{-\infty}^{-x'\beta_2} \left(\int_{-\infty}^{\min(-x'\beta_1-a,x'(\beta_2-\beta_1)+e_2} f(e_1, e_2) de_1 \right) de_2$$
$$P_{10}(\theta \mid x) = \int_{-\infty}^{-x'\beta_2-a} \left(\int_{\max[-x'\beta_1,x'(\beta_2-\beta_1)+e_2]}^{\infty} f(e_1, e_2) de_1 \right) de_2$$
$$P_{00}(\theta \mid x) = \int_{-\infty}^{-x'\beta_2} \left(\int_{-\infty}^{\min(-x'\beta_1,-x'(\beta_1+\beta_2)-a-e_2} f(e_1, e_2) de_1 \right) de_2$$

With n iid draws, log likelihood is

$$\sum_{i=1}^{n} y_{1i} y_{2i} \ln P_{11}(\theta \mid x_i) + (1-y_{1i}) y_{2i} \ln P_{01}(\theta \mid x_i) +$$
$$y_{1i}(1-y_{2i}) \ln P_{10}(\theta \mid x_i) + (1-y_{1i})(1-y_{2i}) \ln P_{00}(\theta \mid x_i$$

Conclusions

Coherency is existence of a reduced form.

Necessary and sufficient conditions for coherency of simultaneous systems containing a binary choice equation were provided.

Coherency generally requires models to be triangular or recursive, similar to Heckman's linear model result, except nonlinearity permits the direction of causality to vary across observations.

Alternatively, coherency can be obtained by nesting the behavioral models that generate each equation separately into a single larger behavioral model that determines both, using the logic of McFadden's Random Utility Model.