Infinitesimal Methods in Mathematical Economics

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Chapter 0

Preface

Nonstandard analysis is a mathematical technique widely used in diverse areas in pure and applied mathematics, including probability theory, mathematical physics, functional analysis. Our primary goal is to provide a careful development of nonstandard methodology in sufficient detail to allow the reader to use it in diverse areas in mathematical economics. This requires some work in mathematical logic. More importantly, it requires a careful study of the nonstandard treatment of real analysis, measure theory, topological spaces, and so on.

Chapter 1 provides an informal description of nonstandard methods which should permit the reader to understand the mathematical Chapters 3, 4 and 9, as well as Chapters 5, 6, 7 and 8 which provide applications in Economics. Chapter 2 provides a precise statement of the fundamental results on nonstandard models, using tools from mathematical logic. Most readers will find it better to browse through the proofs in Chapter 3 before tackling Chapter 2; many readers will choose to skip Chapter 2 entirely! The proof of the existence of nonstandard models is trivial but tedious, and so is deferred to Appendix A.

This is an expanded version of Anderson (1992), a Chapter in Volume IV of the *Handbook of Mathematical Economics*. This first handout contains Chapters 1–3 and the Bibliography. Chapters 4 and 5 and part of Chapter 6 are done, apart from minor revisions to be made based on the experience in the first few weeks of teaching; they will be handed out in a few weeks. The remainder of the the chapters listed in the contents are phantoms at this point; they will be written and distributed over the course of the semester.

Chapter 1

Nonstandard Analysis Lite

1.1 When is Nonstandard Analysis Useful?

Nonstandard analysis can be used to formalize most areas of modern mathematics, including real and complex analysis, measure theory, probability theory, functional analysis, and point set topology; algebra is less amenable to nonstandard treatments, but even there significant applications have been found.

Complicated ϵ—δ arguments can usually be replaced by simpler, more intuitive nonstandard arguments involving infinitesimals. Given the dependence of work in mathematical economics on arguments from real analysis at the level of Rudin(1976) or Royden (1968), a very large number of papers could be significantly simplified using nonstandard arguments. Unfortunately, there is a significant barrier to the widespread adoption of nonstandard arguments for these kinds of problems, a barrier very much akin to the problems associated with the adoption of a new technological standard. Few economists are trained in nonstandard analysis, so papers using the methodology are necessarily restricted to a small audience. Consequently, relatively few authors use the methodology if more familiar methods will suffice. Therefore, the incentives to learn the methodology are limited. Accordingly, the use of nonstandard methods in economics has largely been limited to certain problems in which the advantages of the methodology are greatest.

1.1.1 Large Economies

Most of the work in Economics using nonstandard methods has occurred in the literature on large economies. Suppose $\chi_n : A_n \to P \times \mathbf{R}^k_+$ is a sequence of exchange economies with $|A_n| \to \infty$. In other words, A_n is the set of agents and $|A_n|$ the number of agents in the n^{th} economy, P a set of preferences, and χ_n assigns to each agent a preference and an endowment vector. A natural approach to analyzing the sequence χ_n is to formulate a notion of a limit economy $\chi : A \to P \times \mathbf{R}^k_+$. This limit economy can be formulated with A being either a nonatomic measure space or a hyperfinite set–a nonstandard construction. The properties of measure spaces and hyperfinite sets are closely analogous. Indeed, using the Loeb measure construction which we describe in Chapter 4, a hyperfinite set can be converted into a measure space. The theory of economies with a hyperfinite set of agents is analogous in many respects to the theory of economies with a measure space of agents.

However, there are certain phenomena that can occur in hyperfinite economies which are ruled out by the measure-theoretic formulation. For the most part, these relate to situations in which a small proportion of the agents are endowed with, or consume, a substantial fraction of the goods present in the economy. In the hyperfinite context, certain conditions inherent in the measure-theoretic formulation can be seen to be strong endogenous assumptions. Using hyperfinite exchange economies, we can state exogenous assumptions which imply the endogenous assumptions inherent in the measure-theoretic formulation, as well as explore the behavior of economies in which the endogenous assumptions fail.

The power of the nonstandard methodology is seen most clearly at the next stage, in which one deduces theorems about the sequence χ_n from the theorems about the limit χ. A central result known as the Transfer Principle asserts that any property which can be formalized in a particular formal language and which holds for χ must also hold for χ_n for sufficiently large n. Viewed in this context, the Transfer Principle functions as a sweeping generalization of the convergence theorems that can be formulated using topology and measure theory. The Transfer Principle converts results about the limit economy χ into limiting results about the sequence χ_n in a few lines of argument. Consequently, for those properties which hold both in measure-theoretic and hyperfinite economies, nonstandard analysis provides a very efficient tool to derive limit theorems for large finite economies. On the other hand, in the situations in which the behavior of the measure-theoretic and hyperfinite economies differ, it is the hyperfinite economy rather than the measure-theoretic economy which captures the behavior of large finite economies. The literature on large economies is discussed in Chapter 5.

1.1.2 Continuum of Random Variables

Probability theory is currently the most active field for applications of nonstandard analysis. Since probabilistic constructions are widely used in general equilibrium theory, game theory and finance, these seem fruitful areas for further applications of nonstandard methodology.

Nonstandard analysis provides an easy resolution of the so-called continuum of random variables problem. Given a standard measure space $(A, \mathcal{A}, \mu)$ and an uncountable family of independent identically distributed random variables $X_\lambda : A \to \mathbf{R}$, one would like to assert that there is no aggregate uncertainty; in other words, the empirical distribution of the X_λ's equals the theoretical distributuion with probability one. Typically, however, the set of $a \in A$ for which the empirical distribution of the X_λ's equals the theoretical distribution is not measurable; by extending μ, it can be assigned any measure between 0 and 1. Note that, with a large finite number of random variables, the measurability issue does not even arise. Thus, the problem is a pathology arising from the formulation of measure theory, and not a problem of large finite systems. If A is a hyperfinite set, and μ is the Loeb measure, then the empirical distribution does equal the theoretical distribution with probability one. Moreover, as in the large economies literature, the Transfer Principle can be used to deduce asymptotic properties of large finite systems. The Continuum of Random Variables Problem is discussed in Chapter 6.

1.1.3 Searching For Elementary Proofs

Nonstandard analysis allows one to replace many measure-theoretic arguments by discrete combinatoric arguments. For example, the Shapley-Folkman Theorem plays the same role in hyperfinite sets as Lyapunov's Theorem plays in nonatomic measure spaces. Thus, nonstandard analysis is an effective tool for determining exactly which parts of a given proof really depend on arguments in analysis, and which follow from more elementary considerations. Occasionally, it is possible to replace every step in a proof by an elementary argument; as a consequence, one obtains a proof using neither nonstandard analysis nor measure theory. Examples are discussed in Chapter 9.

1.2 Ideal Elements

Leibniz' formulation of calculus was based on the notion of an infinitesimal. Mathematics has frequently advanced through the introduction of ideal elements to provide solutions to equations. The Greeks were horrified to discover that the equation $x^2 = 2$ has no rational solution; this problem was resolved by the introduction of the ideal element $\sqrt{2}$; ultimately, the real numbers were defined as the completion of the rationals. Similarly, the complex numbers were created by the introduction of the ideal element $i = \sqrt{-1}$. Leibniz introduced infinitesimals as ideal elements which, while not zero, were smaller than any positive real number. Thus, an infinitesimal is an ideal element providing a solution to the family of equations

$$x > 0;\ x < 1,\ x < \frac{1}{2},\ x < \frac{1}{3}, \ldots. \tag{1.1}$$

Infinitesimals played a key role in Leibniz' formulation of calculus. For example, the derivative of a function was defined as the slope of the function over an interval of infinitesimal length. Leibniz asserted that the real numbers, augmented by the addition of infinitesimals, obeyed all the same rules as the ordinary real numbers. Unfortunately, neither Leibniz nor his successors were able to develop a formulation of infinitesimals which was free from contradictions. Consequently, in the middle of the nineteenth century, the ϵ–δ formulation replaced infinitesimals as the generally accepted foundation of calculus and real analysis.

In 1961, Abraham Robinson discovered that model theory, a branch of mathematical logic, provided a satisfactory foundation for the use of infinitesimals in analysis. In the remainder of Section 1, we will provide an informal description of a model of the nonstandard real numbers. In Section 2, we will provide a formal description of nonstandard models, along with a precise statement of the rules of inference which are allowed in reasoning about nonstandard models. The proof of the underlying theorems which justify the rules of inference will be presented in the Appendix.

1.3 Ultraproducts

A very simple construction which produces elements with infinitesimal properties is $\mathbf{R}^{\mathbf{N}}$, the space of real sequences. We can embed $\mathbf{R}$ into $\mathbf{R}^{\mathbf{N}}$ by mapping each $r \in \mathbf{R}$ to the constant sequence $\bar{r} = (r, r, r, \ldots)$. Now consider the sequence defined by $x_n = \frac{1}{n}$. Let $\mathbf{R}_{++}$ denote the set of strictly positive real numbers. Given any $r \in \mathbf{R}_{++}$, observe that $x_n < \bar{r}_n$ for all but a finite number of values of n. In other words, if we were to define a relation $<_F$ on $\mathbf{R}^{\mathbf{N}}$ by

$$x <_F y \iff x_n < y_n \text{ for all but a finite number of } n \in \mathbf{N}, \tag{1.2}$$

then x would be infinitesimal in the sense that $x <_F \bar{r}$ for every positive $r \in \mathbf{R}$. Unfortunately, this very simple construction does not yield a satisfactory theory of infinitesimals. For example, consider y defined by $y_n = 2$ for n odd and $y_n = 0$ for n even. Neither $y <_F \bar{1}$ nor $\bar{1} <_F y$ is true; in other words, $<_F$ is only a partial order on $\mathbf{R}^{\mathbf{N}}$. In order to construct a satisfactory theory of infinitesimals, we consider a slightly more elaborate construction, known as an *ultraproduct*.

Definition 1.3.1 A *free ultrafilter* on $\mathbf{N}$ is a collection $\mathcal{U}$ of subsets of $\mathbf{N}$ satisfying the following properties:

1. if $A, B \in \mathcal{U}$, then $A \cap B \in \mathcal{U}$;
2. if $A \in \mathcal{U}$ and $A \subset B \subset \mathbf{N}$, then $B \in \mathcal{U}$;
3. if A is finite, then $A \notin \mathcal{U}$;
4. if $A \subset \mathbf{N}$, either $A \in \mathcal{U}$ or $\mathbf{N} \setminus A \in \mathcal{U}$.

Remark 1.3.2 Note that by item 4 of Definition 1.3.1, we must have either $\{2, 4, 6, \ldots\} \in \mathcal{U}$ or $\{1, 3, 5, \ldots\} \in \mathcal{U}$, but not both by items 1 and 3.

Proposition 1.3.3 *Suppose* $\mathbf{N} = A_1 \cup \ldots \cup A_n$ *with* $n \in \mathbf{N}$, *and* $A_i \cap A_j = \emptyset$ *for* $i \neq j$. *Then* $A_i \in \mathcal{U}$ *for exactly one* i.

Proof: Let $B_i = \mathbf{N} \setminus A_i$. If there is no i such that $A_i \in \mathcal{U}$, then $B_i \in \mathcal{U}$ for each i. Then $\emptyset = B_1 \cap (B_2 \cap \ldots \cap (B_{n-1} \cap B_n) \ldots) \in \mathcal{U}$ by $n-1$ applications of Property 1 of Definition 1.3.1, which contradicts Property 3 (since $\emptyset$ is finite). Thus, $A_i \in \mathcal{U}$ for some i. If $A_i \in \mathcal{U}$ and $A_j \in \mathcal{U}$ with $i \neq j$, then $\emptyset = A_i \cap A_j \in \mathcal{U}$, again contradicting Property 3. Thus, $A_i \in \mathcal{U}$ for exactly one i. ■

Definition 1.3.4 The equivalence relation $=_{\mathcal{U}}$ on $\mathbf{R}^{\mathbf{N}}$ is defined by

$$x =_{\mathcal{U}} y \iff \{n : x_n = y_n\} \in \mathcal{U}. \tag{1.3}$$

Given $x \in \mathbf{R}^{\mathbf{N}}$, let $[x]$ denote the equivalence class of x with respect to the equivalence relation $=_{\mathcal{U}}$. The set of *nonstandard real numbers*, denoted $^*\mathbf{R}$ and read "star R", is $\{[x] : x \in \mathbf{R}^{\mathbf{N}}\}$.

Any relation on $\mathbf{R}$ can be extended to $^*\mathbf{R}$. In particular, given $[x], [y] \in {}^*\mathbf{R}$, we can define

$$[x] <_{\mathcal{U}} [y] \Longleftrightarrow \{n : x_n < y_n\} \in \mathcal{U}. \tag{1.4}$$

The reader will easily verify that this definition is independent of the particular representatives x and y chosen from the equivalence classes $[x], [y]$.

Proposition 1.3.5 *Suppose* $[x], [y] \in {}^*\mathbf{R}$. *Then exactly one of* $[x] <_{\mathcal{U}} [y]$, $[x] =_{\mathcal{U}} [y]$, *or* $[x] >_{\mathcal{U}} [y]$ *is true.*

Proof: Let $A = \{n \in \mathbf{N} : x_n < y_n\}$, $B = \{n \in \mathbf{N} : x_n = y_n\}$, and $C = \{n \in \mathbf{N} : x_n > y_n\}$. By Proposition 1.3.3, exactly one of A, B or C is in $\mathcal{U}$. ■

Example 1.3.6 Let $x \in \mathbf{R}^{\mathbf{N}}$ be defined by $x_n = \frac{1}{n}$, and $\bar{r} = (r, r, \ldots)$ for $r \in \mathbf{R}$. If $r > 0$, then $\{n : x_n < \bar{r}_n\} \in \mathcal{U}$, since its complement is finite. Thus, $[x] <_{\mathcal{U}} [\bar{r}]$ for all $r \in \mathbf{R}_{++}$, i.e. $[x]$ is an infinitesimal. We write $[x] \simeq [y]$ if $[x] - [y]$ (defined to be $[z]$ where $z_n = x_n - y_n$) is an infinitesimal.

Definition 1.3.7 $[x] \in {}^*\mathbf{R}$ is said to be *infinite* if $[x] >_{\mathcal{U}} \bar{m}$ for all $m \in \mathbf{N}$.

Given any function $f : \mathbf{R} \to \mathbf{R}$, we can define a function $^*f : {}^*\mathbf{R} \to \mathbf{R}$ by

$$^*f([x]) = [(f(x_1), f(x_2), \ldots)]. \tag{1.5}$$

In other words, $^*f$ is defined by evaluating f pointwise on the components of x.

1.4 Internal and External Sets

In order to work with the nonstandard real numbers, we need to be able to talk about subsets of $^*\mathbf{R}$. We extend the ultraproduct construction to sets by considering sequences in $(\mathcal{P}(\mathbf{R}))^{\mathbf{N}}$, where $\mathcal{P}(\mathbf{R})$ is the collection of all subsets of $\mathbf{R}$, and extending the equivalence relation $=_{\mathcal{U}}$ from Definition 1.3.4.

Definition 1.4.1 Suppose $A, B \in (\mathcal{P}(\mathbf{R}))^{\mathbf{N}}$, $[x] \in {}^*\mathbf{R}$. We define an equivalence relation $=_{\mathcal{U}}$ by

$$A =_{\mathcal{U}} B \iff \{n : A_n = B_n\} \in \mathcal{U}. \tag{1.6}$$

Let $[A]$ denote the equivalence class of A. We define

$$[x] \in_{\mathcal{U}} [A] \iff \{n : x_n \in A_n\} \in \mathcal{U}.^1 \tag{1.7}$$

Note that $[A]$ is *not* a subset of $^*\mathbf{R}$; it is an equivalence class of sequences of sets of real numbers, not a set of equivalence classes of sequences of real numbers. However, we can associate it with a subset of $^*\mathbf{R}$ in a natural way, as follows.

Definition 1.4.2 (Mostowski Collapsing Function) Given $A \in (\mathcal{P}(\mathbf{R}))^{\mathbf{N}}$, define a set $M([A]) \subset {}^*\mathbf{R}$ by

$$M([A]) = \{[x] \in {}^*\mathbf{R} : [x] \in_{\mathcal{U}} [A]\}. \tag{1.8}$$

A set $B \subset {}^*\mathbf{R}$ is said to be *internal* if $B = M([A])$ for some $A \in (\mathcal{P}(\mathbf{R}))^{\mathbf{N}}$; otherwise, it is said to be *external*. A function is *internal* if its graph is internal.

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[1]As above, the reader will have no trouble verifying that the definition does not depend on the choice of representatives from the equivalence classes.

Definition 1.4.3 Suppose $B \subset \mathbf{R}$. Define ${}^*B = M([A])$, where $A \in (\mathcal{P}(\mathbf{R}))^{\mathbf{N}}$ is the constant sequence $A_n = B$ for all $n \in \mathbf{N}$.

Example 1.4.4 The set of nonstandard natural numbers is

$$ {}^*\mathbf{N} = \{[x] \in {}^*\mathbf{R} : \{n : x_n \in \mathbf{N}\} \in \mathcal{U}\}. \tag{1.9} $$

Let $\bar{\mathbf{N}} = \{[\bar{n}] : n \in \mathbf{N}\}$. Then $\bar{\mathbf{N}} \subset {}^*\mathbf{N}$. Indeed, $\bar{\mathbf{N}}$ is a *proper* subset of ${}^*\mathbf{N}$, as can be seen by considering $[x]$, where $x_n = n$ for all $n \in \mathbf{N}$. If $m \in \mathbf{N}$, $\{n : x_n = \bar{m}_n\} = \{m\} \notin \mathcal{U}$, so $[x] \neq [\bar{m}]$.

Proposition 1.4.5 *${}^*\mathbf{N} \setminus \bar{\mathbf{N}}$ is external.*

Proof: Suppose ${}^*\mathbf{N} \setminus \bar{\mathbf{N}} = M([A])$. We shall derive a contradiction by constructing $[y] \in {}^*\mathbf{N} \setminus \bar{\mathbf{N}}$ with $[y] \notin M([A])$.

Let $J = \{n : A_n \subset \mathbf{N}\}$. We may choose $x \in \mathbf{R}^{\mathbf{N}}$ such that $x_n \in A_n \setminus \mathbf{N}$ for $n \in \mathbf{N} \setminus J$, and $x_n = 0$ for $n \in J$. Therefore, $\{n \in \mathbf{N} : x_n \in \mathbf{N}\} = \emptyset$, so $[x] \notin {}^*\mathbf{N}$. Since $M([A]) \subset {}^*\mathbf{N}$, $[x] \notin M([A])$, so $\mathbf{N} \setminus J \notin \mathcal{U}$, so $J \in \mathcal{U}$. Without loss of generality, we may assume that $A_n \subset \mathbf{N}$ for all $n \in \mathbf{N}$.

For $m \in \mathbf{N}$, let $T_m = \{n \in \mathbf{N} : m \notin A_n\}$; since $[\bar{m}] \notin M([A])$, $T_m \in \mathcal{U}$. For $m \in \mathbf{N} \cup \{0\}$, let

$$ S_m = \{n \in \mathbf{N} : A_n \subset \{m, m+1, m+2, \ldots\}\}. \tag{1.10} $$

Then $S_m = \cap_{k=1}^{m-1} T_k$, so $S_m \in \mathcal{U}$. $S_0 = \mathbf{N}$. Let $S_\infty = \cap_{m \in \mathbf{N}} S_m = \{n \in \mathbf{N} : A_n = \emptyset\}$. If $S_\infty \in \mathcal{U}$, then $M([A]) = \emptyset$, a contradiction since ${}^*\mathbf{N} \setminus \bar{\mathbf{N}} \neq \emptyset$ by Example 1.4.4. Hence, $S_\infty \notin \mathcal{U}$.

Define a sequence $y \in \mathbf{R}^{\mathbf{N}}$ by $y_n = m$ if $n \in S_{m+1} \setminus S_{m+2}$, $y_n = 0$ for $n \in S_\infty$. Then $\{n : y_n \in \mathbf{N}\} = \mathbf{N} \setminus S_\infty \in \mathcal{U}$, so $[y] \in {}^*\mathbf{N}$. Given $m \in \mathbf{N}$, $\{n : y_n = m\} = S_{m+1} \setminus S_{m+2} \subset$

$\mathbf{N} \setminus S_{m+2} \notin \mathcal{U}$, so $[y] \notin \bar{\mathbf{N}}$, and hence $[y] \in {}^*\mathbf{N} \setminus \bar{\mathbf{N}}$. However, $\{n : y_n \in A_n\} \subset S_\infty \notin \mathcal{U}$, so $[y] \notin M([A])$, so $M([A]) \neq {}^*\mathbf{N} \setminus \mathbf{N}$. ■

Corollary 1.4.6 $\bar{\mathbf{N}}$ *is external.*

Proof: Suppose $\bar{\mathbf{N}} = M([A])$. Let $B_n = \mathbf{N} \setminus A_n$ for each $n \in \mathbf{N}$. $M([B]) \subset {}^*\mathbf{N}$. Suppose $[y] \in {}^*\mathbf{N}$; we may assume without loss of generality that $y_n \in \mathbf{N}$ for all $n \in \mathbf{N}$. Then

$$[y] \in M([B]) \Leftrightarrow \{n \in \mathbf{N} : y_n \in B_n\} \in \mathcal{U}$$

$$\Leftrightarrow \{n \in \mathbf{N} : y_n \in A_n\} \notin \mathcal{U} \Leftrightarrow [y] \notin M([A]). \tag{1.11}$$

Thus, $M([B]) = {}^*\mathbf{N} \setminus \bar{\mathbf{N}}$, so ${}^*\mathbf{N} \setminus \bar{\mathbf{N}}$ is internal, contradicting Proposition 1.4.5. ■

1.5 Notational Conventions

It is customary to omit *'s in many cases. Note first that we can embed $\mathbf{R}$ in ${}^*\mathbf{R}$ by the map $r \to [\bar{r}]$. Thus, it is customary to view $\mathbf{R}$ as a subset of ${}^*\mathbf{R}$, and to refer to $[\bar{r}]$ as r. Thus, we can also write $\mathbf{N}$ instead of the more awkward $\bar{\mathbf{N}}$. Basic relations such as $<, >, \leq, \geq$ are written without the addition of a *. Functions such as sin, cos, log, e^x, $|\cdot|$ (for absolute value or cardinality) are similarly written without *'s.

Consider the function $g(n) = \mathbf{R}^n$. If n is an infinite natural number, then ${}^*\mathbf{R}^n$ is defined to be $({}^*g)(n)$; equivalently, it is the set of all internal functions from $\{1, \ldots, n\}$ to ${}^*\mathbf{R}$. The summation symbol $\sum$ represents a function from $\mathbf{R}^n$ to $\mathbf{R}$. Thus, if n is an infinite natural number and $y \in {}^*\mathbf{R}^n$, $({}^*\sum)_{i=1}^n y_i$ is defined. It is customary to omit the * from summations, products, or Cartesian products.

Thus, the following expressions are acceptable:

$$\forall x \in {}^*\mathbf{R}\ e^x > 0; \tag{1.12}$$

$$\exists n \in {}^*\mathbf{N}\ \sum_{i=1}^{n} x_i = 0. \tag{1.13}$$

1.6 Standard Models

We need to be able to consider objects such as topological spaces or probability measures in addition to real numbers. This is accomplished by considering a *superstructure*. We take a base set X consisting of the union of the point sets of all objects we wish to consider. For example, if we wish to consider real-valued functions on a particular topological space $(T, \mathcal{T})$, we take $X = \mathbf{R} \cup T$. The superstructure is the class of all objects which can be obtained from the base set by iterating the operation of forming subsets; we will refer to it as the *standard model* generated by X.

Definition 1.6.1 Suppose X is a set, all of whose members are atomic, i.e. $\emptyset \notin X$ and no $x \in X$ contains any elements. Let

$$\mathcal{X}_0 = X; \tag{1.14}$$

$$\mathcal{X}_{n+1} = \left[\mathcal{P}(\bigcup_{k=0}^{n} X_k)\right] \cup X_0 \qquad (n = 0, 1, 2, \ldots) \tag{1.15}$$

where $\mathcal{P}$ is the power set operator, which associates to any set S the collection of all the subsets of S. Let

$$\mathcal{X} = \bigcup_{n=0}^{\infty} \mathcal{X}_n; \tag{1.16}$$

$\mathcal{X}$ is called the *superstructure determined by* X. For any set $B \in \mathcal{X}$, let $\mathcal{FP}(B)$ denote the set of all finite subsets of B.

The superstructure determined by X contains representations of subsets of X, functions defined on X, Cartesian products of subsets of X, and indeed essentially all the classical mathematical constructions that can be defined using X as the initial point set.[2] The exact form of the representation can become quite complicated; fortunately, we need never work in detail with the superstructure representations, but only need to know they exist. The following examples illustrate how various mathematical constructions are represented in the superstructure.

Example 1.6.2 An ordered pair $(x, y) \in X^2$ is defined in set theory as $\{\{x\}, \{x, y\}\}$. $x, y \in \mathcal{X}_0$, so $\{x\} \in \mathcal{X}_1$ and $\{x, y\} \in \mathcal{X}_1$, so $\{\{x\}, \{x, y\}\} \in \mathcal{X}_2$.

Example 1.6.3 A function $f: A \to B$, where $A, B \subset X$, can be represented by its graph $G = \{(x, f(x)): x \in A\}$. From the previous example, we know that each ordered pair $(x, f(x))$ in the graph G is an element of $\mathcal{X}_2$, so $G \in \mathcal{X}_3$.

Example 1.6.4 The set of all functions from A to B, with $A, B \subset X$, is thus represented by an element of $\mathcal{X}_4$.

Example 1.6.5 If $\mathbf{N} \subset X$, an n-tuple $(x_1, \ldots, x_n) \in X^n$ can be represented as a function from $\{1, \ldots, n\}$ to X. Thus, if $A \subset X$, then A^n is an element of $\mathcal{X}_4$.

Example 1.6.6 $\mathcal{X}_n$ is an element of $\mathcal{X}_{n+1}$.

Example 1.6.7 Let $(T, \mathcal{T})$ be a topological space, so that T is the set of points and $\mathcal{T}$ the collection of open sets. Take $X = T$. Then $\mathcal{T} \in \mathcal{X}_2$.

[2] Indeed, formally speaking, the *definition* of each of these constructions is expressed in terms of set theory; see for example Bourbaki(1970).

Example 1.6.8 Consider an exchange economy with a set A of agents and commodity space $\mathbf{R}^k_+$. Let $X = A \cup \mathbf{R}$. An element of $\mathbf{R}^k_+$ is a k-tuple of elements of X, and hence is an element of $\mathcal{X}_3$. A pair (x, y) with $x, y \in \mathbf{R}^k_+$ can be viewed as an element of $\mathbf{R}^{2k}$, and so is also an element of $\mathcal{X}_3$. A preference relation is a subset of $\mathbf{R}^k_+ \times \mathbf{R}^k_+$, so it is an element of $\mathcal{X}_4$. A preference-endowment pair $(\succ_a, e(a))$ with $e(a) \in \mathbf{R}^k_+$ is an element of $\mathcal{X}_6$. The exchange economy is a function from A to the set of preference-endowment pairs, so it is an element of $\mathcal{X}_9$.

Remark 1.6.9 If $Z \in \mathcal{X}$, then $Z \in \mathcal{X}_n$ for some n; thus, there is an upper bound on the number of nested set brackets, uniform over all elements $z \in Z$. In particular, the set $\{x, \{x\}, \{\{x\}\}, \{\{\{x\}\}\}, \ldots\}$ is *not* an element of the superstructure $\mathcal{X}$. Moreover, $\mathcal{X}$ is *not* an element of $\mathcal{X}$.

1.7 Superstructure Embeddings

Given a standard model $\mathcal{X}$, we want to construct a *nonstandard extension*, i.e. a superstructure $\mathcal{Y}$ and a function $* : \mathcal{X} \to \mathcal{Y}$ satisfying certain properties.

Definition 1.7.1 Consider a function $*$ from a standard model $\mathcal{X}$ to a superstructure $\mathcal{Y}$. $A \in \mathcal{Y}$ is said to be *internal* if $A \in {}^*B$ for some $B \in \mathcal{X}$, and external otherwise. The function $* : \mathcal{X} \to \mathcal{Y}$ is called a *superstructure embedding*[3] if

1. $*$ is an injection;

2. $\mathcal{X}_0 \subset \mathcal{Y}_0$; moreover $x \in \mathcal{X}_0 \Rightarrow {}^*x = x$.

3. ${}^*\mathcal{X}_0 = \mathcal{Y}_0$;

[3] Some of the properties listed can be derived from others.

4. ${}^*\mathcal{X}_n \subset \mathcal{Y}_n$;

5. ${}^*(\mathcal{X}_{n+1} \setminus \mathcal{X}_n) \subset \mathcal{Y}_{n+1} \setminus \mathcal{Y}_n \qquad (n = 0, 1, 2, \ldots)$;

6. $x_1, \ldots, x_n \in \mathcal{X} \Rightarrow {}^*\{x_1, \ldots, x_n\} = \{{}^*x_1, \ldots, {}^*x_n\}$;

7. $A, B \in \mathcal{X} \Longrightarrow \{A \in B \Leftrightarrow {}^*A \in {}^*B\}$;

8. $A, B \in \mathcal{X} \Longrightarrow$

   (a) ${}^*(A \cap B) = {}^*A \cap {}^*B$;
   (b) ${}^*(A \cup B) = {}^*A \cup {}^*B$;
   (c) ${}^*(A \setminus B) = {}^*A \setminus {}^*B$;
   (d) ${}^*(A \times B) = {}^*A \times {}^*B$;

9. If Γ is the graph of a function from A to B, with $A, B \in \mathcal{X}$, then ${}^*\Gamma$ is the graph of a function from ${}^*A$ to ${}^*B$;

10. $A \in {}^*\mathcal{X}_n, B \in A \Rightarrow B \in {}^*\mathcal{X}_{n-1}$;

11. A internal, $A \subset B$, $B \in {}^*(\mathcal{P}(C)) \Rightarrow A \in {}^*(\mathcal{P}(C))$.

$A \in \mathcal{Y}$ is said to be *hyperfinite* if $A \in {}^*(\mathcal{FP}(B))$ for some set $B \in \mathcal{X}$ (recall $\mathcal{FP}(B)$ is the set of all finite subsets of B). Let ${}^*\mathcal{X}$ denote $\{y \in \mathcal{Y} : y \text{ is internal}\}$. A function whose domain and range belong to $\mathcal{Y}$ is said to be *internal* if its graph is internal.

Example 1.7.2 Suppose $X = \mathbf{R}$. Take $Y = {}^*\mathbf{R}$, defined via the ultraproduct construction. Let $\mathcal{Y}$ be the superstructure constructed with Y as the base set. Then $*$ as defined by the ultraproduct construction is a superstructure embedding. Note that $\mathcal{Y}_1$ contains both internal and external sets; thus, the embedding $*$ is not onto.

1.8 A Formal Language

> "I shall not today attempt further to define the kinds of material I understand to be embraced within that shorthand description; and perhaps I could never succeed in intelligibly doing so. But I know it when I see it." *Justice Potter Stewart*, concurring in *Jacobellis v. Ohio*, 378 U.S. 184 at 197.

In order to give a precise definition of a nonstandard extension, one must define a formal language $\mathcal{L}$; see Chapter 2 for details. In practice, one quickly learns to recognize which formulas belong to $\mathcal{L}$. The formal language $\mathcal{L}$ is rich enough to allow us to express any formula of conventional mathematics concerning the standard model $\mathcal{X}$, with one caveat: all quantifiers must be bounded, i.e. they are of the form $\forall x \in B$ or $\exists x \in B$ where B refers to an object at a specific level $\mathcal{X}_n$ in the superstructure $\mathcal{X}$. Thus, the quantifier $\forall f \in \mathcal{F}(\mathbf{R}, \mathbf{R})$, where $\mathcal{F}(\mathbf{R}, \mathbf{R})$ denotes the set of functions from $\mathbf{R}$ to $\mathbf{R}$, is allowed; the quantifiers $\forall x \in \mathcal{X}$ and $\forall x$ are not allowed.

1.9 Transfer Principle

Leibniz asserted, roughly speaking, that the nonstandard real numbers obey all the same properties as the ordinary real numbers. The Transfer Principle gives a precise statement of Leibniz' assertion. The key fact which was not understood until Robinson's work is that the Transfer Principle cannot be applied to external sets. Thus, the distinction between internal and external sets is crucial in nonstandard analysis. Given a sentence $F \in \mathcal{L}$ which describes the standard superstructure $\mathcal{X}$, we can form a sentence ${}^*F$ by making the following substitutions:

1. For any set $A \in \mathcal{X}$, substitute ${}^*A$;

2. For any function $f : A \to B$ with $A, B \in \mathcal{X}$, substitute ${}^*f$.

3. For any quantifier over sets such as $\forall A \in \mathcal{P}(B)$ or $\exists A \in \mathcal{P}(B)$, where $B \in \mathcal{X}$, substitute the quantifier $\forall A \in {}^*(\mathcal{P}(B))$ or $\exists A \in {}^*(\mathcal{P}(B))$ which ranges over all *internal* subsets of ${}^*B$.

4. For any quantifier over functions such as $\forall f \in \mathcal{F}(A, B)$ or $\exists f \in \mathcal{F}(A, B)$, where $\mathcal{F}(A, B)$ denotes the set of functions from A to B for $A, B \in \mathcal{X}$, substitute the quantifier $\forall f \in {}^*(\mathcal{F}(A, B))$ or $\exists f \in {}^*(\mathcal{F}(A, B))$ which ranges over all *internal* functions from ${}^*A$ to ${}^*B$.

We emphasize that quantifiers in ${}^*F$ range only over internal entities. The *Transfer Principle* asserts that F is a true statement about the real numbers if and only if ${}^*F$ is a true statement about the nonstandard real numbers.

Example 1.9.1 Consider the following sentence F:

$$\forall S \in \mathcal{P}(\mathbf{N}) \;\; [S = \emptyset \vee \exists n \in S \;\; \forall m \in S \;\; m \geq n]. \qquad (1.17)$$

F asserts that every nonempty subset of the natural numbers has a first element. ${}^*F$ is the sentence

$$\forall S \in {}^*(\mathcal{P}(\mathbf{N})) \;\; [S = \emptyset \vee \exists n \in S \;\; \forall m \in S \;\; m \geq n]. \qquad (1.18)$$

${}^*F$ asserts that every nonempty internal subset of ${}^*\mathbf{N}$ has a first element. External subsets of ${}^*\mathbf{N}$ need not have a first element. Indeed, ${}^*\mathbf{N} \setminus \mathbf{N}$ has no first element; if it did have a first element n, then $n-1$[4] would of necessity be an element of $\mathbf{N}$, but then n would be an element of $\mathbf{N}$.

1.10 Saturation

Saturation was introduced to nonstandard analysis by Luxemburg (1969).

Definition 1.10.1 A superstructure embedding $*$ from $\mathcal{X}$ to $\mathcal{Y}$ is *saturated*[5] if, for every collection $\{A_\lambda : \lambda \in \Lambda\}$ with A_λ internal and $|\Lambda| < |\mathcal{X}|$,

$$\bigcap_{\lambda \in \Lambda} A_\lambda = \emptyset \Longrightarrow \exists \lambda_1, \ldots, \lambda_n \bigcap_{i=1}^{n} A_{\lambda_i} = \emptyset. \qquad (1.19)$$

One can construct saturated superstructure embeddings using an elaboration of the ultraproduct construction described above. To make the saturation property plausible, we present the following proposition.

Proposition 1.10.2 *Suppose* ${}^*\mathbf{R}$ *is constructed via the ultraproduct construction of Section 1.3. If* $\{A_n : n \in \mathbf{N}\}$ *is a collection of internal subsets of* ${}^*\mathbf{R}$*, and* $\cap_{n \in \mathbf{N}} A_n = \emptyset$*, then* $\cap_{n=1}^{n_0} A_n = \emptyset$ *for some* $n_0 \in \mathbf{N}$.

Proof: Since A_n is internal, there is a sequence B_{nm} ($m \in \mathbf{N}$) such that $A_n = M([B_{n\cdot}])$. If $A_1 \cap \cdots \cap A_n \neq \emptyset$ for all $n \in$

[4] We can define a function $f : \mathbf{N} \to \mathbf{N} \cup \{0\}$ by $f(m) = m - 1$. Then $n-1$ is defined to be ${}^*f(n)$. It is easy to see that, if $n = [x]$ for $x \in \mathbf{R}^{\mathbf{N}}$, $n - 1 = [(x_1 - 1, x_2 - 1, \ldots)]$.

[5] Our use of the term "saturated" is at variance with standard usage in nonstandard analysis or model theory. Commonly used terms are "polysaturated" (Stroyan and Luxemburg (1976)) or $|\mathcal{X}|$-saturated. Model theorists use the term "saturated" to mean that equation 1.19 holds provided $|\Lambda| < |\mathcal{X}|$.

$\mathbf{N}$, we can find $[x_n] \in {}^*\mathbf{R}$ with $[x_n] \in A_1 \cap \cdots \cap A_n$ for each n. Note that $x_n \in \mathbf{R}^{\mathbf{N}}$, so let x_{nm} denote the m'th component of x_n. Then $\{m : x_{nm} \in B_{1m} \cap \cdots \cap B_{nm}\} \in \mathcal{U}$. We may assume without loss of generality that $x_{nm} \in B_{1m} \cap \cdots \cap B_{nm}$ for all n and m. Define $[z] \in {}^*\mathbf{R}$ by $z_m = x_{mm}$. Then $\{m : z_m \in B_{nm}\} \supset \{n, n+1, \cdots\} \in \mathcal{U}$. Thus, $[z] \in A_n$ for all n, so $\cap_{n \in \mathbf{N}} A_n \neq \emptyset$. ■

Theorem 1.10.3 *Suppose* $^* : \mathcal{X} \to \mathcal{Y}$ *is a saturated superstructure embedding. If* B *is internal and* $x_1, x_2, \ldots$ *is a sequence with* $x_n \in B$ *for each* $n \in \mathbf{N}$*, there is an internal sequence* y_n *with* $y_n \in B$ *for all* $n \in {}^*\mathbf{N}$ *such that* $y_n = x_n$ *for* $n \in \mathbf{N}$.

Proof: Let $A_n = \{$ internal sequences $y : y_i = x_i$ $(1 \leq i \leq n)$, $y_i \in B (i \in {}^*\mathbf{N})\}$. Fix $b \in B$. If we consider y defined by $y_i = x_i (1 \leq i \leq n)$, and $y_i = b$ for $i > n$, we see that $A_n \neq \emptyset$. By saturation, we may find $y \in \cap_{n \in \mathbf{N}} A_n$. Then y is an internal sequence, $y_n \in B$ for all $n \in {}^*\mathbf{N}$, and $y_n = x_n$ for all $n \in \mathbf{N}$. ■

1.11 Internal Definition Principle

One consequence of the Transfer Principle, the Internal Definition Principle, is used sufficiently often that it is useful to present it separately. The informal statement of the Internal Definition Principle is as follows: any object in the nonstandard model which is describable using a formula which does not contain any external expressions is internal. For a formal statement, see Definition 2.7.1.

Example 1.11.1 The following examples will help to clarify the use of the Internal Definition Principle.

1. If $n \in {}^*\mathbf{N}$, $\{m \in {}^*\mathbf{N} : m > n\}$ is internal.

2. If f is an internal function and B is internal, then $f^{-1}(B)$ is internal.

3. If A, B are internal sets with $A \subset B$, then $\{C \in \mathcal{P}(B) : C \supset A\}$, the class of all internal subsets of B which contain A, is internal.

4. $\{x \in {}^*\mathbf{R} : x \simeq 0\}$ is not internal; the presence of the external expression $x \simeq 0$ renders the Internal Definition Principle inapplicable.

1.12 Nonstandard Extensions, or Enough Already with the Ultraproducts

Definition 1.12.1 A *nonstandard extension* of a standard model $\mathcal{X}$ is a saturated superstructure embedding $^* : \mathcal{X} \to \mathcal{Y}$ satisfying the Transfer Principle and the Internal Definition Principle.

As we noted above, the real numbers $\mathbf{R}$ are defined as the completion of the rational numbers $\mathbf{Q}$. The completion is constructed in one of two ways: Dedekind cuts or Cauchy sequences. In practice, mathematical arguments concerning $\mathbf{R}$ never refer to the details of the construction. Rather, the construction is used once to establish the existence of a set $\mathbf{R}$ satisfying certain axioms. All further arguments are given in terms of the axioms.

In the same way, the ultraproduct construction is used to demonstrate the existence of nonstandard extensions. Nonstandard proofs are then stated wholly in terms of those properties, without reference to the details of the ultraproduct construction.

1.13 Hyperfinite Sets

Definition 1.13.1 Suppose that $A \in \mathcal{X}$ and $^* : \mathcal{X} \to \mathcal{Y}$ is a nonstandard extension. Let $\mathcal{FP}(A)$ denote the set of finite subsets of A. A set $B \subset {}^*A$ is said to be *hyperfinite* if $B \in {}^*(\mathcal{FP}(A))$.

Example 1.13.2 Suppose m is an infinite natural number. Consider $B = \{k \in {}^*\mathbf{N} : k \leq m\}$. The sentence

$$\forall m \in \mathbf{N}\ \{k \in \mathbf{N} : k \leq m\} \in \mathcal{FP}(\mathbf{N}) \tag{1.20}$$

is true in the standard model $\mathcal{X}$. By the Transfer Principle, the sentence

$$\forall m \in {}^*\mathbf{N}\ \{k \in {}^*\mathbf{N} : k \leq m\} \in {}^*(\mathcal{FP}(\mathbf{N})) \tag{1.21}$$

is true, so B is hyperfinite.

Remark 1.13.3 The Transfer Principle implies that hyperfinite sets possess all the formal properties of finite sets.

Theorem 1.13.4 *Suppose $^* : \mathcal{X} \to \mathcal{Y}$ is a nonstandard extension. If $B \in \mathcal{X}$ and $n \in {}^*\mathbf{N} \setminus \mathbf{N}$, then there exists a hyperfinite set D with $|D| < n$ such that $x \in B \Rightarrow {}^*x \in D$.*

Proof: See Theorem 2.8.3. ■

Proposition 1.13.5 *Suppose that $^* : \mathcal{X} \to \mathcal{Y}$ is a nonstandard extension. Suppose that B is hyperfinite and $A \subset B$, A internal. Then A is hyperfinite.*

Proof: See Proposition 2.6.5. ■

1.14 Nonstandard Theorems Have Standard Proofs

Although nonstandard proofs never make use of the details of the ultraproduct construction, the construction shows that the existence of nonstandard models with the assumed properties follows from the usual axioms of mathematics. Any nonstandard proof can be rephrased as a proof from the usual axioms by reinterpreting each line in the proof as a statement about ultraproducts. Consequently, any theorem about the standard world which has a nonstandard proof is guaranteed to have a standard proof, although the proof could be exceedingly complex and unintuitive. The important point is that, if we present a nonstandard proof of a standard statement, we know that the statement follows from the usual axioms of mathematics.

Chapter 2

Nonstandard Analysis Regular

2.1 Warning: Do Not Read this Chapter

There is nothing innately difficult about the mathematical logic presented in this Chapter. However, learning it does require a perspective that is a little foreign to many mathematicians and economists, and this may produce a certain degree of intimidation. Do not worry! At the first sign of queasiness, rip this Chapter and Appendix A out of the book, and go on to Chapter 3!

2.2 A Formal Language

In this Section, we specify a formal language $\mathcal{L}$ in which we can make statements about the superstructure $\mathcal{X}$. One must first specify the *atomic symbols*, the vocabulary in which the language is written. Next, one specifies grammar rules which determine whether a given string of atomic symbols is a valid *formula* in the language. The grammar rules are described inductively, showing how complicated formulas can be built up from simpler formulas. Implicit in the grammar rules is a unique way to parse each formula, so that there is only one sequence of applications of the rules that yields the given formula. Together, the atomic symbols and the grammar rules determine the *syntax* of the language. All formulas will be given the conventional mathematical interpretation; the formal specification of the interpretation process is given in Section 2.4.

Definition 2.2.1 The *atomic symbols* of $\mathcal{L}$ are the following:

1. logical connectives $\vee$ $\wedge$ $\Rightarrow$ $\Leftrightarrow$ $\neg$
2. variables v_1 v_2 $\ldots$
3. quantifiers $\forall$ $\exists$
4. brackets [ ]
5. basic predicates $\in$ $=$
6. constant symbols C_x (one for each $x \in \mathcal{X}$)
7. function symbols C_f (one for each function f with domain x, range y and $x, y \in \mathcal{X}$)
8. set description symbols { : }
9. n-tuple symbols (,)

Definition 2.2.2 A finite string of atomic symbols is a *formula* of $\mathcal{L}$ if it can be derived by iterative application of the following rules:

1. If F is a variable or constant symbol, then F is a term;

2. If F is a term and C_f is a function symbol, then $C_f(F)$ is a term;

3. If F is a formula and G is a term, F does not contain the quantifier $\forall v_i$, the quantifier $\exists v_i$, or a term of the form $\{v_i \in H : J\}$, and G does not contain the variable v_i, then $\{v_i \in G : F\}$ is a term;

4. If $F_1, \ldots, F_n$ are terms, then $\{F_1, \ldots, F_n\}$ and $(F_1, \ldots, F_n)$ are terms;

5. If F and G are terms, then $[F \in G]$ and $[F = G]$ are formulas;

6. If F is a formula, $[\neg F]$ is a formula;

7. If F and G are formulas, $[F \vee G]$, $[F \wedge G]$, $[F \Rightarrow G]$, and $[F \Leftrightarrow G]$ are formulas;

8. If F is a formula and G is a variable, a constant symbol, or a term, and neither F nor G contains $\exists v_i$, $\forall v_i$, or a term of the form $\{v_i \in H : J\}$, then $[\exists v_i \in G[F]]$ and $[\forall v_i \in G[F]]$ are formulas.

2.3 Extensions of $\mathcal{L}$

In practice, mathematics is always written with a certain degree of informality, since strict adherence to the formulas of a formal language like $\mathcal{L}$ would make even simple arguments impenetrable. One of the key ways a formal language is extended is through the use of abbreviations. It is important to be able to recognize whether or not a given formula, expressed with the degree of informality commonly used in mathematics, is in fact expressible as a formula in the formal language. In the following examples, we show how to construct formulas in $\mathcal{L}$ using the grammar rules and abbreviations; we also give some examples of strings of atomic symbols which are not formulas.

Example 2.3.1 The symbol string $\forall v_1[v_1 = v_1]$ is not a formula, because the quantifier is not of the correct form. However, $\forall v_1 \in C_x[v_1 = v_1]$ is a formula of $\mathcal{L}$ provided x is an element of the superstructure $\mathcal{X}$. It is important that, every time a quantifier is used, we specify an element of the superstructure $\mathcal{X}$ over which it ranges.

Example 2.3.2 The symbol strings

$$\wedge[v_1 \forall C_X \tag{2.1}$$

$$v_1 \in v_2 \Leftrightarrow \neg \tag{2.2}$$

$$\forall[v_2 \in v_3] \tag{2.3}$$

are not formulas because they cannot be constructed by iterative application of the grammar rules.

Example 2.3.3 The induction axiom for the natural numbers $\mathbf{N}$ is the following:

$$\forall v_1 \in C_{\mathcal{P}(\mathbf{N})}\,[[v_1 = \emptyset] \vee [\exists v_2 \in v_1\,[\forall v_3 \in v_1[(v_2, v_3) \in C_x]]]] \tag{2.4}$$

where $x = \{(n, m) \in \mathbf{N}^2 : n \leq m\}$. Note that, although we could describe the set x in our language $\mathcal{L}$, there is no need for us to do so; since $x \in \mathcal{X}$, we know there is a constant symbol C_x corresponding to it already present in our language.

Given $x \in \mathcal{X}$, we can substitute x for the more cumbersome C_x, the constant symbol which corresponds to x. We can omit all brackets $[,]$ except those which are needed for clarity. We can substitute function and relation symbols such as $f(x)$ or $m \leq n$ for the more awkward $C_f(x)$ or $(m,n) \in C_x$ where x is the graph of the relation "$\leq$". Finally, we can substitute more descriptive variable names (by using conventions such as small letters to denote individual elements, capital letters sets, etc.).

Example 2.3.4 Using the above abbreviations, the induction axiom can be written as

$$\forall S \in \mathcal{P}(\mathbf{N})\ \exists n \in S\ \forall m \in S\ n \leq m; \tag{2.5}$$

since we know this is an abbreviation for a formula in $\mathcal{L}$, we know that we can treat it as if it were in $\mathcal{L}$.

Example 2.3.5 The grammar of the language $\mathcal{L}$ does not permit us to quantify directly over functions. However, we can easily get around this restriction by representing functions by their graphs. Given $A, B \in \mathcal{X}$, let $\mathcal{F}(A,B)$ denote the set of all functions from A to B, $\mathcal{G}(A,B)$ the set of graphs of functions in $\mathcal{F}(A,B)$. $\mathcal{G}(A,B) \in \mathcal{X}$. Consider the function $F_{AB} : \mathcal{G}(A,B) \times A \to B$ defined by

$$F_{AB}(\Gamma, a) = f(a) \text{ where } f \text{ is the function whose graph is } \Gamma. \tag{2.6}$$

Note that $F_{AB} \in \mathcal{F}(\mathcal{G}(A,B) \times A, B)$, so there is a function symbol in $\mathcal{L}$ corresponding to F_{AB}. The formula $\forall f \in \mathcal{F}(A,B)[G(f)]$ is an abbreviation for

$$\forall \Gamma \in \mathcal{G}(A,B)[G(F_{AB}(\Gamma, \cdot)]. \tag{2.7}$$

For example, the sentence

$$\forall f \in \mathcal{F}([0,1], \mathbf{R}) \exists x \in [0,1] \forall y \in [0,1] f(x) \geq f(y) \tag{2.8}$$

which asserts (falsely) that every function from $[0,1]$ to $\mathbf{R}$ assumes its maximum is an abbreviation for the following sentence in $\mathcal{L}$:

$$\forall\Gamma \in \mathcal{G}([0,1],\mathbf{R})\exists x \in [0,1]\forall y \in [0,1] F_{[0,1]\mathbf{R}}(\Gamma, x) \geq F_{[0,1]\mathbf{R}}(\Gamma, y) \tag{2.9}$$

Example 2.3.6 As noted in Section 1.5, summation is a function defined on n-tuples or sequences. Thus, expressions like $\sum_{n\in\mathbf{N}} x_n$ are abbreviations for formulas in L.

Example 2.3.7 Suppose $x, y \in \mathcal{X}_n$. We can define a relation $\subset$ by saying that $x \subset y$ is an abbreviation for $\forall z \in \mathcal{X}_n[z \in x \Rightarrow z \in y]$. In the induction axiom discussed in the previous example, we did not write

$$\forall S \subset \mathbf{N}\ \exists n \in S\ \forall m \in S\ \ n \leq m \tag{2.10}$$

because the quantifier is required to be of the form $\forall S \in C$ for some set $C \in \mathcal{X}$. A key issue in nonstandard models is the interpretation of quantifiers over subsets; for this reason, we will not use the abbreviation $\subset$ within quantifiers.

2.4 Assigning Truth Value to Formulas

We shall be interested in interpreting formulas in the standard universe, as well as in nonstandard models. For this reason, it is important that we specify precisely the procedure by which formulas are interpreted. In the standard world, every formula F in $\mathcal{L}$ is interpreted according to the conventional rules for interpreting mathematical formulas. Thus, the logical symbol $\vee$ will have the conventional interpretation "or," while the quantifier $\forall$ has the conventional interpretation "for all." If all the variables appearing in F appear within the scope of a quantifier, then F can be assigned an unambiguous truth value; if not, then the truth value of F depends on how we choose to interpret the variables which are not quantified. This leads to the following set of definitions.

Definition 2.4.1 An occurrence of a variable v_i in a formula F is *bound* if there is a formula E such that $F = \ldots E \ldots$,

$$\begin{gathered} E = [\forall v_i \in G[H]] \text{ or } E = [\exists v_i \in G[H]] \\ \text{or } F = \{v_i \in G : H\} \end{gathered} \tag{2.11}$$

and the occurrence of v_i is inside E. If the occurrence of v_i is not bound, it is said to be *free*. If every occurrence of each variable in F is bound, F is said to be a *sentence*.

Definition 2.4.2 An *interpretation* of $\mathcal{L}$ in $\mathcal{X}$ is a function I mapping $\{v_1, v_2, \ldots\}$ to $\mathcal{X}$.

In the next definition, we show how to extend an interpretation so that it assigns an element of $\mathcal{X}$ to each term and a truth value **t** ("true") or **f** ("false") to each formula.

Definition 2.4.3 The *value of a constant symbol or set description J under the interpretation I*, denoted $I(J)$, and the *truth value of a formula F under the interpretation I*, denoted $I(F)$, are defined inductively as follows:

1. For any constant symbol C_x, $I(C_x) = x$ (i.e. every constant symbol is interpreted as the corresponding element of $\mathcal{X}$).

2. If F is $C_f(G)$, where C_f is a function symbol and G is a term, then $I(F) = f(I(G))$ if $I(G)$ is defined and $I(G)$ is an element of the domain of f; otherwise, $I(F)$ is undefined.

3. If J is a term $\{v_i \in H : K\}$, then

$$I(J) = \{x \in I(H) : I'(K) = \mathbf{t} \text{ where } I'(v_i) = x \text{ and } I'(v_j) = I(v_j) \text{ for } j \neq i\} \quad (2.12)$$

   if $I(H)$ is defined, and $I(J)$ is undefined otherwise.

4. (a) If F is $(F_1, \ldots, F_n)$, then

$$I(F) = \begin{cases} (I(F_1), \ldots, I(F_n)) & \text{if } I(F_1), \ldots, I(F_n) \text{ are all defined;} \\ \text{undefined} & \text{otherwise.} \end{cases} \quad (2.13)$$

   (b) If F is $\{F_1, \ldots, F_n\}$, then

$$I(F) = \begin{cases} \{I(F_1), \ldots, I(F_n)\} & \text{if } I(F_1), \ldots, I(F_n) \text{ are all defined;} \\ \text{undefined} & \text{otherwise.} \end{cases} \quad (2.14)$$

5. (a) If F is $[G \in H]$, where G and H are terms, then $I(F) = \mathbf{t}$ if $I(G)$ and $I(H)$ are defined and $I(G) \in I(H)$, while $I(F) = \mathbf{f}$ otherwise.

   (b) If F is $[G = H]$, where G and H are terms, then $I(F) = \mathbf{t}$ if $I(G)$ and $I(H)$ are defined and $I(G) = I(H)$, while $I(F) = \mathbf{f}$ otherwise.

6. If F is $[\neg G]$, then $I(F) = \mathbf{t}$ if $I(G) = \mathbf{f}$ and $I(F) = \mathbf{f}$ if $I(G) = \mathbf{t}$.

7. (a) If F is $[G \vee H]$ then $I(F) = \mathbf{t}$ if $I(G) = \mathbf{t}$ or $I(H) = \mathbf{t}$ and $I(F) = \mathbf{f}$ otherwise;

   (b) If F is $[G \wedge H]$ then $I(F) = \mathbf{t}$ if $I(G) = \mathbf{t}$ and $I(H) = \mathbf{t}$ and $I(F) = \mathbf{f}$ otherwise;

(c) If F is $[G \Rightarrow H]$ then $I(F) = \mathbf{t}$ if $I(G) = \mathbf{f}$ or $I(H) = \mathbf{t}$ and $I(F) = \mathbf{f}$ otherwise;

(d) If F is $[G \Leftrightarrow H]$ then $I(F) = \mathbf{t}$ if $I(G) = \mathbf{t}$ and $I(H) = \mathbf{t}$ or $I(G) = \mathbf{f}$ and $I(H) = \mathbf{f}$ and $I(F) = \mathbf{f}$ otherwise.

8. (a) If F is $[\forall v_i \in G[H]]$, then $I(F) = \mathbf{t}$ if $I(G)$ is defined and, for every $x \in \mathcal{X}$, $I'(H) = \mathbf{t}$, where $I'(v_i) = x$ and $I'(v_j) = I(v_j)$ for every $j \neq i$; $I(F) = \mathbf{f}$ otherwise.

(b) If F is $[\exists v_i \in G[H]]$, then $I(F) = \mathbf{t}$ if $I(G)$ is defined and there exists some $x \in I(G)$ such that $I'(H) = \mathbf{t}$, where $I'(v_i) = x$ and $I'(v_j) = I(v_j)$ for every $j \neq i$; $I(F) = \mathbf{f}$ otherwise.

Remark 2.4.4 (The King of France Has a Beard) It would be possible, but exceedingly tedious, to restrict our language so that it is impossible to write formulas that contain expressions of the form $f(x)$ where x is outside the domain of f. We have instead taken the route of allowing them in the language, and have specified a more or less arbitrary assignment of truth value. In practice, such nonsensical formulas are easily spotted, and so no problems will arise. Note that, assuming for concreteness that $f : \mathbf{N} \to \mathbf{N}$, our assignment of truth value has the following consequences:

- $I(f(\pi) = 0) = \mathbf{f}$.
- The formula $f(\pi) \neq 0$ is ambiguous. It is not an element of $\mathcal{L}$.
  - If it is an abbreviation for $\neg[f(\pi) = 0]$, then $I(f(\pi) \neq 0) = \mathbf{t}$.
  - If it is an abbreviation for $(f(x), 0) \in C$ where $C = \{(y, z) \in \mathbf{N}^2 : y \neq z\}$, then $I(f(\pi) \neq 0) = \mathbf{f}$.

Proposition 2.4.5 *If F is a sentence, then $I(F)$ is independent of the interpretation I.*

Proof: Let v_i be a variable which occurs in F. Since every occurrence of v_i is bound, $I(F)$ is independent of $I(v_i)$ by items 3 and 8 of Definition 2.4.3. Thus, $I(F)$ is independent of I. ■

Definition 2.4.6 Suppose F is a sentence. We say F *holds in* $\mathcal{X}$ if $I(F) = \mathbf{t}$ for some (and thus every) interpretation I of $\mathcal{L}$ in $\mathcal{X}$, and F *fails in* $\mathcal{X}$ if $I(F) = \mathbf{f}$ for some (and thus every) interpretation I of $\mathcal{L}$ in $\mathcal{X}$.

2.5 Interpreting Formulas in Superstructure Embeddings

Suppose that * is a superstructure embedding from $\mathcal{X}$ to $\mathcal{Y}$. We shall define the first of two languages which we will use to describe $\mathcal{Y}$.

Definition 2.5.1 The *internal language* ${}^*\mathcal{L}$ is defined in exactly the same way as the language $\mathcal{L}$, except that the set of constant symbols is $\{C_x : x \in {}^*\mathcal{X}\} = \{C_y : y \in \mathcal{Y}, y \text{ internal}\}$ and the set of function symbols is $\{C_f : f$ is an internal function from x to y for some $x, y \in {}^*\mathcal{X}\}$. An *interpretation of* ${}^*\mathcal{L}$ in ${}^*\mathcal{X}$ is a function $I : \{v_1, v_2, \ldots\} \to {}^*\mathcal{X}$.

Given an interpretation I of ${}^*\mathcal{L}$ in ${}^*\mathcal{X}$, we can extend it to assign truth values to formulas in ${}^*\mathcal{L}$ in exactly the same way as in Definition 2.4.3.

Remark 2.5.2 Definition 2.5.1 forces the interpretation of a variable to be internal. The astute reader may wonder whether, given an interpretation I, $I(J)$ is internal for complex terms J such as $\{v_i \in G : F\}$ or $C_f(F)$. If the superstructure embedding satisfies the Internal Definition Principle, and I is an interpretation of ${}^*\mathcal{L}$ in ${}^*\mathcal{X}$, then $I(J)$ is internal for every term in ${}^*\mathcal{L}$. However, since the assignment of truth values in the previous definition makes sense whether or not $I(J)$ is internal, we will defer the formal discussion of the Internal Definition Principle to Section 2.7.

Proposition 2.5.3 *If F is a sentence in ${}^*\mathcal{L}$, then $I(F)$ is independent of the interpretation I.*

Definition 2.5.4 Suppose F is a sentence in ${}^*\mathcal{L}$. We say F *holds in* ${}^*\mathcal{X}$ if $I(F) = \mathbf{t}$ for some interpretation I of ${}^*\mathcal{L}$ in ${}^*\mathcal{X}$, and F *fails in* $\mathcal{Y}$ if ${}^*I(F) = \mathbf{f}$ for some interpretation I of ${}^*\mathcal{L}$ in ${}^*\mathcal{X}$.

2.6 Transfer Principle

Definition 2.6.1 Given an interpretation I of $\mathcal{L}$ in $\mathcal{X}$, we can associate an interpretation ${}^*I$ of ${}^*\mathcal{L}$ in ${}^*\mathcal{X}$ by specifying $({}^*I)(v_i) = {}^*(I(v_i))$ for each variable v_i. Given a formula $F \in \mathcal{L}$, we associate a formula ${}^*F \in {}^*\mathcal{L}$ by replacing each constant symbol C_x ($x \in \mathcal{X}$) with the constant symbol $C_{{}^*x}$ and each function symbol C_f with the function symbol $C_{{}^*f}$.

Definition 2.6.2 A superstructure embedding $*$ from $\mathcal{X}$ to $\mathcal{Y}$ satisfies the *Transfer Principle* if, for every formula $F \in \mathcal{L}$ and every interpretation I of $\mathcal{L}$ in $\mathcal{X}$,

$$I(F) = \mathbf{t} \text{ if and only if} {}^*I({}^*F) = \mathbf{t}. \tag{2.15}$$

Proposition 2.6.3 *If a superstructure embedding $*$ from $\mathcal{X}$ to $\mathcal{Y}$ satisfies the Transfer Principle, and F is a sentence in $\mathcal{L}$, then F holds in $\mathcal{X}$ if and only if ${}^*F$ holds in ${}^*\mathcal{X}$.*

Remark 2.6.4 The astute reader may have noticed a very important point arising in the case that F involves quantifiers over sets or other objects not in $\mathcal{X}_0$. Assume for concreteness that we are considering a formula $F = [\forall v_1 \in C_{\mathcal{P}(X)}[G]]$; $I(F) = \mathbf{t}$ if the property specified by G holds for every subset of X. ${}^*I(F) = \mathbf{t}$ if the property specified by G holds for every *element of* ${}^*(\mathcal{P}(X)) \subset \mathcal{P}({}^*X) = \mathcal{P}(\mathcal{Y}_0)$; thus, the property specified by G need hold only for *internal* subsets of $\mathcal{Y}_0$; it need not hold for external subsets of $\mathcal{Y}_0$.

Proposition 2.6.5 *Suppose that the superstructure embedding * from $\mathcal{X}$ to $\mathcal{Y}$ satisfies the Transfer Principle, B is hyperfinite, and $A \subset B$, A internal. Then A is hyperfinite.*

Proof: Since B is hyperfinite, $B \in {}^*(\mathcal{FP}(C))$ for some $C \in \mathcal{X}$. Since A is internal, and $A \subset {}^*C$, $A \in {}^*\mathcal{P}(C)$ by item 11 of Definition 1.7.1. The sentence

$$\forall B \in \mathcal{FP}(C) \forall A \in \mathcal{P}(C) \;\; [[\forall x \in C[x \in A \Rightarrow x \in B]] \Longrightarrow [A \in \mathcal{FP}(C)]], \tag{2.16}$$

which asserts that a subset of a finite subset is finite, holds in $\mathcal{X}$; by transfer, the sentence

$$\forall B \in {}^*\mathcal{FP}(C) \forall A \in {}^*\mathcal{P}(C) \;\; [[\forall x \in {}^*C[x \in A \Rightarrow x \in B]] \Longrightarrow [A \in {}^*\mathcal{FP}(C)]], \tag{2.17}$$

holds in ${}^*\mathcal{X}$. Thus, $A \in {}^*\mathcal{FP}(C)$, so A is hyperfinite. ■

2.7 Internal Definition Principle

Definition 2.7.1 A superstructure embedding * from $\mathcal{X}$ to ${}^*\mathcal{X}$ satisfies the *internal definition principle* if, for every term J in ${}^*\mathcal{L}$, and every interpretation I of ${}^*\mathcal{L}$ in ${}^*\mathcal{X}$, $I(J)$ is internal or undefined.

2.8 Nonstandard Extensions

Definition 2.8.1 A *nonstandard extension* of a superstructure $\mathcal{X}$ is a superstructure embedding $^*\colon \mathcal{X} \to \mathcal{Y}$ which is saturated (Definition 1.10.1) and satisfies the Transfer Principle (Definition 2.6.1) and the Internal Definition Principle (Definition 2.7.1).

Theorem 2.8.2 *Suppose* $^* : \mathcal{X} \to \mathcal{Y}$ *is a nonstandard extension, and* $\mathbf{N} \subset X$*. Then* $^*\mathbf{N} \setminus \mathbf{N} \neq \emptyset$*.* $n \in {}^*\mathbf{N} \setminus \mathbf{N} \Rightarrow n > m$ *for every* $m \in \mathbf{N}$*.*

Proof: Given $m \in \mathbf{N}$, let $A_m = \{n \in {}^*\mathbf{N} : n > m\}$. A_m is internal by the Internal Definition Principle; $A_m \supset \{n \in \mathbf{N} : n > m\}$, so $A_m \neq \emptyset$. Given $m_1, \ldots, m_k \in \mathbf{N}$ with $k \in \mathbf{N}$, $\cap_{i=1}^k A_{m_i} = A_{\max\{m_1,\ldots,m_k\}} \neq \emptyset$. By Saturation $\cap_{m\in\mathbf{N}} A_m \neq \emptyset$. Take any $n \in \cap_{m\in\mathbf{N}} A_m$; then $n \in {}^*\mathbf{N}$, but $n > m$ for every $m \in \mathbf{N}$, so $n \in {}^*\mathbf{N} \setminus \mathbf{N}$.

The sentence

$$\forall n \in \mathbf{N}\ \forall m \in \mathbf{N}\ \ [[m < n] \vee [m = n] \vee [m > n]] \tag{2.18}$$

holds in $\mathcal{X}$; by Transfer, the sentence

$$\forall n \in {}^*\mathbf{N}\ \forall m \in {}^*\mathbf{N}\ \ [[m < n] \vee [m = n] \vee [m > n]] \tag{2.19}$$

holds in $^*\mathcal{X}$. Now suppose n is any element of $^*\mathbf{N} \setminus \mathbf{N}$, and fix $m \in \mathbf{N}$. Since * is a superstructure embedding, $m \in {}^*\mathbf{N}$. Since $n \in {}^*\mathbf{N} \setminus \mathbf{N}$, $m \neq n$. Suppose $n < m$. The sentence

$$\forall n \in \mathbf{N}\ \ [m < n \Rightarrow [n = 1 \vee n = 2 \vee \ldots \vee n = m - 1]] \tag{2.20}$$

holds in $\mathbf{N}$. By Transfer, we must have $n = 1 \vee n = 2 \vee \ldots \vee n = m - 1$, so $n \in \mathbf{N}$. Accordingly, $n \in {}^*\mathbf{N} \setminus \mathbf{N}$ implies $n > m$ for every $m \in \mathbf{N}$. ∎

Theorem 2.8.3 *Suppose* $^* : \mathcal{X} \to \mathcal{Y}$ *is a nonstandard extension. If* $B \in \mathcal{X}$ *and* $n \in {}^*\mathbf{N} \setminus \mathbf{N}$*, then there exists a hyperfinite set* D *with* $|D| < n$ *such that* $x \in B \Rightarrow {}^*x \in D$*.*

Proof: Since $n \in {}^*\mathbf{N} \setminus \mathbf{N}$, $n > m$ for each $m \in \mathbf{N}$ by Theorem 2.8.2. Let $\Lambda = B$, $A_\lambda = \{D \in {}^*\mathcal{FP}(B) : {}^*\lambda \in D, |D| < n\}$. A_λ is internal by the Internal Definition Principle and $|\Lambda| = |B| < |\mathcal{X}|$. Given $\lambda_1, \ldots, \lambda_m$ with $m \in \mathbf{N}$, $\{{}^*\lambda_1, \ldots, {}^*\lambda_m\} \in \cap_{i=1}^m A_{\lambda_i}$, so the intersection is not empty. Accordingly, $\cap_{\lambda \in \Lambda} A_\lambda \neq \emptyset$ by saturation; if D is any element of $\cap_{\lambda \in \Lambda} A_\lambda$, then $D \in {}^*\mathcal{FP}(B)$, so D is hyperfinite, $|D| < n$, and $D \supset \{{}^*x : x \in B\}$. ■

2.9 The External Language

We shall define a language $\hat{\mathcal{L}}$ which permits us to refer to external objects, including the set of ordinary natural numbers $\mathbf{N}$, viewed as a subset of its nonstandard extension ${}^*\mathbf{N}$, or the set of all nonstandard real numbers infinitely close to a given real number. We can use all the usual axioms of conventional mathematics to make arguments involving formulas in $\hat{\mathcal{L}}$; however, the Transfer Principle *cannot* be applied to formulas in $\hat{\mathcal{L}}$.

Definition 2.9.1 The *external language* $\hat{\mathcal{L}}$ is defined in exactly the same way as the language $\mathcal{L}$, except that the set of constant symbols is $\{C_y : z \in \mathcal{Y}\}$ and the set of function symbols is $\{C_f : f$ is a function from x to y for some $x, y \in \mathcal{Y}\}$. An *interpretation* of $\hat{\mathcal{L}}$ in $\mathcal{Y}$ is a function from $\{v_1, v_2, \ldots\}$ to $\mathcal{Y}$. Given a formula F in $\hat{\mathcal{L}}$ and such an interpretation I, the truth value $I(F)$ is defined exactly as in Definition 2.4.3. If F is a sentence of $\hat{\mathcal{L}}$, we say that F *holds* in $\mathcal{Y}$ if $I(F) = \mathbf{t}$ for some interpretation of $I(F)$ in $\mathcal{Y}$, and F *fails* in $\mathcal{Y}$ otherwise.

2.10 The Conservation Principle

The usual axiomatization of set theory is due to Zermelo and Frankel. The Zermelo-Frankel axioms, with the addition of the Axiom of Choice, are collectively referred to as ZFC.

Theorem 2.10.1 (Los,Robinson,Luxemburg) *Let $\mathcal{X}$ be a superstructure in a model of ZFC. Then there exists a nonstandard extension* $*: \mathcal{X} \to \mathcal{Y}$.

Proof: See Appendix A. ■

Corollary 2.10.2 (The Conservation Principle) *Let $\mathcal{A}$ be a set of axioms containing the axioms of ZFC. Suppose F is a sentence in $\mathcal{L}$ which can be deduced from $\mathcal{A}$ and the assumption that there exists a nonstandard extension $* : \mathcal{X} \to \mathcal{Y}$. Then F has a proof using the axioms of $\mathcal{A}$ alone.*

Chapter 3

Euclidean, Metric and Topological Spaces

In this Chapter, we explore the nonstandard formulation of the basic results in Euclidean, Metric and Topological Spaces. The results are due for the most part to Robinson (1966) and Luxemburg (1969). The results stated here are of considerable use in applications of nonstandard analysis to economics. In addition, the proofs given here illustrate how the properties of nonstandard extensions are used in writing proofs. We form a superstructure by taking $\mathcal{X}_0$ to be the union of the point sets of all the spaces under consideration, and suppose that $^* : \mathcal{X} \to \mathcal{Y}$ is a nonstandard extension.

3.1 Monads

Definition 3.1.1 Suppose $(X, \mathcal{T})$ is a topological space. If $x \in X$, the *monad* of x, denoted $\mu(x)$, is $\cap_{x \in T \in \mathcal{T}} {}^*T$. If $y \in {}^*X$ and $y \in \mu(x)$, we write $y \simeq x$ (read "y is *infinitely close to* x").

Definition 3.1.2 Suppose (X, d) is a metric space. If $x \in$

$^*X$, the *monad* of x, denoted $\mu(x)$, is $\{y \in {}^*X : {}^*d(x,y) \simeq 0\}$. If $x, y \in {}^*X$ and $y \in \mu(x)$, we write $y \simeq x$ (read "y is *infinitely close to* x").

Proposition 3.1.3 *Suppose (X,d) is a metric space, and $x \in X$. Then the monad of x (viewing X as a metric space) equals the monad of x (viewing X as a topological space).*

Proof: Suppose y is in the metric monad of x. Then ${}^*d(x,y) \simeq 0$. Suppose $x \in T \in \mathcal{T}$. Then there exists $\delta \in \mathbf{R}_{++}$ such that the formula

$$d(z,x) < \delta \Rightarrow z \in T \tag{3.1}$$

holds in $\mathcal{X}$. By Transfer,

$${}^*d(z,x) < \delta \Rightarrow z \in {}^*T \tag{3.2}$$

holds in ${}^*\mathcal{X}$. Since this holds for each T satisfying $x \in T \in \mathcal{T}$, y is in the topological monad of x.

Conversely, suppose y is in the topological monad of x. Choose $\delta \in \mathbf{R}_{++}$ and let $T = \{z \in X : d(z,x) < \delta\}$. $x \in T \in \mathcal{T}$, so $y \in {}^*T$. Therefore, ${}^*d(x,y) < \delta$ for each $\delta \in \mathbf{R}_{++}$, so ${}^*d(x,y) \simeq 0$. Thus, y is in the metric monad of x. ■

Remark 3.1.4 The topological monad of x can be defined for an arbitrary $x \in {}^*X$, not just for $x \in X$. However, it is not very well behaved.

Proposition 3.1.5 (Overspill) *Suppose $(X,\mathcal{T})$ is a topological space, $x \in X$, and A is an internal subset of ${}^*X$.*

1. *If $x \in A \subset \mu(x)$, there exists $S \in {}^*\mathcal{T}$ with $A \subset S \subset \mu(x)$.*

2. *If $A \supset \mu(x)$, then $A \supset {}^*T$ for some T satisfying $x \in T \in \mathcal{T}$.*

3. *If $(X, \mathcal{T})$ is a T_1 space and $\mu(x)$ is internal, then $\mu(x) = \{x\} \in \mathcal{T}$.*

Proof: Let $\mathcal{T}' = \{T \in \mathcal{T} : x \in T\}$.

(1)Suppose $A \subset \mu(x)$. By Theorem 1.13.4, there exists a hyperfinite set $\mathcal{S} \subset {}^*\mathcal{T}'$ such that $T \in \mathcal{T}'$ implies ${}^*T \in \mathcal{S}$. Let $\mathcal{S}' = \{T \in \mathcal{S} : T \supset A\}$; S' is internal by the Internal Definition Principle, and hyperfinite by Proposition 1.13.5. Let $S = \cap_{T \in \mathcal{S}'} T$. Then $A \subset S \subset \mu(x)$. Since $\mathcal{T}$ is closed under finite intersections, ${}^*\mathcal{T}$ is closed under hyperfinite intersections, by Transfer. Therefore $S \in {}^*\mathcal{T}$.

(2) Suppose $A \supset \mu(x)$. Given $T \in \mathcal{T}'$, let $A_T = {}^*T \setminus A$. A_T is internal by the Internal Definition Principle. $\cap_{T \in \mathcal{T}'} A_T = (\cap_{T \in \mathcal{T}'} {}^*T) \setminus A = \mu(x) \setminus A = \emptyset$. By Saturation, there exist $T_1, \ldots, T_n$ ($n \in \mathbf{N}$) such that $\cap_{i=1}^n A_{T_i} = \emptyset$, so $A \supset \cap_{i=1}^n {}^*T_i = {}^*(\cap_{i=1}^n T_i)$. Since $x \in \cap_{i=1}^n T_i \in \mathcal{T}$, the proof of (2) is complete.

(3) Suppose $\mu(x)$ is internal and $(X, \mathcal{T})$ is a T_1 space. By (2), there exists $T \in {}^*\mathcal{T}$ such that $\mu(x) \subset {}^*T \subset \mu(x)$, so $\mu(x) = {}^*T$. If $y \in X$, $y \neq x$, then there exists $S \in \mathcal{T}$ with $x \in S$ and $y \notin S$. By Transfer, $y \notin {}^*S$, so $y \notin \mu(x) = {}^*T$. By Transfer, $y \notin T$. Since y is an arbitrary element of X different from x, $T = \{x\}$. Then $\mu(x) = {}^*T = \{x\}$ by Transfer. ■

Proposition 3.1.6 (Overspill) *Suppose A is an internal subset of ${}^*\mathbf{N}$.*

1. *If $A \supset {}^*\mathbf{N} \setminus \mathbf{N}$, then $A \supset \{n, n+1, \ldots\}$ for some $n \in \mathbf{N}$.*

2. *If $A \supset \mathbf{N}$, then $A \supset \{1, 2, \ldots, n\}$ for some $n \in {}^*\mathbf{N} \setminus \mathbf{N}$.*

Proof: We could prove this as a corollary of Proposition 3.1.5 by considering $X = \mathbf{N} \cap \{\infty\}$ with the one-point compactification metric. However, we shall present a direct proof.

Every nonempty subset of $\mathbf{N}$ has a first element; by transfer, every nonempty internal subset of $^*\mathbf{N}$ has a first element.

(1) Let $B = \{n \in {}^*\mathbf{N} : \forall m \in {}^*\mathbf{N}[m \geq n \Rightarrow m \in A]\}$. B is internal by the Internal Definition Principle. Let n be the first element of B. Since $A \supset {}^*\mathbf{N} \setminus \mathbf{N}$, $B \supset {}^*\mathbf{N} \setminus \mathbf{N}$, so $n \in \mathbf{N}$.

(2) Let $B = \{n \in {}^*\mathbf{N} : \forall m \in {}^*\mathbf{N}[m \leq n \Rightarrow m \in A]\}$. B and ${}^*\mathbf{N} \setminus B$ are internal by the Internal Definition Principle. If ${}^*\mathbf{N} \setminus B = \emptyset$, we are done. Otherwise, let n be the first element of ${}^*\mathbf{N} \setminus B$. Since $A \supset \mathbf{N}$, $B \supset \mathbf{N}$, so $n \in {}^*\mathbf{N} \setminus \mathbf{N}$. ∎

Proposition 3.1.7 *Suppose $(X, \mathcal{T})$ is a topological space. Then X is Hausdorff if and only if for every every $x, y \in X$ with $x \neq y$, $\mu(x) \cap \mu(y) = \emptyset$.*

Proof: Suppose X is Hausdorff. If $x, y \in X$ and $x \neq y$, we may find S, T with $x \in S \in \mathcal{T}$, $y \in T \in \mathcal{T}$, and $S \cap T = \emptyset$. ${}^*S \cap {}^*T = {}^*(S \cap T) = \emptyset$, by Transfer. $\mu(x) \cap \mu(y) \subset {}^*S \cap {}^*T = \emptyset$.

Comversely, suppose $\mu(x) \cap \mu(y) = \emptyset$. By Proposition 3.1.5, we may find $S, T \in {}^*\mathcal{T}$ with $x \in S \subset \mu(x)$ and $y \in T \subset \mu(y)$, and thus $S \cap T = \emptyset$. Thus, the sentence

$$\exists S \in {}^*\mathcal{T} \ \ \exists T \in {}^*\mathcal{T} \ \ [x \in S \ \wedge \ y \in T \ \wedge \ S \cap T = \emptyset] \qquad (3.3)$$

holds in ${}^*\mathcal{X}$. By Transfer, the sentence

$$\exists S \in \mathcal{T} \ \ \exists T \in \mathcal{T} \ \ [x \in S \ \wedge \ y \in T \ \wedge \ S \cap T = \emptyset] \qquad (3.4)$$

holds in $\mathcal{X}$, so X is Hausdorff. ∎

Definition 3.1.8 If $(X, \mathcal{T})$ is a Hausdorff space, $y \in {}^*X$, and $y \in \mu(x)$, we write $x = {}^\circ y$ (read "x is the *standard part* of y").

Proposition 3.1.9 *Suppose* $\{x_n : n \in {}^*\mathbf{N}\}$ *is an internal sequence of elements of* ${}^*\mathbf{R}$. *Then the standard sequence* $\{{}^\circ x_n : n \in \mathbf{N}\}$ *converges to* $x \in \mathbf{R}$ *if and only if there exists* $n_0 \in {}^*\mathbf{N} \setminus \mathbf{N}$ *such that* $x_n \simeq x$ *for every* $n \leq n_0, n \in {}^*\mathbf{N} \setminus \mathbf{N}$.

Proof: Suppose ${}^\circ x_n$ converges to x. Fix $\delta \in \mathbf{R}_{++}$. There exists $n_\delta \in \mathbf{N}$ such that $n \geq n_\delta, n \in \mathbf{N}$ implies $|{}^\circ x_n - x| < \delta/2$; thus, $|x_n - x| < \delta$. Thus, given $\delta \in \mathbf{R}_{++}$ and $k \in \mathbf{N}$, let $A_{\delta k} = \{n \in {}^*\mathbf{N} : n \geq k$ and $|x_m - x| < \delta$ for each $m \in \{k, \ldots, n\}\}$. For any finite collection $\{(\delta_1, k_1), \ldots (\delta_n, k_n)\}$ with $k_i \geq n_{\delta_i}$, $\cap_{i=1}^n A_{\delta_i k_i} \neq \emptyset$. Let $\Lambda = \{(\delta, k) \in \mathbf{R}_{++} \times \mathbf{N} : k \geq n_\delta\}$. By Saturation, $\cap_{(\delta,k)\in\Lambda} A_{\delta k} \neq \emptyset$. Choose $n_0 \in \cap_{(\delta,k)\in\Lambda} A_{\delta k}$. Then $n_0 \in {}^*\mathbf{N} \setminus \mathbf{N}$; given $n \in {}^*\mathbf{N} \setminus \mathbf{N}$ with $n \leq n_0$, $|x_n - x| \simeq 0$.

Conversely, suppose there exists $n_0 \in {}^*\mathbf{N} \setminus \mathbf{N}$ such that $n \in {}^*\mathbf{N}$, $n \leq n_0$ implies $x_n \simeq x$. Given $\delta \in \mathbf{R}_{++}$, let $A = \{n \in {}^*\mathbf{N} : |x_n - x| < \delta/2\} \cup \{n \in {}^*\mathbf{N} : n > n_0\}$. A is internal and contains ${}^*\mathbf{N} \setminus \mathbf{N}$. By Proposition 3.1.6, $A \supset \{n, n+1, \ldots\}$ for some $n \in \mathbf{N}$. Thus, $|{}^\circ x_m - x| < \delta$ for $m \in \mathbf{N}$ satisfying $m \geq n$. Therefore ${}^\circ x_n$ converges to x. ■

Proposition 3.1.10 *Suppose* $\{x_n : n \in \mathbf{N}\}$ *is a sequence of elements of* $\mathbf{R}$. *Then* $x_n \to x \in \mathbf{R}$ *if and only if* $x_n \simeq x$ *for every* $n \in {}^*\mathbf{N} \setminus \mathbf{N}$.

Proof: Suppose $x_n \to x$. Given $\epsilon \in \mathbf{R}_{++}$, there exists $n_0 \in \mathbf{N}$ such that the sentence

$$\forall n \in \mathbf{N}[n \geq n_0 \Rightarrow |x_n - x| < \epsilon] \tag{3.5}$$

holds in $\mathcal{X}$. By Transfer, the sentence

$$\forall n \in {}^*\mathbf{N}[n \geq n_0 \Rightarrow |x_n - x| < \epsilon] \tag{3.6}$$

holds in ${}^*\mathcal{X}$. If $n \in {}^*\mathbf{N} \setminus \mathbf{N}$, then $|x_n - x| < \epsilon$; since ϵ is an arbitrary element of $\mathbf{R}_{++}$, $x_n \simeq x$.

Conversely, suppose $x_n \simeq x$ for all $n \in {}^*\mathbf{N} \setminus \mathbf{N}$. For $n \in \mathbf{N}$, ${}^\circ x_n = x_n$, so $x_n \to x$ by Proposition 3.1.9. ■

3.2 Open and Closed Sets

Proposition 3.2.1 *Suppose $(X, \mathcal{T})$ is a topological space. Then $A \subset X$ is open if and only if $\mu(x) \subset {}^*A$ for every $x \in A$.*

Proof: If A is open and $x \in A$, then $\mu(x) \subset {}^*A$ by Definition 3.1.1. Conversely, suppose $\mu(x) \subset {}^*A$ for every $x \in A$. By Proposition 3.1.5, we may find $S \in {}^*\mathcal{T}$ with $x \in S \subset \mu(x)$. Thus, the sentence

$$\exists S \in {}^*\mathcal{T} \;\; x \in S \subset {}^*A \tag{3.7}$$

holds in ${}^*\mathcal{X}$, so the sentence

$$\exists S \in \mathcal{T} \;\; x \in S \subset A \tag{3.8}$$

holds in $\mathcal{X}$ by Transfer. Thus, A is open. ■

Proposition 3.2.2 *Suppose $(X, \mathcal{T})$ is a topological space. Then $A \subset X$ is closed if and only if $y \in {}^*A$ implies $x \in A$ for every $x \in X$ such that $y \in \mu(x)$.*

Proof: Let $B = X \setminus A$.

Suppose A is closed. If $y \in {}^*A$ and $y \in \mu(x)$ with $x \in X \setminus A$, then $x \in B$. Since A is closed, B is open. Since $y \in \mu(x)$, $y \in {}^*B$ by Proposition 3.2.1. ${}^*B \cap {}^*A = {}^*(B \cap A) = \emptyset$, by Transfer. Thus, $y \notin {}^*A$, a contradiction.

Conversely, suppose $y \in {}^*A$ implies $x \in A$ for every $x \in X$ such that $y \in \mu(x)$. Suppose $x \in B$. Then we must have $y \in {}^*X \setminus {}^*A = {}^*B$ for every $y \in \mu(x)$. Accordingly, B is open by Proposition 3.2.1, so A is closed. ■

Proposition 3.2.3 *Suppose$(X, \mathcal{T})$ is a topological space, and $A \subset {}^*X$ is internal. Then $\{x \in X : \exists y \in A \; [y \in \mu(x)]\}$ is closed.*

Proof: Let $C = \{x \in X : \exists y \in A\, [y \in \mu(x)]\}$; we shall show that $B = X \setminus C$ is open. Let $D = {}^*X \setminus A$; D is internal by the Internal Definition Principle. If $x \in B$, then $D \supset \mu(x)$, so $D \supset {}^*T$ for some T satisfying $x \in T \in \mathcal{T}$, by Proposition 3.1.5. If $y \in T$, then $\mu(y) \subset {}^*T$, so $D \supset \mu(y)$, so $y \in B$. Thus, B is open, so C is closed. ■

3.3 Compactness

Definition 3.3.1 Let $(X, \mathcal{T})$ be a topological space and $y \in {}^*X$. We say y is *nearstandard* if there exists $x \in X$ such that $y \simeq x$. We let $ns({}^*X)$ denote the set of nearstandard points in ${}^*X$.

Theorem 3.3.2 *Let $(X, \mathcal{T})$ be a topological space. Then $(X, \mathcal{T})$ is compact if and only if every $y \in {}^*X$ is nearstandard.*

Proof: Suppose $(X, \mathcal{T})$ is compact, and there is some $y \in {}^*X$ which is not nearstandard. Then for every $x \in X$, there exists T_x with $x \in T_x \in \mathcal{T}$ and $y \notin {}^*T_x$. $\{T_x : x \in X\}$ is thus an open cover of X; let $\{T_{x_1}, \ldots, T_{x_n}\}$ be a finite subcover (so $n \in \mathbf{N}$). Since * is a superstructure embedding, $\cup_{i=1}^n {}^*T_{x_i} = {}^*(\cup_{i=1}^n T_{x_i}) = {}^*X$, so $y \notin {}^*X$, a contradiction.

Conversely, suppose that every $y \in {}^*X$ is nearstandard. Let $\{T_\lambda : \lambda \in \Lambda\}$ be an open cover of X. Let $C_\lambda = X \setminus T_\lambda$. If there is no finite subcover, then for every collection $\{\lambda_1, \ldots, \lambda_n\}$ with $n \in \mathbf{N}$, $\cap_{i=1}^n C_{\lambda_i} \neq \emptyset$. $\cap_{i=1}^n {}^*C_{\lambda_i} = {}^*(\cap_{i=1}^n C_{\lambda_i}) \neq \emptyset$. $|\Lambda| \leq |\mathcal{X}_1|$, so by saturation, $C = \cap_{\lambda \in \Lambda} {}^*C_\lambda \neq \emptyset$. Choose any $y \in C$. Given $x \in X$, there exists λ such that $x \in T_\lambda$. Since $y \in C \subset {}^*C_\lambda$, $y \notin {}^*T_\lambda$, so $y \not\simeq x$. Since x is an arbitrary element of X, y is not nearstandard, a contradiction. Thus, $\{T_\lambda : \lambda \in \Lambda\}$ has a finite subcover, so X is compact. ■

Definition 3.3.3 $x \in {}^*\mathbf{R}$ is said to be *finite* if there is some $n \in \mathbf{N}$ such that $x \leq n$.

Proposition 3.3.4 *Suppose $y \in {}^*\mathbf{R}$, and y is finite. Then y is nearstandard.*

Proof: Let $A = \{z \in \mathbf{R} : z < y\}$, $x = \sup A$. Given $\delta \in \mathbf{R}_{++}$, we can find $z \in A$ with $z > x - \delta$. But $z < y$, so $x - \delta < y$. On the other hand, $x + \delta > y$ by the definitions of A and x. Therefore $x - \delta < y < x + \delta$. Since δ is an arbitrary element of $\mathbf{R}_{++}$, $y \simeq x$, so y is nearstandard. ■

Theorem 3.3.5 (Bolzano-Weierstrass) *If C is a closed and bounded subset of $\mathbf{R}^k$ ($k \in \mathbf{N}$), then C is compact.*

Proof: Suppose $y \in {}^*C$. Since C is bounded, there exists $n \in \mathbf{N}$ such that

$$\forall z \in C |z| \leq n. \tag{3.9}$$

By Transfer

$$\forall z \in {}^*C |z| \leq n \tag{3.10}$$

and so each component y_i of y is finite. By Proposition 3.3.4, y_i is nearstandard, with $y_i \simeq x_i$ for some $x_i \in \mathbf{R}$. Let $x = (x_1, \ldots, x_k)$. Then $y \simeq x$. Since C is closed, $x \in C$. Thus, C is compact by Theorem 3.3.2. ■

Theorem 3.3.6 *Suppose $\succ$ is a binary relation on a compact toplogical space $(X, \mathcal{T})$ satisfying*

1. *irreflexivity (for all $x \in X$, $x \not\succ x$);*
2. *transitivity (for all $x, y, z \in X$, $x \succ y, y \succ z \Rightarrow x \succ z$);*
3. *continuity ($\{(x, y) \in X^2 : x \succ y\}$ is open).*

Then X contains a maximal element with respect to $\succ$, i.e. there is some $x \in X$ such that there is no $z \in X$ with $z \succ x$.

Proof: By Theorem 1.13.4, there exists a hyperfinite set A such that $T \in \mathcal{T} \Rightarrow \exists x \in A\ [x \in {}^*T]$. Since $\succ$ is irreflexive and transitive, any finite set $B \subset X$ contains a maximal element with respect to $\succ$. By Transfer, any hyperfinite set contains a maximal element with respect to ${}^*\!\succ$. Let y be such a maximal element of A. Since X is compact, there exists $x \in X$ such that $y \simeq x$ by Theorem 3.3.2.

Suppose $z \in X$ and $z \succ x$. Then there exists S, T with $x \in T \in \mathcal{T}$ and $z \in S \in \mathcal{T}$ such that $v \succ w$ for each $v \in S$ and $w \in T$. By transfer, $v {}^*\!\succ w$ for each $v \in {}^*S$ and each $w \in {}^*T$. But there exists $v \in {}^*S \cap A$, and so $v {}^*\!\succ y$, a contradiction. Thus, x is maximal in X with respect to $\succ$.
■

Proposition 3.3.7 *Suppose$(X, \mathcal{T})$ is a regular topological space, and $A \subset {}^*X$ is internal. Suppose further that every $y \in A$ is nearstandard. Then $\{x \in X : \exists y \in A\ [y \in \mu(x)]\}$ is compact.*

Proof: Let $C = \{x \in X : \exists y \in A\ [y \in \mu(x)]\}$. Suppose $\{C_\lambda : \lambda \in \Lambda\}$ is a collection of relatively closed subsets of C, with $\cap_{\lambda \in \Lambda} C_\lambda = \emptyset$, but $\cap_{i=1}^n C_{\lambda_i} \neq \emptyset$ for every finite collection $\{\lambda_1, \ldots, \lambda_n\}$; C is closed by Proposition 3.2.3, so C_λ is closed in X. Given $x \in C$ with $x \notin C_\lambda$, we may find sets $S_{\lambda x}, T_{\lambda x} \in \mathcal{T}$ such that $C_\lambda \subset S_{\lambda x}$, $x \in T_{\lambda x}$, and $S_{\lambda x} \cap T_{\lambda x} = \emptyset$. Let $\Lambda' = \{(\lambda, x) : x \notin C_\lambda\}$. Given any finite collection $\{(\lambda_1, x_1), \ldots, (\lambda_n, x_n)\} \subset \Lambda'$, $\cap_{i=1}^n S_{\lambda_i x_i}$ is an open set; because it contains $\cap_{i=1}^n C_{\lambda_i} \neq \emptyset$, it is not empty. Choose $c \in \cap_{i=1}^n C_{\lambda_i}$. Then $c \in C$, so there exists $a \in A$ with $a \in \mu(c)$. $\cap_{i=1}^n {}^*S_{\lambda_i x_i} = {}^* \cap_{i=1}^n S_{\lambda_i x_i} \supset \mu(c)$ by Proposition 3.2.1. Therefore, $A \cap (\cap_{i=1}^n {}^*S_{\lambda_i x_i}) \neq \emptyset$. By saturation, $A \cap (\cap_{(\lambda,x) \in \Lambda'} {}^*S_{\lambda_i x_i}) \neq \emptyset$; choose $y \in A \cap (\cap_{(\lambda,x) \in \Lambda'} {}^*S_{\lambda_i x_i})$.

Since $y \in A$, y is nearstandard, so $y \in \mu(x)$ for some $x \in X$. By the definition of C, $x \in C$. Since $\cap_{\lambda\in\Lambda} C_\lambda = \emptyset$, there exists $\lambda \in \Lambda$ with $x \notin \lambda$. Since ${}^*T_{\lambda x} \supset \mu(x)$, ${}^*S_{\lambda x} \cap {}^*T_{\lambda x} = {}^*(S_{\lambda x} \cap T_{\lambda x}) = \emptyset$, we get $y \notin \mu(x)$, a contradiction. Therefore, C is compact. ■

3.4 Products

Proposition 3.4.1 *Let $(X_\lambda, \mathcal{T}_\lambda)$ be a family of topological spaces, and let $(X, \mathcal{T})$ be the product topological space. Then*

$$ {}^*X = \{y : y \text{ is an internal function from } {}^*\Lambda \text{ to } \cup_{\lambda \in {}^*\Lambda} {}^*X_\lambda \text{ and } \forall \lambda \in {}^*\Lambda \ y_\lambda \in {}^*X_\lambda\}. \tag{3.11} $$

Given $x \in X$,

$$ \mu(x) = \{y \in {}^*X : \forall \lambda \in \Lambda \ \ y_\lambda \simeq x_\lambda\}. \tag{3.12} $$

Proof: The formal definition of the product is

$$ X = \{f \in \mathcal{F}(\Lambda, \cup_{\lambda\in\Lambda} X_\lambda) : \forall \lambda \in \Lambda \ \ f(\lambda) \in X_\lambda\} \tag{3.13} $$

where $\mathcal{F}(A, B)$ denotes the set of all functions from A to B. By the Transfer Principle,

$$ {}^*X = \{f \in {}^*(\mathcal{F}(\Lambda, \cup_{\lambda\in\Lambda} X_\lambda)) : \forall \lambda \in {}^*\Lambda \ \ f(\lambda) \in {}^*X_\lambda\} \tag{3.14} $$

$$ = \{y : {}^*\Lambda \to \cup_{\lambda\in{}^*\Lambda} {}^*X_\lambda : y \text{ is internal}, \forall \lambda \in {}^*\Lambda \ y_\lambda \in {}^*X_\lambda\}. \tag{3.15} $$

Suppose $y \in \mu(x)$ with $x \in X$. Fix $\lambda \in \Lambda$. Given $T \in \mathcal{T}_\lambda$ with $x_\lambda \in T$, let $S = \{z \in X : z_\lambda \in T\}$. $S \in \mathcal{T}$ and $x \in S$, so $y \in {}^*S$. Therefore, $y_\lambda \in {}^*T$, so $y_\lambda \simeq x_\lambda$.

Conversely, suppose $y \in {}^*X$ and $y_\lambda \simeq x_\lambda$ for all $\lambda \in \Lambda$. If $x \in T \in \mathcal{T}$, then there exist $\lambda_1, \ldots, \lambda_n \in \Lambda$ with $n \in \mathbf{N}$ and

$T_{\lambda_i} \in \mathcal{T}_i$ such that if $S = \{z \in X : z_{\lambda_i} \in T_{\lambda_i}(1 \le i \le n)\}$, then $x \in S \subset T$. But ${}^*S = \{z \in {}^*X : z_{\lambda_i} \in {}^*T_{\lambda_i}(1 \le i \le n\})$ by Transfer, so $y \in {}^*S \subset {}^*T$. Therefore $y \simeq x$. ∎

Theorem 3.4.2 (Tychonoff) *Let $(X_\lambda, \mathcal{T}_\lambda)$ ($\lambda \in \Lambda$) be a family of topological spaces, and $(X, \mathcal{T})$ the product topological space. If $(X_\lambda, \mathcal{T}_\lambda)$ is compact for each $\lambda \in \Lambda$, then $(X, \mathcal{T})$ is compact.*

Proof: Suppose $y \in {}^*X$. For each $\lambda \in \Lambda$, there exists $x_\lambda \in X_\lambda$ such that $y_\lambda \simeq x_\lambda$. By the Axiom of Choice, this defines an element $x \in X$ such that $y_\lambda \simeq x_\lambda$ for each $\lambda \in \Lambda$. Therefore, $y \simeq x$ by Proposition 3.4.1. Thus, every $y \in {}^*X$ is nearstandard, so X is compact by Theorem 3.3.2. ∎

3.5 Continuity

Proposition 3.5.1 *Suppose $(X, \mathcal{S})$ and $(Y, \mathcal{T})$ are topological spaces and $f : X \to Y$. Then f is continuous if and only if ${}^*f(\mu(x)) \subset \mu(f(x))$ for every $x \in X$.*

Proof: Suppose f is continuous. If $y = f(x)$ and $y \in T \in \mathcal{T}$, then $S = f^{-1}(T) \in \mathcal{S}$. Hence, the sentence $\forall z \in S\ f(z) \in T$ holds in $\mathcal{X}$. By Transfer, the sentence $\forall z \in {}^*S\ {}^*f(z) \in {}^*T$ holds in ${}^*\mathcal{X}$. If $z \in \mu(x)$, then $z \in {}^*S$, so ${}^*f(z) \in {}^*T$. Since this holds for every T satisfying $f(x) \in T \in \mathcal{T}$, ${}^*f(z) \in \mu(f(x))$. Thus, ${}^*f(\mu(x)) \subset \mu(f(x))$.

Conversely, suppose ${}^*f(\mu(x)) \subset \mu(f(x))$ for every $x \in X$. Choose T such that $f(x) \in T \in \mathcal{T}$. By Proposition 3.1.5, we may find $S \in {}^*\mathcal{S}$ such that $x \in S \subset \mu(x)$. Accordingly, the sentence

$$\exists S \in {}^*\mathcal{S}\ \ [x \in S\ \wedge\ {}^*f(S) \subset {}^*T] \tag{3.16}$$

holds in ${}^*\mathcal{X}$; by Transfer, the sentence

$$\exists S \in \mathcal{S}\ \ [x \in S\ \wedge\ f(S) \subset T] \tag{3.17}$$

holds in $\mathcal{X}$, so f is continuous. ■

Corollary 3.5.2 *If $f : \mathbf{R} \to \mathbf{R}$, then f is continuous if and only if $y \simeq x \in \mathbf{R}$ implies ${}^*f(y) \simeq f(x)$.*

Definition 3.5.3 Suppose $(X, \mathcal{S})$ and $(Y, \mathcal{T})$ are topological spaces with $(Y, \mathcal{T})$ Hausdorff, and suppose $f : {}^*X \to {}^*Y$ is internal. f is said to be *S-continuous* if $f(x)$ is nearstandard and $f(\mu(x)) \subset \mu({}^\circ f(x))$ for every $x \in X$.

Definition 3.5.4 A topological space $(X, \mathcal{T})$ is *regular* if it is Hausdorff and, given $x \in X$ and $C \subset X$ with $x \notin C$ and C closed, there exist $S, T \in \mathcal{T}$ with $x \in S$, $C \subset T$, and $S \cap T = \emptyset$.

Proposition 3.5.5 *Suppose $(X, \mathcal{S})$ and $(Y, \mathcal{T})$ are topological spaces with $(Y, \mathcal{T})$ regular, and $f : {}^*X \to {}^*Y$ is S-continuous. Define ${}^\circ f : X \to Y$ by $({}^\circ f)(x) = {}^\circ(f(x))$ for each $x \in X$. Then ${}^\circ f$ is a continuous function.*

Proof: Because $f(x)$ is nearstandard for each $x \in X$, there exists $y \in Y$ such that $f(x) \in \mu(y)$; since $(Y, \mathcal{T})$ is Hausdorff, this y is unique by Proposition 3.1.7. Thus, the formula for ${}^\circ f$ defines a function.

Suppose $x \in X$, $y = {}^\circ f(x)$. If $y \in V \in \mathcal{T}$, then $X \setminus V$ is closed. Since $(Y, \mathcal{T})$ is regular, we may find $S, T \in \mathcal{T}$ with $y \in S$, $X \setminus V \subset T$, and $S \cap T = \emptyset$. Since f is S-continuous, $f^{-1}({}^*S) \supset \mu(x)$. $f^{-1}({}^*S)$ is internal by the Internal Definition Principle, so it contains ${}^*W$ for some W satisfying $x \in W \in \mathcal{S}$, by Proposition 3.1.5. If $w \in W$, then $w \in {}^*W$, so $f(w) \in {}^*S$. If ${}^\circ f(w) \notin V$, then ${}^\circ f(w) \in X \setminus V \subset T$. Since $T \in \mathcal{T}$, $f(w) \in {}^*T$ by Proposition 3.2.1. But ${}^*S \cap {}^*T = {}^*(S \cap T) = \emptyset$, so $f(w) \notin {}^*S$, a contradiction which shows ${}^\circ f(w) \in V$ for $w \in W$. Thus, ${}^\circ f$ is continuous. ■

Remark 3.5.6 In the proof of Proposition 3.5.5, one is tempted to consider the function $g = {}^*({}^\circ f)$ and apply Proposition 3.5.1. However, since ${}^\circ f$ is constructed using the nonstandard extension, using the properties of f propels us into a second nonstandard extension ${}^*({}^*\mathcal{X})$, creating more problems than we solve. Hence, the argument must proceed without invoking the nonstandard characterization of continuity presented in Proposition 3.5.1.

Definition 3.5.7 Suppose $(X, \mathcal{T})$ is a topological space and (Y, d) is a metric space. A function $f : X \to Y$ is *bounded* if $\sup_{x,y \in X} d(f(x), f(y)) < \infty$. $(C(X, Y), \bar{d})$ denotes the metric space of bounded continuous functions from X to Y, where $\bar{d}(f, g) = \sup_{x \in X} d(f(x), g(x))$.

Theorem 3.5.8 (Nonstandard Ascoli's Theorem) *Let $(X, \mathcal{T})$ be a compact topological space and (Y, d) a metric space. If f is an S-continuous function from ${}^*X$ to ${}^*Y$, then ${}^*\bar{d}(f, {}^\circ f) \simeq 0$, i.e. f is nearstandard as an element of ${}^*(C(X, Y), \bar{d})$, and ${}^\circ f$ is its standard part.*

Proof: By Proposition 3.5.5, ${}^\circ f$ is a continuous function from $(X, \mathcal{T})$ to (Y, d). Let $g = {}^\circ f$. Given $z \in {}^*X$, $z \in \mu(x)$ for some $x \in X$. Transferring the triangle inequality,

$$\begin{gathered} {}^*d({}^*g(z), f(z)) \leq \\ {}^*d({}^*g(z), g(x)) + {}^*d(g(x), f(x)) + {}^*d(f(x), f(z)). \end{gathered} \tag{3.18}$$

The first term is infinitesimal by Propositions 3.5.1 and 3.5.5; the second by the definition of $g = {}^\circ f$; and the third because f is S-continuous. Therefore ${}^*d(g(z), f(z)) \simeq 0$ for every $z \in {}^*X$. Therefore, ${}^*\bar{d}(f, g) < \epsilon$ for every $\epsilon \in \mathbf{R}_{++}$, and thus ${}^*\bar{d}(f, g) \simeq 0$. ■

Corollary 3.5.9 (Ascoli) *Suppose $A \subset C([0,1],\mathbf{R})$ is closed, bounded and equicontinuous. Then A is compact.*

Proof: Given $\epsilon \in \mathbf{R}_{++}$, there exists $\delta \in \mathbf{R}_{++}$ and $M \in \mathbf{R}$ such that the sentence

$$\forall f \in A\, \forall x,y \in [0,1] \;\; [[|f(x)| < M]$$

$$\wedge\, [|y-x| < \delta \Rightarrow |f(x) - f(y)| < \epsilon]] \tag{3.19}$$

holds in $\mathcal{X}$. By Transfer, the sentence

$$\forall f \in {}^*A\, \forall x,y \in {}^*[0,1] \;\; [[|f(x)| < M]$$

$$\wedge\, [|y-x| < \delta \Rightarrow |{}^*f(x) - {}^*f(y)| < \epsilon]] \tag{3.20}$$

holds in ${}^*\mathcal{X}$. Suppose $f \in {}^*A$. $f(x)$ is finite for all $x \in {}^*[0,1]$. Moreover, if $y \in \mu(x)$, then $|f(y) - f(x)| < \epsilon$; since ϵ is arbitrary, $|f(y) - f(x)| \simeq 0$. Therefore ${}^*f$ is S-continuous, so $f \in \mu({}^\circ f)$. Since A is closed, ${}^\circ f \in A$ by Proposition 3.2.2. Thus, every element $f \in {}^*A$ is nearstandard, so A is compact by Theorem 3.3.2. ■

3.6 Differentiation

Definition 3.6.1 Suppose $x, y \in {}^*\mathbf{R}$. We write $y = o(x)$ if there is some $\delta \simeq 0$ such that $|y| \leq \delta|x|$ and $y = O(x)$ if there is some $m \in \mathbf{N}$ such that $|y| \leq M|x|$.

Proposition 3.6.2 *Suppose $f : \mathbf{R}^m \to \mathbf{R}^n$ and $x \in \mathbf{R}^m$. Then f is differentiable at x if and only if there exists a linear function $J : \mathbf{R}^m \to \mathbf{R}^n$ such that ${}^*f(y) = f(x) + {}^*J(y-x) + o(y-x)$ for all $y \simeq x$.*

Proof: Let L be the set of all linear maps from $\mathbf{R}^m$ to $\mathbf{R}^n$. Suppose f is differentiable at x. Then there exists $J \in L$ such that for each $\epsilon \in \mathbf{R}_{++}$, there exists $\delta \in \mathbf{R}_{++}$ such that the sentence

$$\forall y \in \mathbf{R}^m [|y - x| < \delta \Rightarrow |f(y) - f(x) - J(y - x)| \leq \epsilon |y - x|] \tag{3.21}$$

holds in $\mathcal{X}$; by Transfer, the setence

$$\forall y \in {}^*\mathbf{R}^m \ [|y - x| < \delta \Rightarrow |f(y) - f(x) - {}^*J(y - x)| \leq \epsilon |y - x|] \tag{3.22}$$

holds in ${}^*\mathcal{X}$. Therefore, if $y \simeq x$, then $|f(y) - f(x) - {}^*J(y - x)| \leq \epsilon |y - x|$. Since ϵ is an arbitrary element of $\mathbf{R}_{++}$, $|f(y) - f(x) - {}^*J(y - x)| = o(|y - x|)$ for all $y \simeq x$.

Conversely, suppose that there exists $J \in L$ such that $y \simeq x$ implies $|f(y) - f(x) - {}^*J(y - x)| = o(|y - x|)$. Fix $\epsilon \in \mathbf{R}_{++}$. Then the sentence

$$\begin{aligned} &\exists \delta \in {}^*\mathbf{R}_{++} \ [|y - x| < \delta \\ &\Rightarrow |f(y) - f(x) - {}^*J(y - x)| \leq \epsilon |y - x|] \end{aligned} \tag{3.23}$$

holds in ${}^*\mathcal{X}$. By Transfer, the sentence

$$\begin{aligned} &\exists \delta \in \mathbf{R}_{++} \ [|y - x| < \delta \\ &\Rightarrow |f(y) - f(x) - J(y - x)| \leq \epsilon |y - x|] \end{aligned} \tag{3.24}$$

holds in $\mathcal{X}$. Since ϵ is an arbitrary element of $\mathbf{R}_{++}$, f is differentiable at x. ■

3.7 Riemann Integration

Theorem 3.7.1 *Suppose $[a, b] \subset \mathbf{R}$ and $f : [a, b] \to \mathbf{R}$ is continuous. Given $n \in {}^*\mathbf{N} \setminus \mathbf{N}$,*

$$\int_a^b f(t)dt = {}^\circ \left(\frac{1}{n} \sum_{k=1}^{n} {}^*f(a + \frac{k(b-a)}{n}) \right). \tag{3.25}$$

Proof: Let $I_n = \frac{1}{n}\sum_{k=1}^{n} f(a + \frac{k(b-a)}{n})$ for $n \in \mathbf{N}$. By Transfer, $I_n = \frac{1}{n}\sum_{k=1}^{n} {}^*f(a+\frac{k(b-a)}{n})$ for $n \in {}^*\mathbf{N}$. Since f is continuous, $I_n \to \int_a^b f(t)dt$. By Proposition 3.1.10, ${}^*I_n \simeq \int_{t=a}^b f(t)dt$ for all $n \in {}^*\mathbf{N} \setminus \mathbf{N}$. ■

3.8 Differential Equations

Nonstandard analysis permits the construction of solutions of ordinary differential equations by means of a hyperfinite polygonal approximation; the standard part of the polygonal approximation is a solution of the differential equation.

Construction 3.8.1 Suppose $F : \mathbf{R}^k \times [0,1] \to \mathbf{R}^k$ is continuous, there exists $M \in \mathbf{N}$ such that $|F(x,t)| \leq M$ for all $(x,t) \in \mathbf{R}^k \times [0,1]$, and $y_0 \in \mathbf{R}^k$. Choose $n \in {}^*\mathbf{N} \setminus \mathbf{N}$. By the Transfer Principle, we can define inductively

$$\begin{aligned} z\left(\tfrac{0}{n}\right) &= y_0 \\ z\left(\tfrac{k+1}{n}\right) &= z\left(\tfrac{k}{n}\right) + \tfrac{1}{n}{}^*F\left(z\left(\tfrac{k}{n}\right), \tfrac{k}{n}\right) \end{aligned} \tag{3.26}$$

and then extend z to a function with domain ${}^*[0,1]$ by linear interpolation

$$z(t) = ([nt]+1-nt)z\left(\frac{[nt]}{n}\right)+(nt-[nt])z\left(\frac{[nt]+1}{n}\right) \tag{3.27}$$

where $[nt]$ denotes the greatest (nonstandard) integer less than or equal to nt. Let

$$y(t) = {}^\circ(z(t)) \text{ for } z \in [0,1]. \tag{3.28}$$

Theorem 3.8.2 *With the notation in Construction 3.8.1, z is S-continuous and y is a solution of the ordinary differential equation*

$$\begin{aligned} y(0) &= y_o \\ y'(t) &= F(y(t),t). \end{aligned} \tag{3.29}$$

Proof: Given $r, s \in {}^*[0,1]$ with $r \simeq s$, $|z(r) - z(s)| \leq M|r - s| \simeq 0$, so z is S-continuous. By Theorem 3.5.8, y is continuous and there exists $\delta \simeq 0$ such that $|z(t) - {}^*y(t)| \leq \delta$ for all $t \in {}^*[0,1]$. Then

$$\begin{aligned} y(t) - y_0 \simeq z(t) - z(0) \simeq \textstyle\sum_{k=0}^{[nt]-1} \left(z\left(\frac{k+1}{n}\right) - z\left(\frac{k}{n}\right)\right) \\ = \textstyle\sum_{k=0}^{[nt]-1} \frac{1}{n} {}^*F\left(z\left(\frac{k}{n}\right), \frac{k}{n}\right) \simeq \textstyle\sum_{k=0}^{[nt]-1} \frac{1}{n} {}^*F\left({}^*y\left(\frac{k}{n}\right), \frac{k}{n}\right) \\ \simeq \textstyle\int_0^t F(y(s), s) ds \end{aligned} \tag{3.30}$$

by Theorem 3.7.1. Since $y(t) - y_0$ and $\int_0^t F(y(s), s)ds$ are both standard, they are equal. By the Fundamental Theorem of Calculus, $y'(t) = F(y(t), t)$ for all $t \in [0,1]$, so y is a solution of the ordinary differential equation 3.29. ■

Chapter 4

Loeb Measure

The Loeb measure was developed originally by Peter Loeb (1975) to solve a problem in potential theory. Loeb's construction allows one to convert nonstandard summations on hyperfinite spaces to measures in the usual standard sense. It has been used very widely in probability theory, and is an important tool in nonstandard mathematical economics.

4.1 Existence of Loeb Measure

Definition 4.1.1 An *internal probability space* is a triple $(A, \mathcal{A}, \nu)$ where

1. A is an internal set,
2. $\mathcal{A} \subset ({}^*\mathcal{P})(A)$ is an internal σ-algebra, i.e.
    (a) $A \in \mathcal{A}$;
    (b) $B \in \mathcal{A}$ implies $A \setminus B \in \mathcal{A}$; and
    (c) If $\{B_n : n \in {}^*\mathbf{N}\}$ is an internal sequence with $B_n \in \mathcal{A}$, then $\cap_{n \in {}^*\mathbf{N}} B_n \in \mathcal{A}$ and $\cup_{n \in {}^*\mathbf{N}} B_n \in \mathcal{A}$; and

3. $\nu : \mathcal{A} \rightarrow {}^*[0,1]$ is an internal *-countably additive probability measure, i.e.

   (a) $\nu(A) = 1$; and

   (b) if $\{B_n : n \in {}^*\mathbf{N}\}$ is an internal sequence and $B_n \cap B_m = \emptyset$ whenever $n \neq m$, then $\nu(\cup_{n\in {}^*\mathbf{N}} B_n) = \sum_{n\in {}^*\mathbf{N}} \nu(B_n)$.

Remark 4.1.2 The Loeb measure construction also works if we merely assume that $\mathcal{A}$ is closed under finite unions and ν is finitely additive. We shall be primarily interested in hyperfinite spaces, in which integration is just summation.

Definition 4.1.3 An internal probability space is *hyperfinite* if A is a hyperfinite set, $\mathcal{A} = ({}^*\mathcal{P})(A)$ (i.e. $\mathcal{A}$ is the class of all internal subsets of A), and there is an internal set of probability weights $\{\lambda_a : a \in A\}$ such that $\nu(B) = \sum_{a\in B} \lambda_a$ for all $B \in \mathcal{A}$.

Example 4.1.4 The canonical example of a hyperfinite probability space is $(A, \mathcal{A}, \nu)$, where $A = \{1, \ldots, n\}$ for some $n \in {}^*\mathbf{N} \setminus \mathbf{N}$, $\mathcal{A} = ({}^*\mathcal{P})(A)$, and $\nu(B) = \frac{|B|}{n}$ for all $B \in \mathcal{A}$.

Construction 4.1.5 (Loeb Measure) Suppose $(A, \mathcal{A}, \nu)$ is an internal probability space. The Loeb measure construction creates an (external) probability measure in the standard sense, defined on an (external) σ-algebra of (internal and external) subsets of the internal set A. Define

$$\begin{aligned}\bar{\mathcal{A}} = \{B \subset A : \forall \epsilon \in \mathbf{R}_{++}\ \exists C \in \mathcal{A}\ \exists D \in \mathcal{A} \\ [C \subset B \subset D,\ \ \nu(D \setminus C) < \epsilon]\}\end{aligned} \tag{4.1}$$

and

$$\begin{aligned}\bar{\nu}(B) &= \inf\{{}^\circ\nu(D) : B \subset D \in \mathcal{A}\} \\ &= \sup\{{}^\circ\nu(C) : C \subset B,\ C \in \mathcal{A}\}\end{aligned} \tag{4.2}$$

for $B \in \bar{\mathcal{A}}$. $\bar{\nu}$ is called the *Loeb measure* generated by ν.

Theorem 4.1.6 (Loeb) *Suppose* $(A, \mathcal{A}, \nu)$ *is an internal probability space. Then*

1. $\bar{\mathcal{A}}$ *is a* σ*-algebra,* $\bar{\mathcal{A}} \supset \mathcal{A}$*;*
2. $\bar{\nu}$ *is a countably additive probability measure;*
3. $\bar{\mathcal{A}}$ *is complete with respect to* $\bar{\nu}$*;*
4. $\bar{\nu}(B) = {}^\circ\nu(B)$ *for every* $B \in \mathcal{A}$*; and*
5. *for each* $B \in \bar{\mathcal{A}}$*, there exists* $A \in \mathcal{A}$ *such that*

$$\bar{\nu}((A \setminus B) \cup (B \setminus A)) = 0. \tag{4.3}$$

Proof:

1. First, we note that part 4 is obvious.
2. Second, we prove part 5. If $B \in \bar{\mathcal{A}}$, then for each $n \in \mathbf{N}$, we may find $C_n, D_n \in \mathcal{A}$ with $C_1 \subset C_2 \subset \ldots \subset B \ldots D_2 \subset D_1$ and $\nu(D_n \setminus C_n) < 1/n$. By Theorem 1.10.3, we can extend C_n andD_n to internal sequences in $\mathcal{A}$. $\{n \in {}^*\mathbf{N} : [C_m \subset C_{m+1} \subset D_{m+1} \subset D_m \quad \wedge \nu(D_m \setminus C_m) < 1/m] \ (1 \leq m \leq n)\}$ is internal and contains $\mathbf{N}$, so it contains some $n \in {}^*\mathbf{N} \setminus \mathbf{N}$ by Proposition 3.1.6. $C_n \in \mathcal{A}$. If $m \in \mathbf{N}$, $\bar{\nu}((C_n \setminus B) \cup (B \setminus C_n)) \leq \bar{\nu}((C_n \setminus C_m) \cup (D_m \setminus C_n)) \leq \bar{\nu}(D_m \setminus C_m) < 1/m$. Since m is an arbitrary element of $\mathbf{N}$, $\bar{\nu}((C_n \setminus B) \cup (B \setminus C_n)) = 0$.
3. Next, we establish parts 1 and 2.
   (a) Suppose $B, B' \in \bar{\mathcal{A}}$. Fix $\epsilon \in \mathbf{R}_{++}$, and find $C \subset B \subset D$, $C' \subset B' \subset D'$ with $C, C', D, D' \in \mathcal{A}$ and $\nu(D \setminus C) < \epsilon/2, \nu(D' \setminus C') < \epsilon/2$. $C \setminus D' \subset B \setminus B' \subset D \setminus C'$ and $((D \setminus C') \setminus (C \setminus D')) \subset ((D \setminus C) \cup (D' \setminus C'))$, so $\nu((D \setminus C') \setminus (C \setminus D')) < \epsilon/2 + \epsilon/2 = \epsilon$. Thus, $B \setminus B' \in \bar{\mathcal{A}}$.

(b) Now suppose $B_1, B_2, \ldots \in \bar{\mathcal{A}}$ and let $B = \cup_{n \in \mathbf{N}} B_n$. By considering $B'_n = B_n \setminus \cup_{i=1}^{n-1} B_i$, we may assume without loss of generality that the B_i's are disjoint. Given $\epsilon \in \mathbf{R}_{++}$, we may find $C_n \subset B_n \subset D_n$ with $C_n, D_n \in \mathcal{A}$ satisfying $\nu(D_n \setminus C_n) < \epsilon/2^{n+1}$. Extend C_n and D_n to internal sequences with $C_n, D_n \in \mathcal{A}$ by Theorem 1.10.3. If we let $\alpha_n = {}^\circ(\nu(\cup_{i=1}^n C_i))$, then α_n is a nondecreasing sequence in $[0,1]$, so it converges to some $\alpha \in \mathbf{R}$. By Proposition 3.1.9, we may find some $n_0 \in {}^*\mathbf{N} \setminus \mathbf{N}$ such that $\nu(\cup_{i=1}^n C_i) \simeq \alpha$ for $n \in {}^*\mathbf{N} \setminus \mathbf{N}$, $n \le n_0$. Moreover, $\{n : \nu(\cup_{i=1}^n D_i) \le \nu(\cup_{i=1}^n C_i) + \epsilon/2\}$ is an internal set which contains $\mathbf{N}$, so by Proposition 3.1.6, it contains all $n \le n_1$ for some $n_1 \in {}^*\mathbf{N} \setminus \mathbf{N}$. Taking $n = \min\{n_0, n_1\}$, we see there exists $n \in {}^*\mathbf{N} \setminus \mathbf{N}$ such that $\nu(\cup_{i=1}^n C_i) \simeq \alpha$ and $\nu(\cup_{i=1}^n D_i) \le \nu(\cup_{i=1}^n C_i) + \epsilon/2$. Moreover, we can find $m \in \mathbf{N}$ such that $\alpha_m > \alpha - \epsilon/2$. Let $C = \cup_{i=1}^m C_i$ and $D = \cup_{i=1}^n D_i$. Then $C, D \in \mathcal{A}$, $C \subset B \subset D$, and ${}^\circ(\nu(D \setminus C)) < \epsilon$, so $\nu(D \setminus C) < \epsilon$. Therefore, $B \in \mathcal{A}$. $\alpha - \epsilon/2 < {}^\circ(\nu(C)) \le \bar{\nu}(B) \le {}^\circ(\nu(D)) \le \alpha + \epsilon/2$. $|\bar{\nu}(B_n) - (\alpha_n - \alpha_{n-1})| < \epsilon/2^{n+1}$, so $|\sum_{n \in \mathbf{N}} \bar{\nu}(B_n) - \alpha| < \epsilon/2$. Since ϵ is arbitrary, $\bar{\nu}(B) = \sum_{n \in \mathbf{N}} \bar{\nu}(B_n)$, so $\bar{\nu}$ is countably additive.

4. Finally, we establish part 3. Suppose $B \subset B'$ with $B' \in \bar{\mathcal{A}}$ and $\bar{\nu}(B') = 0$. Then given $\epsilon \in \mathbf{R}_{++}$, there exists $D \in \mathcal{A}$ such that $B' \subset D$ and ${}^\circ\nu(D) < \epsilon$. Then $\emptyset \subset B \subset D$, so $B \in \bar{\mathcal{A}}$, so $\bar{\mathcal{A}}$ is complete with respect to $\bar{\nu}$.

■

4.2 Lebesgue Measure

In this section, we present a construction of Lebesgue measure in terms of the Loeb measure on a natural hyperfinite probability space.

Construction 4.2.1 (Anderson) Fix $n \in {}^*\mathbf{N} \setminus \mathbf{N}$, and let $A = \{\frac{k}{n} : k \in {}^*\mathbf{N},\ 1 \leq k \leq n\}$. Let $\mathcal{A} = ({}^*\mathcal{P})(A)$, the set of all internal subsets of A, and $\nu(B) = \frac{|B|}{n}$ for $B \in \mathcal{A}$. Let $(A, \bar{\mathcal{A}}, \bar{\nu})$ be the Loeb probability space generated by $(A, \mathcal{A}, \nu)$. Let $^\circ$ denote the restriction of the standard part map to A, i.e. $^\circ(a) = {}^\circ a$ for $a \in A$, and let st^{-1} denote the inverse image of st, i.e. $\mathrm{st}^{-1}(C) = \{a \in A : {}^\circ a \in C\}$ for $C \subset [0,1]$. Let $\mathcal{C} = \{C \subset [0,1] : \mathrm{st}^{-1}(C) \in \bar{\mathcal{A}}\}$ and $\mu(C) = \bar{\nu}(\mathrm{st}^{-1}(C))$ for all $C \in \mathcal{C}$.

Theorem 4.2.2 (Anderson, Henson) $([0,1], \mathcal{C}, \mu)$ *is the Lebesgue measure space on* $[0,1]$.

Proof: Since $\bar{\mathcal{A}}$ is a σ-algebra, so is $\mathcal{C}$; since $\bar{\nu}$ is countably additive, so is μ. Consider the closed interval $[a,b] \subset [0,1]$. Then $\mathrm{st}^{-1}([a,b]) = \cap_{m \in \mathbf{N}} {}^*(a - \frac{1}{m}, b + \frac{1}{m}) \cap A$. ${}^*(a - \frac{1}{m}, b + \frac{1}{m}) \cap A \in \mathcal{A}$ by the Internal Definition Principle, so $\mathrm{st}^{-1}([a,b]) \in \bar{\mathcal{A}}$. $\mu([a,b]) = \bar{\nu}(\mathrm{st}^{-1}([a,b])) = \lim_{m \to \infty} {}^\circ \frac{|{}^*(a - \frac{1}{m}, b + \frac{1}{m}) \cap A|}{n} = \lim_{m \to \infty} b - a + \frac{2}{m} = b - a$. Thus, $\mu([a,b]) = \mu([a,b])$.

Let $\mathcal{B}$ be the class of Borel sets. Since $\mathcal{B}$ is the smallest σ-algebra containing the closed intervals, $\mathcal{C} \supset \mathcal{B}$. Since μ and Lesbesgue measure are countably additive, and agree on closed intervals, they agree on $\mathcal{B}$. $\mathcal{C}$ is complete with respect to μ because $\bar{\mathcal{A}}$ is complete with respect to $\bar{\nu}$. Therefore, $\mathcal{C}$ contains the class of Lebesgue measurable sets and μ agrees with Lebesgue measure on that class.

Finally, we show that $\mathcal{C}$ is contained in the class of Lebesgue measurable sets. [1] Suppose $C \in \mathcal{C}$. Given $\epsilon \in \mathbf{R}_{++}$, there ex-

[1] The proof of this part given here is due to Edward Fisher.

ist $B, D \in \mathcal{A}$ with $B \subset \mathrm{st}^{-1}(C) \subset D$ and $^\circ\nu(D) - {}^\circ\nu(B) < \epsilon$. Let $\hat{B} = \{^\circ b : b \in B\}$, $\hat{D} = [0,1] \setminus \{^\circ a : a \in A - D\}$. Then $\hat{B} \subset C \subset \hat{D}$. $\hat{B}$ and $[0,1] \setminus \hat{D}$ are closed by Proposition 3.2.3, so $\hat{D}$ is open; thus, $\hat{B}, \hat{D} \in \mathcal{B}$. $\mu(\hat{D}) - \mu(\hat{B}) = \bar{\nu}(\mathrm{st}^{-1}(\hat{D})) - \bar{\nu}(\mathrm{st}^{-1}(\hat{B})) < \bar{\nu}(D) - \bar{\nu}(B)$ (since $\mathrm{st}^{-1}(\hat{B}) \supset B$ and $\mathrm{st}^{-1}(\hat{D}) \subset D$) $= {}^\circ\nu(D) - {}^\circ\nu(B) < \epsilon$. Since the Lebesgue measure space is complete, C is Lebesgue measurable. ■

4.3 Representation of Radon Measures

Definition 4.3.1 A *Radon* probability space is a probability space $(X, \mathcal{B}, \mu)$ where $(X, \mathcal{T})$ is a Hausdorff space, $\mathcal{B}$ is the σ-algebra of Borel sets (i.e. the smallest σ-algebra containing $\mathcal{T}$), and

$$\mu(B) = \sup\{\mu(C) : C \subset B,\ C \text{ compact}\}$$
$$= \inf\{\mu(T) : B \subset T,\ T \in \mathcal{T}\} \tag{4.4}$$

for every $B \in \mathcal{B}$.

Example 4.3.2 Let (X, d) be any complete separable metric space, $\mathcal{B}$ the σ-algebra of Borel sets. Then any probability measure μ on $\mathcal{B}$ is Radon (see Billingsley (1968)).

Theorem 4.3.3 (Anderson) *Let $(X, \mathcal{B}, \mu)$ be a Radon probability space, and $\bar{\mathcal{B}}$ is the completion of $\mathcal{B}$ with respect to μ. Then there is a hyperfinite probability space $(A, \mathcal{A}, \nu)$ and a function $S : A \to X$ such that $B \in \bar{\mathcal{B}}$ if and only if $S^{-1}(B) \in \bar{\mathcal{A}}$. For every $B \in \bar{\mathcal{B}}$, $\mu(B) = \bar{\nu}(S^{-1}(B))$.*

Proof: The proof is similar to the proof of Theorem 4.2.2; for details, see Anderson (1982). ■

4.4 Lifting Theorems

In this section, we state a number of "Lifting Theorems" relating integration theory in Loeb probability spaces to the internal integration theory in internal measure spaces. Note that, in the case of a hyperfinite measure space, the internal integration theory reduces to finite summations. In particular, feasibility conditions in hyperfinite exchange economies can be formulated as conditions on summations, in exactly the same way as they are formulated in finite exchange economies. However, the Loeb measure construction can be used to convert a hyperfinite exchange economy to an economy with a measure space of agents in the sense introduced by Aumann. The lifting theorems provide the link between the two constructions.

Definition 4.4.1 Suppose $(A, \mathcal{A}, \nu)$ is an internal probability space and $(X, \mathcal{T})$ is a topological space. A function $f : A \to {}^*X$ is said to be ν-measurable if it is internal and $f^{-1}(T) \in \mathcal{A}$ for every $T \in {}^*\mathcal{T}$.

Theorem 4.4.2 (Loeb, Moore, Anderson) *Let $(A, \mathcal{A}, \nu)$ be an internal probability space.*

1. *If $(X, \mathcal{T})$ is a regular topological space, $f : A \to {}^*X$ is $\nu-$measurable, and $f(a)$ is nearstandard for $\bar{\nu}$-almost all $A \in \mathcal{A}$, then ${}^\circ f : A \to X$ is $\bar{\nu}$-measurable.*

2. *If $(X, \mathcal{T})$ is a Hausdorff topological space with a countable base of open sets and $F : A \to X$ is $\bar{\nu}$-measurable, then there exists a ν-measurable function $f : A \to {}^*X$ such that ${}^\circ f(a) = F(a)$ $\bar{\nu}$-almost surely.*

Proof: This will be provided in the monograph. For now, see Anderson (1982).■

Definition 4.4.3 Suppose $(A, \mathcal{A}, \nu)$ is an internal probability space and $f : A \to {}^*\mathbf{R}$ is ν-measurable. Suppose $A \in {}^*\mathcal{X}_n$. Let I be the function which assigns to every pair $((B, \mathcal{B}, \mu), g)$, where $(B, \mathcal{B}, \mu)$ is a standard probability space, $B \in \mathcal{X}_n$, and g is a μ-measurable real-valued function, the integral $I((B, \mathcal{B}, \mu), g) = \int_B g d\mu$. The *internal integral* of f, denoted $\int_A f d\nu$, is defined by $\int_A f d\nu = ({}^*I)((A, \mathcal{A}, \nu), f)$.

Example 4.4.4 Let $(A, \mathcal{A}, \nu)$ be as in the construction of Lebesgue measure (Construction 4.2.1). If $f : A \to {}^*\mathbf{R}$ is any internal function, then f is ν-measurable by the Internal Definition Principle. Moreover,

$$\int_A f d\nu = \frac{1}{n} \sum_{a \in A} f(a) \tag{4.5}$$

by the Transfer Principle.

Definition 4.4.5 Suppose $(A, \mathcal{A}, \nu)$ is an internal probability space and $f : A \to {}^*\mathbf{R}$. We say f is *S-integrable* if

1. f is a ν-measurable function;
2. ${}^\circ \int_A |f| d\nu < \infty$;
3. $B \in \mathcal{A},\ \nu(B) \simeq 0 \Rightarrow \int_B |f| d\nu \simeq 0$.

Theorem 4.4.6 (Loeb, Moore, Anderson) *Let $(A, \mathcal{A}, \nu)$ be an internal probability space.*

1. *If $f : A \to {}^*\mathbf{R}$ is ν-measurable and $\int_A |f(a)| d\nu$ is finite, then ${}^\circ f$ is integrable with respect to $\bar{\nu}$ and $\int_A |{}^\circ f| d\bar{\nu} \leq {}^\circ \int_A |f| d\nu$;*
2. *If $f : A \to {}^*\mathbf{R}$ is S-integrable, then ${}^\circ f$ is integrable with respect to $\bar{\nu}$; moreover, ${}^\circ \int_A f d\nu = \int_A {}^\circ f d\bar{\nu}$;*

3. *If $F : A \to \mathbf{R}$ is integrable with respect to $\bar{\nu}$, then there exists an S-integrable function $f : A \to {}^*\mathbf{R}$ such that ${}^\circ f = F$ $\bar{\nu}$-almost surely. Moreover, ${}^\circ\int_A f d\nu = \int_A {}^\circ f d\bar{\nu}$.*

Proof: Parts 2 and 3 are proved in Anderson(1976). To see part 1, suppose $m \in \mathbf{N}$, and define $f_m(a) = \min\{|f(a)|, m\}$. f_m is obviously S-integrable, so by (2), ${}^\circ f_m$ is integrable and ${}^\circ\int_A f_m d\nu = \int_A {}^\circ f_m d\bar{\nu}$. By the definition of the standard integral, $\int_A {}^\circ|f| d\bar{\nu} = \lim_{m\to\infty} \int_A {}^\circ f_m d\bar{\nu} = \lim_{m\to\infty} {}^\circ\int_A f_m d\nu$; note that this last sequence is increasing and bounded above by ${}^\circ\int_A |f| d\nu < \infty$. Therefore ${}^\circ f$ is integrable and $\int_A |{}^\circ f| d\bar{\nu} \leq {}^\circ\int_A |f| d\nu$. ■

Definition 4.4.7 Suppose A_n is a sequence of finite sets. A sequence of functions $f_n : A_n \to \mathbf{R}^k_+$ is said to be *uniformly integrable* if, for every sequence of sets $E_n \subset A_n$ satisfying $|E_n|/|A_n| \to 0$,

$$\frac{1}{|A_n|} \sum_{a \in E_n} f_n(a) \to 0. \tag{4.6}$$

Proposition 4.4.8 *Suppose $\{A_n : n \in \mathbf{N}\}$ is a sequence of finite sets and $f_n : A_n \to \mathbf{R}^k_+$. Then $\{f_n : n \in \mathbf{N}\}$ is uniformly integrable if and only if for all $n \in {}^*\mathbf{N}$, f_n is S-integrable with respect to the normalized counting measure $\nu_n(B) = |B|/|A_n|$ for each internal $B \subset A_n$.*

Proof: See Anderson (1982), Theorem 6.5. ■

4.5 Weak Convergence

Definition 4.5.1 A sequence of probability measures μ_n on a complete separable metric space (X, d) is said to *converge* *weakly* to a probability measure μ (written $\mu_n \Rightarrow \mu$) if

$$\int_X F d\mu_n \to \int_X F d\mu \tag{4.7}$$

for every bounded continuous function $F : X \to \mathbf{R}$.

The standard theory of weak convergence of probability measures is developed in Billingsley (1968). Because the theory is widely used in the large economies literature (see for example Hildenbrand (1974,1982) and Mas-Colell (1985)), it is useful to have the following nonstandard characterization in terms of the Loeb measure.

Theorem 4.5.2 (Anderson, Rashid) [2] *Suppose* ν_n $(n \in \mathbf{N})$ *is a sequence of Borel probability measures on a complete separable metric space* (X, d)*. Let* $\mu_n(B) = \overline{{}^*\nu_n}(\mathrm{st}^{-1}(B))$ *for each Borel set* $B \subset X$ *define a Borel measure on* X *for each* $n \in {}^*\mathbf{N}$*. Then* ν_n *converges weakly if and only if*

1. $\overline{{}^*\nu_n}(ns({}^*X)) = 1$ *for all* $n \in {}^*\mathbf{N} \setminus \mathbf{N}$*; and*
2. $\mu_n = \mu_m$ *for all* $n, m \in {}^*\mathbf{N} \setminus \mathbf{N}$*.*

In this case, the weak limit is the common value μ_n *for* $n \in {}^*\mathbf{N} \setminus \mathbf{N}$*.*

Proof:

1. Suppose $\nu_n \Rightarrow \nu$.
   (a) Since ν is a probability measure and X is complete separable metric, $\{\nu_n : n \in \mathbf{N}\}$ is tight, i.e. given $\epsilon \in \mathbf{R}_+$, there exists K compact such

[2] The result holds for spaces much more general than the complete separable metric spaces considered here. See Anderson and Rashid (1978) for details.

that $\nu_n(K) > 1 - \epsilon$ for all $n \in \mathbf{N}$ (Billingsley (1968)). By transfer, ${}^*\nu_n({}^*K) > 1 - \epsilon$ for all $n \in {}^*\mathbf{N} \setminus \mathbf{N}$. ${}^*K \subset ns({}^*X)$ by Theorem 3.3.2, so $\overline{{}^*\nu_n}(ns({}^*X)) = 1$.

(b) Given $F : X \to \mathbf{R}$, F bounded and continuous, $\int_X F d\nu_n \to \int_X F d\nu$ by assumption. Therefore, $\int_X {}^*F d\nu_n \simeq \int_X F d\nu$ for all $n \in {}^*\mathbf{N} \setminus \mathbf{N}$ by Proposition 3.1.10. Then

$$\int_X F d\mu_n = \int_{ns({}^*X)} F({}^\circ x) d\overline{\nu_n} \tag{4.8}$$

$$= \int_{ns({}^*X)} {}^\circ({}^*F(x)) d\overline{\nu_n} \tag{4.9}$$

by Proposition 3.5.1

$$= \int_{{}^*X} {}^\circ({}^*F(x)) d\overline{\nu_n} \simeq \int_{{}^*X} {}^*F(x) d\nu_n \tag{4.10}$$

by Theorem 4.4.6

$$\simeq \int_X F d\nu. \tag{4.11}$$

Therefore, $\int_X F d\mu_n = \int_X F d\nu$ for all $n \in {}^*\mathbf{N} \setminus \mathbf{N}$. Since this holds for every bounded continuous F, $\mu_n = \nu$ by Billingsley (1968). In particular, if $m, n \in {}^*\mathbf{N} \setminus \mathbf{N}$, then $\mu_n = \mu_m$.

2. Suppose $\overline{\nu_n}(ns({}^*X)) = 1$ for all $n \in {}^*\mathbf{N} \setminus \mathbf{N}$ and $\mu_n = \mu_m$ for all $n, m \in {}^*\mathbf{N} \setminus \mathbf{N}$. Let ν be the common value of μ_n for $n \in {}^*\mathbf{N} \setminus \mathbf{N}$.

$$\int_{{}^*X} {}^*F d\nu_n \simeq \int_{{}^*X} {}^\circ({}^*F(x)) d\overline{\nu_n} \tag{4.12}$$

by Theorem 4.4.6

$$\begin{aligned} &= \textstyle\int_{ns({}^*X)} {}^\circ({}^*F(x)) d\overline{\nu_n} = \int_{ns({}^*X)} F({}^\circ x)) d\overline{\nu_n} \\ &= \textstyle\int_X F d\mu_n = \int_X F d\nu. \end{aligned} \tag{4.13}$$

Thus, for $n \in {}^*\mathbf{N} \setminus \mathbf{N}$, $\int_{{}^*X} {}^*F d\nu_n \simeq \int_X F d\nu$. Then $\int_X F d\nu_n \to \int_X F d\nu$ as $n \to \infty$ by Proposition 3.1.10. Therefore $\nu_n \Rightarrow \nu$.

■

Chapter 5

Large Economies

The subject of large economies was introduced briefly in Section 1.1.1. We are interested in studying properties of sequences of exchange economies $\chi_n : A_n \to P \times \mathbf{R}_+^k$, where A_n is a set of agents, $\mathbf{R}_+^k$ the commodity space, and P the space of preference relations on $\mathbf{R}_+^k$. In this section, we examine price decentralization issues for the core, the bargaining set, the value, and the set of Pareto optima, as well as the question of existence of approximate Walrasian equilibria.

We begin by studying the properties of hyperfinite exchange economies. The Transfer Principle then gives a very simple derivation of analogous properties for sequences of finite economies.

Much of the work on large economies using nonstandard analysis concerns the core. For this reason, we have chosen to devote considerable attention to the core, in order to illustrate the use of the nonstandard methodology, and to contrast that methodology to measure-theoretic methods. In Section 5.5, we focus on the following issues:

1. The properties of the cores of hyperfinite exchange economies, in Theorem 5.5.2.

2. The use of the Transfer Principle to give very simple derivations of asymptotic results about the cores of large finite economies from results about hyperfinite economies, in Theorem 5.5.10.

3. The close relationship between hyperfinite economies and Aumann continuum economies, which are linked using the Loeb measure construction, in Proposition 5.5.6.

4. The ability of the nonstandard methodology to capture behavior of large finite economies which is not captured in Aumann continuum economies, in Remark 5.5.3 and Examples 5.5.5, 5.5.7, 5.5.8 and 5.5.9.

5.1 Preferences

Given $x, y \in \mathbf{R}_+^k$, x^i denotes the i^{th} component of x; $x > y$ means $x^i \geq y^i$ for all i and $x \neq y$; $x >> y$ means $x^i > y^i$ for all i. If $1 \leq p \leq \infty$, $||x||_p = (\sum_{i=1}^k |x^i|^p)^{1/p}$; $||x||_\infty = \max\{|x^i| : i = 1, \ldots, k\}$.

Definition 5.1.1 A *preference* is a binary relation on $\mathbf{R}_+^k$. Let P denote the set of preferences. A preference $\succ$

1. is *transitive* if $\forall x, y, z \in \mathbf{R}_+^k [x \succ y, y \succ z \Rightarrow x \succ z]$;

2. is *continuous* if $\{(x, y) \in \mathbf{R}_+^k \times \mathbf{R}_+^k : x \succ y\}$ is relatively open in $\mathbf{R}_+^k \times \mathbf{R}_+^k$;

3. is

   (a) *monotonic* if $\forall x, y \in \mathbf{R}_+^k [x >> y \Rightarrow x \succ y]$;

   (b) *strongly monotonic* if $\forall x, y \in \mathbf{R}_+^k [x > y \Rightarrow x \succ y]$;

4. is *irreflexive* if $\forall x \in \mathbf{R}^k_+ [x \not\succ x]$;

5. is

   (a) *convex* if $\forall x \in \mathbf{R}^k_+$, $\{y : y \succ x\}$ is convex;

   (b) *strongly convex* if

$$\forall x, y \in \mathbf{R}^k_+, \left[x \neq y \Rightarrow [\frac{x+y}{2} \succ x \vee \frac{x+y}{2} \succ y] \right]; \tag{5.1}$$

6. *satisfies free disposal* if

$$\forall x, y, z \in \mathbf{R}^k_+, [[x > y] \wedge [y \succ z] \Rightarrow x \succ z]. \tag{5.2}$$

Let P_c denote the space of continuous preferences.

Definition 5.1.2 We define a metric on P_c as follows: Let d_1 be the one-point compactification metric on $\mathbf{R}^{2k}_+ \cup \{\infty\}$. Given any compact metric space (X, d), the *Hausdorff metric* d^H is defined on the space of closed sets of X by

$$\begin{aligned} d^H(B, C) = \inf\{\delta : [\forall x \in B\ \exists y \in C\ d(x, y) < \delta] \\ \wedge [\forall y \in C\ \exists x \in B\ d(x, y) < \delta]\}. \end{aligned} \tag{5.3}$$

Let d_2 be the Hausdorff metric $(d_1)^H$. Given $\succ \in P_c$, define $C_\succ = \{(x, y) \in \mathbf{R}^{2k}_+ : x \not\succ y\} \cup \{\infty\}$. Then define

$$d(\succ, \succ') = d_2(C_\succ, C_{\succ'}). \tag{5.4}$$

Proposition 5.1.3 (Brown, Robinson, Rashid) *If* $\succ \in P_c$*,*

$$\mu(\succ) = \{\succ' \in {}^*P_c : \forall x, y \in \mathbf{R}^k_+\ [x \succ y \Leftrightarrow \mu(x) \succ' \mu(y)]\} \tag{5.5}$$

where $\mu(x), \mu(y)$ are taken with respect to the Euclidean metric on $\mathbf{R}^k_+$. (P_c, d) is compact.

Proof:

1. Recall that the one-point compactification metric induces the usual Euclidean topology on $\mathbf{R}^k_+$, so that if $x \in \mathbf{R}^k_+$, the d_1-monad of x, $\mu_{d_1}(x)$ coincides with the Euclidean monad $\mu(x)$. Suppose $\succ \in P_c$. We will show that equation 5.5 holds.
   (a) Suppose $\succ' \in {}^*P_c$, $\succ' \in \mu(\succ)$. Fix $x, y \in \mathbf{R}^k_+$. We show $x \succ y$ if and only if $\mu(x) \succ' \mu(y)$.
      i. Suppose $x \succ y$. If there exist $u \in \mu(x)$, $v \in \mu(y)$ with $u \not\succ' v$, then $(u,v) \in {}^*C_{\succ'}$, so there exist $(w,z) \in {}^*C_{\succ}$ such that ${}^*d_1((u,v),(w,z)) \leq {}^*d_2(\succ',\succ) \simeq 0$. ${}^*d_1((w,z),(x,y)) \leq {}^*d_1((w,z),(u,v)) + {}^*d_1((u,v),(x,y)) \simeq 0$, so $w \in \mu(x)$, $z \in \mu(y)$. Since $(w,z) \in {}^*C_{\succ}$, $w {}^*\!\not\succ z$. Since $x \succ y$ and $\succ \in P_c$, $\mu(x) {}^*\!\succ \mu(y)$, so $w {}^*\!\succ z$, a contradiction which shows $\mu(x) \succ' \mu(y)$.
      ii. If $x \not\succ y$, then $(x,y) \in C_{\succ}$, so there exists $(u,v) \in {}^*C_{\succ'}$ such that ${}^*d_1((u,v),(x,y)) \simeq 0$. Therefore $u \in \mu(x)$ and $v \in \mu(y)$, so $\mu(x) \not\succ \mu(y)$.
   (b) Conversely, suppose for every $x, y \in \mathbf{R}^k_+$, $x \succ y \Leftrightarrow \mu(x) \succ' \mu(y)$. We will show that every $w \in {}^*C_{\succ'}$ is infinitely close to some $z \in {}^*C_{\succ}$, and vice versa.
      i. Suppose $w \in {}^*C_{\succ'}$. We will show there exists $z \in {}^*C_{\succ}$ with ${}^*d_1(w,z) \simeq 0$. We consider two cases:
         A. Suppose $w \in \mu_{d_1}(\infty)$. In this case $\infty \in {}^*C_{\succ'}$ and ${}^*d_1(w,\infty) \simeq 0$.
         B. $w = (u,v) \in \mu_{d_1}(x,y)$ for some $x, y \in \mathbf{R}^k_+$. In this case, $u \not\succ' v$, so $\mu(x) \not\succ' \mu(y)$, so $x \not\succ y$, so $(x,y) \in C_{\succ}$.

Accordingly, for every w in ${}^*C_{\succ'}$, there exists $z \in {}^*C_{\succ}$ such that ${}^*d_1(w,z) \simeq 0$.

ii. Suppose $w \in {}^*C_{\succ}$. We will show there exists $z \in {}^*C_{\succ'}$ with ${}^*d_1(w,z) \simeq 0$. Again there are two cases.

   A. The case $w \in \mu_{d_1}(\infty)$ is handled as in item 1(b)iA above.

   B. Suppose $w = (u,v) \in \mu_{d_1}(x,y)$ for some $x, y \in \mathbf{R}^k_+$. In this case, $u {}^*\not\succ v$, so $\mu(x) {}^*\not\succ \mu(y)$, so $x \not\succ y$ (since $\succ$ is continuous), so $(x,y) \in C_{\succ}$.

Therefore,

$$\{n \in {}^*\mathbf{N} : [\forall x \in B\ \exists y \in C\ d(x,y) < 1/n]$$

$$\wedge\ [\forall y \in C\ \exists x \in B\ d(x,y) < 1/n]\} \qquad (5.6)$$

contains $\mathbf{N}$. The set is internal by the Internal Definition Principle. Hence, it includes some infinite n by Proposition 3.1.6, so ${}^*d(\succ, \succ') = {}^*d_2({}^*C_{\succ}, {}^*C_{\succ'}) \simeq 0$. Therefore, $\succ' \in \mu(\succ)$.

We have thus verified equation 5.5.

2. It remains to show that (P_c, d) is compact. Given $\succ' \in {}^*P_c$, define $\succ$ by $x \succ y \Leftrightarrow \mu(x) \succ' \mu(y)$. If $x \succ y$, then $\mu(x) \succ' \mu(y)$. Let $B = \{(u,v) : u \succ' v\}$. B is internal and contains $\mu(x,y)$, so it contains ${}^*T$ for some open set T with $(x,y) \in T$. If $(w,z) \in T$, then $\mu(w,z) \subset {}^*T$, so $\mu(w) \succ' \mu(z)$, so $w \succ z$. Thus, $\succ \in P_c$. By equation 5.5, $\succ' \in \mu(\succ)$. By Theorem 3.3.2, (P_c, d) is compact.

■

5.2 Hyperfinite Economies

Definition 5.2.1 A *hyperfinite exchange economy* is an internal function $\chi : A \to {}^*(P \times \mathbf{R}^k_+)$, where A is a hyperfinite set. We define the *endowment e(a)* and *preference* $\succ_a$ of a by $(\succ_a, e(a)) = \chi(a)$.

5.3 Loeb Measure Economies

Definition 5.3.1 Let $(A, \mathcal{B}, \mu)$ be a standard probability space. An *Aumann continuum economy* is a function $\chi : A \to P_c \times \mathbf{R}^k_+$ such that

1. χ is measurable;
2. $e(a)$ is integrable.

Construction 5.3.2 Suppose $\chi : A \to {}^*(P_c \times \mathbf{R}^k_+)$ is a hyperfinite exchange economy. Let $\mathcal{A}$ denote the set of all internal subsets of A, and $\nu(B) = \frac{|B|}{|A|}$ for $B \in \mathcal{A}$. Let $(A, \bar{\mathcal{A}}, \bar{\nu})$ be the Loeb measure space generated by $(A, \mathcal{A}, \nu)$. Define ${}^\circ\chi : A \to P_c \times \mathbf{R}^k_+$ by ${}^\circ\chi(a) = ({}^\circ{\succ_a}, {}^\circ e(a))$.

Theorem 5.3.3 (Rashid) *If $\chi : A \to {}^*(P_c \times \mathbf{R}^k_+)$ is a hyperfinite exchange economy with $n = |A|$ infinite and $\frac{1}{n}\sum_{a\in A} e(a)$ is finite, then ${}^\circ\chi$ as defined in Construction 5.3.2 is an Aumann continuum economy. $\int_A {}^\circ e(a) d\bar{\nu} \leq {}^\circ(\frac{1}{n}\sum_{a\in A} e(a))$, with equality if e is S-integrable.*

Proof: Since P_c is compact by Proposition 5.1.3, $\succ_a$ is nearstandard for all $a \in A$. ${}^\circ\nu(\{a : ||e(a)||_\infty \geq M\}) \leq {}^\circ(\frac{\sum_{a\in A} ||e(a)||_\infty}{Mn}) \leq {}^\circ(\frac{k||\sum_{a\in A} e(a)||_\infty}{Mn})$, so $\bar{\nu}(\{a : e(a)$ is finite $\}) = 1$. Thus, ${}^\circ\chi(a)$ is defined for $\bar{\nu}$-almost all $a \in A$; it is measurable by Theorem 4.4.2. $\int_A {}^\circ e(a) d\bar{\nu} \leq {}^\circ(\frac{1}{n}\sum_{a\in A} e(a))$ (with equality in case e is S-integrable) by Theorem 4.4.6. ■

5.4 Budget, Support and Demand Gaps

Definition 5.4.1 Let $\Delta = \{p \in \mathbf{R}^k : ||p||_1 = 1\}$, $\Delta_+ = \Delta \cap \mathbf{R}^k_+$, and $\Delta_{++} = \Delta \cap \mathbf{R}^k_{++}$.

Definition 5.4.2 Define $D, Q : \Delta \times P \times \mathbf{R}^k_+ \to \mathcal{P}(\mathbf{R}^k_+)$ by

$$D(p, \succ, e) = \{x \in \mathbf{R}^k_+ : p \cdot x \leq p \cdot e, y \succ x \Rightarrow p \cdot y > p \cdot e\}, \tag{5.7}$$

$$Q(p, \succ, e) = \{x \in \mathbf{R}^k_+ : p \cdot x \leq p \cdot e, y \succ x \Rightarrow p \cdot y \geq p \cdot e\}. \tag{5.8}$$

D and Q are called the *demand set* and the *quasidemand set* respectively.

Definition 5.4.3 Define $\phi_B : \mathbf{R}^k_+ \times \Delta \times \mathbf{R}^k_+ \to \mathbf{R}_+$, $\phi_S : \mathbf{R}^k_+ \times \Delta \times P \to \mathbf{R}_+$, and $\phi : \mathbf{R}^k_+ \times \Delta \times P \times \mathbf{R}^k_+ \to \mathbf{R}_+$ by

$$\phi_B(x, p, e) = |p \cdot (x - e)|; \tag{5.9}$$

$$\phi_S(x, p, \succ) = \sup\{p \cdot (x - y) : y \succ x\}; \text{and} \tag{5.10}$$

$$\phi(x, p, \succ, e) = \phi_B(x, p, e) + \phi_S(x, p, \succ). \tag{5.11}$$

ϕ_B, ϕ_S and ϕ are referred to as the *budget gap*, the *support gap*, and the *demand gap* respectively.

Proposition 5.4.4 *Suppose $x, e \in {}^*\mathbf{R}^k_+$ are finite, $\succ \in {}^*P_c$, and $p \in {}^*\Delta$.*

1. *If ${}^*\phi_S(x, p, \succ) \simeq 0$, then ${}^\circ x \in Q({}^\circ p, {}^\circ\succ, {}^\circ x)$. If in addition ${}^\circ p \in \Delta_{++}$ and $0 \not\succ 0$, then ${}^\circ x \in D({}^\circ p, {}^\circ\succ, {}^\circ x)$.*

2. *If ${}^*\phi(x, p, \succ, e) \simeq 0$, then ${}^\circ x \in Q({}^\circ p, {}^\circ\succ, {}^\circ e)$. If in addition ${}^\circ p \in \Delta_{++}$, then ${}^\circ x \in D({}^\circ p, {}^\circ\succ, {}^\circ e)$.*

Proof:

1. Suppose the hypotheses of (1) are satisfied. If $y \in \mathbf{R}^k_+$ and $y^\circ {}^\circ\!\succ {}^\circ x$, then $y \succ \mu({}^\circ x)$ by Proposition 5.1.3, so $y \succ x$. Therefore, ${}^\circ p \cdot y \simeq p \cdot y \geq p \cdot x - \phi_S(p, x, \succ) \simeq p \cdot x \simeq {}^\circ p \cdot {}^\circ x$, so ${}^\circ p \cdot y \geq {}^\circ p \cdot {}^\circ x$, hence ${}^\circ x \in Q({}^\circ p, {}^\circ\!\succ, {}^\circ x)$. If ${}^\circ p \in \Delta_{++}$, we show that ${}^\circ x \in D({}^\circ p, {}^\circ\!\succ, {}^\circ x)$ by considering two cases:

   (a) If ${}^\circ p \cdot {}^\circ x = 0$, then ${}^\circ x = 0$. Since $0 \not\succ 0$, $0^\circ \not\succ 0$, so $D({}^\circ p, {}^\circ\!\succ, {}^\circ x) = \{0\}$. Therefore ${}^\circ x \in D({}^\circ p, {}^\circ\!\succ, {}^\circ x)$.

   (b) If ${}^\circ p \cdot {}^\circ x > 0$, suppose $y \in \mathbf{R}^k_+$, $y^\circ {}^\circ\!\succ {}^\circ x$ and ${}^\circ p \cdot y = {}^\circ p \cdot {}^\circ x$. Since ${}^\circ\!\succ$ is continuous, we may find $w \in \mathbf{R}^k_+$ with ${}^\circ p \cdot {}^\circ w < {}^\circ p \cdot y = {}^\circ p \cdot {}^\circ x$ with $w^\circ {}^\circ\!\succ {}^\circ x$. By Proposition 5.1.3, $w \succ x$, so $\phi_S(x, p, \succ) \not\simeq 0$, a contradiction. Hence ${}^\circ x \in D({}^\circ p, {}^\circ\!\succ, {}^\circ x)$.

2. If the hypotheses of (2) are satisfied, then (1) holds and in addition ${}^\circ p \cdot {}^\circ x = {}^\circ(p \cdot x) = {}^\circ(p \cdot e) = {}^\circ p \cdot {}^\circ e$, so the conclusions of (2) follow from those of (1).

■

5.5 Core

Definition 5.5.1 Suppose $\chi : A \to P \times \mathbf{R}^k_+$ is a finite exchange economy or an Aumann continuum economy. The *Core*, the set of *Walrasian allocations*, and the set of *quasi-Walrasian allocations*, of χ, denoted $\mathcal{C}(\chi)$, $\mathcal{W}(\chi)$ and $\mathcal{Q}(\chi)$ respectively, are as defined in Chapter 18 of this Handbook Hildenbrand (1982). In case χ is a finite exchange economy, $\mathcal{C}(\chi)$, $\mathcal{W}(\chi)$, and $\mathcal{Q}(\chi)$ are defined by the following sentences:

$$\mathcal{C}(\chi) = \{f \in \mathcal{F}(A, \mathbf{R}^k_+) : \sum_{a \in A} f(a) = \sum_{a \in A} e(a)$$

$$\wedge\ \forall S \in \mathcal{P}(A)\ \forall g \in \mathcal{F}(S, \mathbf{R}^k_+)[\sum_{a \in S} g(a) = \sum_{a \in S} e(a)$$

$$\Rightarrow [S = \emptyset \vee \exists a \in S g(a) \not\succ_a f(a)]]\} \tag{5.12}$$

$$\mathcal{W}(\chi) = \{f \in \mathcal{F}(A, \mathbf{R}^k_+) : \sum_{a \in A} f(a) = \sum_{a \in A} e(a)$$
$$\wedge\ \exists p \in \Delta\ \forall a \in A\ f(a) \in D(p, \succ_a, e(a))\} \tag{5.13}$$

and

$$\mathcal{Q}(\chi) = \{f \in \mathcal{F}(A, \mathbf{R}^k_+) : \sum_{a \in A} f(a) = \sum_{a \in A} e(a)$$
$$\wedge\ \exists p \in \Delta\ \forall a \in A\ f(a) \in Q(p, \succ_a, e(a))\}. \tag{5.14}$$

Given $\delta \in \mathbf{R}_{++}$, define

$$\mathcal{W}_\delta(\chi) = \{f \in \mathcal{F}(A, \mathbf{R}^k_+) : \frac{1}{|A|} |\sum_{a \in A} f(a) - e(a)| < \delta$$
$$\wedge\ \exists p \in \Delta\ \forall a \in A\ f(a) \in D(p, \succ_a, e(a))\}. \tag{5.15}$$

Because $\mathcal{C}$, $\mathcal{Q}$, and $\mathcal{W}$ are defined by sentences, if χ is a hyperfinite exchange economy, we can form ${}^*\mathcal{C}(\chi)$, ${}^*\mathcal{W}(\chi)$, and ${}^*\mathcal{Q}(\chi)$; each is internal by the Internal Definition Principle. Define

$$\mathcal{W}_{\simeq 0}(\chi) = \cap_{\delta \in \mathbf{R}_{++}} {}^*\mathcal{W}_\delta(\chi). \tag{5.16}$$

Theorem 5.5.2 (Brown, Robinson, Khan, Rashid, Anderson) *Let $\chi : A \to {}^*(P_c \times \mathbf{R}^k_+)$ be a hyperfinite exchange economy.*

1. *If*
   (a) *$n \in {}^*\mathbf{N} \setminus \mathbf{N}$;*
   (b) *for each $a \in A$, $\succ_a$*
      i. *is *-monotonic;*
      ii. *satisfies *-free disposal;*

(c) $^\circ(\frac{1}{n}\sum_{a\in A} e(a)) \in \mathbf{R}^k_{++}$.

(d) $e(a)/n \simeq 0$ *for all* $a \in A$.

then for every $f \in {}^*\mathcal{C}(\chi)$, *there exists* $p \in {}^*\Delta_+$ *such that* $^\circ f(a) \in Q(^\circ p, {}^\circ{\succ_a}, {}^\circ e(a))$ *for* $\bar{\nu}$*-almost all* $a \in A$. *If* $^\circ p \in \Delta_{++}$ *and for each* $a \in A$, $0 \not\succ 0$, *then* $^\circ f(a) \in D(^\circ p, {}^\circ{\succ_a}, {}^\circ e(a))$ *for* $\bar{\nu}$*-almost all* $a \in A$.

2. *If the assumptions in 1 hold and in addition for each commodity* i, $\bar{\nu}(\{a \in A : {}^\circ{\succ_a}$ *is strongly monotonic,* $^\circ e(a)^i > 0\}) > 0$, *then* $^\circ p \in \Delta_{++}$ *and hence* $^\circ f(a) \in D(^\circ p, {}^\circ{\succ_a}, {}^\circ e(a))$ *for* $\bar{\nu}$*-almost all* a.

3. *If the assumptions in (1) and (2) hold and in addition* e *is S-integrable, then* f *is S-integrable and* $(^\circ p, {}^\circ f) \in \mathcal{W}(^\circ \chi)$.

4. *If the assumptions in (1) hold and in addition*

   (a) $^\circ \succ_a$ *is strongly convex for* $\bar{\nu}$*-almost all* $a \in A$;

   (b) *for each commodity* i, $\bar{\nu}(\{a : {}^\circ e(a)^i > 0\}) > 0$;

   (c) $\succ_a$ *is *-irreflexive, *-convex, and *-strongly convex for all* $a \in A$;

   then $f(a) \simeq {}^*D(p, \succ_a, e(a))$ *for* $\bar{\nu}$*-almost all* $a \in A$.

5. *If the assumptions in (1) and (4) hold and, in addition,* e *is S-integrable, then there exists* $g \in \mathcal{W}_{\simeq 0}(\chi)$ *such that*

$$\frac{1}{n}\sum_{a\in A} |f(a) - g(a)| \simeq 0. \tag{5.17}$$

Proof:

1. Suppose χ satisfies the assumptions in part (1) of the Theorem. By Anderson (1978) (see also Dierker (1975)) and the Transfer Principle, there exists $p \in {}^*\Delta_+$ such that

$$\frac{1}{n}\sum_{a\in A} {}^*\phi(f(a), p, e(a)) \leq \frac{6k \max_{a\in A} ||e(a)||_\infty}{n} \simeq 0 \tag{5.18}$$

since $\max_{a\in A} |e(a)|/n \simeq 0$. $\frac{1}{n}\sum_{a\in A} f(a) = \frac{1}{n}\sum_{a\in A} e(a)$ is finite, so $f(a)$ and $e(a)$ are finite for $\bar{\nu}$-almost all $a \in A$. ${}^\circ f(a) \in Q({}^\circ p, {}^\circ\succ_a, e(a))$ by Proposition 5.4.4. If ${}^\circ p \in \Delta_{++}$, then ${}^\circ f(a) \in D({}^\circ p, {}^\circ\succ_a, e(a))$ by Proposition 5.4.4.

2. Suppose in addition that for each commodity i, $\bar{\nu}(\{a \in A : {}^\circ\succ_a$ is strongly monotonic, ${}^\circ e(a)^i > 0\}) > 0$. We will show that ${}^\circ p \in \Delta_{++}$ by deriving a contradiction. If ${}^\circ p \notin \Delta_{++}$, we may assume without loss of generality that ${}^\circ p^1 = 0, {}^\circ p^2 > 0$. By assumption (1)(c), $|e(a)|$ is finite for $\bar{\nu}$-almost all $a \in A$. Let

$$S = \{a \in A : {}^\circ\succ_a \text{ is strongly monotonic,}$$
$${}^\circ e(a)^2 > 0,\ {}^\circ f(a) \in Q({}^\circ p, {}^\circ\succ_a, {}^\circ e(a))\}. \tag{5.19}$$

$\bar{\nu}(S) > 0$ by the conclusion of (1) and the additional assumption in (2), so in particular $S \neq \emptyset$. Suppose $a \in S$. Then ${}^\circ p \cdot {}^\circ e(a) \geq {}^\circ p^2 \cdot {}^\circ e(a)^2 > 0$. Let $x = {}^\circ f(a) + (1, 0, 0, \ldots, 0)$. Since ${}^\circ\succ_a$ is strongly monotonic, $x {}^\circ\succ_a {}^\circ f(a)$. ${}^\circ p \cdot x = {}^\circ p \cdot {}^\circ f(a) \leq {}^\circ p \cdot {}^\circ e(a)$. There are two cases to consider.

   (a) ${}^\circ p \cdot x < {}^\circ p \cdot {}^\circ e(a)$: Then ${}^\circ f(a) \notin Q({}^\circ p, {}^\circ\succ_a, {}^\circ e(a))$, a contradiction.

   (b) ${}^\circ p \cdot x = {}^\circ p \cdot {}^\circ e(a) > 0$. Since ${}^\circ\succ_a$ is continuous, there exists $\delta \in \mathbf{R}_{++}$ such that $y \in \mathbf{R}^k_+$, $|y - x| < \delta$

implies $y {}^\circ\!\succ_a {}^\circ f(a)$. We may find $y \in \mathbf{R}^k_+$ such that $|y - x| < \delta$ and ${}^\circ p \cdot y < {}^\circ p \cdot x = {}^\circ p \cdot {}^\circ e(a)$, so ${}^\circ f(a) \notin Q({}^\circ p, {}^\circ\!\succ_a, {}^\circ e(a))$, again a contradiction.

Consequently, ${}^\circ p \in \Delta_{++}$, so ${}^\circ f(a) \in D({}^\circ p, {}^\circ\!\succ_a, {}^\circ e(a))$ by the conclusion of (1).

3. We show first that f is S-integrable. Suppose $S \subset A$ is internal and $\nu(S) \simeq 0$.

$$\frac{1}{n}||\sum_{a\in S} f(a)||_\infty \leq \frac{1}{n \min_i p^i} \sum_{a \in S} p \cdot f(a)$$

$$\leq \frac{1}{n \min_i p^i} \sum_{a \in S} p \cdot e(a) + {}^*\phi_B(f(a), p, e(a))$$

$$\simeq \frac{1}{n \min_i p^i} \sum_{a\in S} p \cdot e(a) \simeq 0 \tag{5.20}$$

since e is S-integrable. Thus, f is S-integrable. By Theorem 4.4.6,

$$\int_{a\in A} {}^\circ f d\bar{\nu} = {}^\circ \left(\frac{1}{n} \sum_{a\in A} f(a) \right)$$

$$= {}^\circ \left(\frac{1}{n} \sum_{a\in A} e(a) \right) = \int_{a\in A} {}^\circ e d\bar{\nu}, \tag{5.21}$$

and so $({}^\circ p, {}^\circ f) \in \mathcal{W}({}^\circ \chi)$.

4. (a) Suppose ${}^\circ\!\succ_a$ is strongly convex. We show first that ${}^\circ\!\succ_a$ is strongly monotonic. Suppose $x, y \in \mathbf{R}^k_+$ and $x > y$. Let $z = 2x - y$. Then $\frac{z+y}{2} = x$. Since $z \neq y$, either $x {}^\circ\!\succ_a x$ or $x {}^\circ\!\succ_a y$. If $x {}^\circ\!\succ_a x$, then $x \succ_a x$ by Proposition 5.1.3, which contradicts irreflexivity. Therefore, we must have $x {}^\circ\!\succ_a y$, so ${}^\circ\!\succ_a$ is strongly monotonic. Consequently,

the assumptions in (4) imply the assumptions in (2), so ${}^\circ p \in \Delta_{++}$ and ${}^\circ f(a) \in D({}^\circ p, {}^\circ\succ_a, {}^\circ e(a))$ for $\bar{\nu}$-almost all a.

(b) Suppose $a \in A$. Transferring Theorem 1 of Anderson (1981), ${}^*D(p, \succ_a, e(a))$ contains exactly one element. Define $g(a) = {}^*D(p, \succ_a, e(a))$. For $\bar{\nu}$-almost all $a \in A$, we have $p \cdot f(a) \simeq p \cdot e(a) \simeq \inf\{p \cdot x : x \succ_a f(a)\}$ and $e(a)$ is finite; consider any such $a \in A$. We will show that $f(a) \simeq g(a)$. We consider two cases:

i. If $e(a) \simeq 0$, then $p \cdot f(a) \simeq 0 \simeq p \cdot g(a)$. Since ${}^\circ p >> 0$, $f(a) \simeq 0 \simeq g(a)$, so $f(a) \simeq g(a)$.

ii. If $e(a) \not\simeq 0$, then $p \cdot e(a) \not\simeq 0$. If $f(a) \not\simeq g(a)$, then either

$$\frac{{}^\circ f(a) + {}^\circ g(a)}{2} {}^\circ\succ_a {}^\circ f(a) \tag{5.22}$$

or

$$\frac{{}^\circ f(a) + {}^\circ g(a)}{2} {}^\circ\succ_a {}^\circ g(a). \tag{5.23}$$

A. If equation 5.22 holds, then since ${}^\circ\succ_a$ is continuous and $p \cdot e(a) \not\simeq 0$, we can find $w \in \mathbf{R}^k_+$ with $p \cdot w < p \cdot e(a)$, $p \cdot w \not\simeq p \cdot e(a)$, such that $w {}^\circ\succ_a {}^\circ f(a)$. By Proposition 5.1.3, $w \succ_a f(a)$, which contradicts $\inf\{p \cdot x : x \succ_a f(a)\} \simeq p \cdot e(a)$.

B. If equation 5.23 holds, we may find $w \in \mathbf{R}^k_+$ with $p \cdot w < p \cdot e(a)$, $p \cdot w \not\simeq p \cdot e(a)$, such that $w {}^\circ\succ_a {}^\circ g(a)$. By Proposition 5.1.3, $w \succ_a g(a)$, which contradicts $g(a) = {}^*D(p, \succ_a, e(a))$.

Accordingly, $f(a) \simeq g(a)$.

Therefore, we have $f(a) \simeq g(a)$ for $\bar{\nu}$-almost all $a \in A$.

5. Suppose the assumptions in (1) and (4) hold and in addition e is S-integrable. The assumptions in (4) have been shown to imply the assumptions in (2), so f is S-integrable by (3). As in (4), let $g(a) = {}^*D(p, \succ_a, e(a))$. An easier version of the argument in (3) proves that g is S-integrable. Therefore

$$\frac{1}{n}\sum_{a\in A}|f(a)-g(a)| \simeq \int_A {}^\circ|f(a)-g(a)|d\bar{\nu} = 0 \quad (5.24)$$

by Theorem 4.4.6. Therefore

$$\begin{aligned}\tfrac{1}{n}\left|\textstyle\sum_{a\in A} g(a)-e(a)\right| &= \tfrac{1}{n}\left|\textstyle\sum_{a\in A} g(a)-f(a)\right| \\ &\leq \tfrac{1}{n}\textstyle\sum_{a\in A}|g(a)-f(a)| \simeq 0,\end{aligned} \quad (5.25)$$

so $g \in \mathcal{W}_{\simeq 0}(\chi)$.

■

Remark 5.5.3 Theorem 5.5.2 reveals some significant differences between the hyperfinite and continuum formulations of large economies.

1. One can introduce atoms into both the hyperfinite and continuum (as in Shitovitz (1973,1974)) formulations. However, as noted by Hildenbrand on page 846 of this Handbook, this leads to problems in interpreting the preferences in the continuum formulation. In essence, the consumption set of a trader represented by an atom cannot be $\mathbf{R}^k_+$; it must allow consumptions infinitely

large compared to those of other traders. In asymptotic analogues of the theorems, key assumptions[1] are required to hold under rescalings of preferences; the economic content is then unclear, except in the special case of homothetic preferences. In the nonstandard formulation, this problem does not arise. Preferences over the nonstandard orthant ${}^*\mathbf{R}^k_+$ are rich enough to deal with atoms, although we do not cover this case in Theorem 5.5.2.

2. Now, let us compare how the nonstandard and continuum formulations treat the atomless case. In the continuum formulation, the endowment map is required to be integrable with respect to the underlying population measure. One could of course consider an endowment measure which is singular with respect to the underlying population measure. In this case, however, the representation of preferences becomes problematic. Specifically, if one considers a consumption measure μ which is singular with respect to the population measure, then μ has no Radon-Nikodym derivative with respect to the population measure, so one cannot identify the consumption of individual agents as elements of $\mathbf{R}^k_+$. Moreover, an allocation measure μ' may allocate a coalition consumption which is infinitely large compared to the consumption allocated that coalition by another measure μ''. As in the case with atoms, the consumption space over which preferences need be defined must be larger than $\mathbf{R}^k_+$. Asymptotic formulations require assumptions about rescaled preferences which are hard to interpret except in the case of ho-

[1] For example, strong monotonicity, in conjunction with compactness conditions inherent in the measure-theoretic formulation of convergence for sequences of economies, becomes a uniform monotonicity condition.

mothetic preferences. In the nonstandard framework, replacing the assumption that e is S-integrable with the much weaker assumption that $e(a)/|A| \simeq 0$ for all $a \in A$ poses no technical problems. Part (1) of Theorem 5.5.2 analyzes precisely what happens in that case, while part (3) indicates how the result is strengthened if we assume that e is S-integrable and ${}^\circ\succ_a$ is strongly monotonic for a set of agents of positive $\bar{\nu}$-measure. Example 5.5.4 provides an example of a hyperfinite economy satisfying the hypotheses of part (1), but not those of part (3).

3. Suppose that the endowment map e is S-integrable, which corresponds to the integrability of endowment inherent in the definition of the continuum economy. In a continuum economy, allocations (including core allocations) are by definition required to be integrable. In the hyperfinite context, allocations may fail to be S-integrable. If $f \in {}^*\mathcal{C}(\chi)$ is not S-integrable, then

$$\int_A {}^\circ f d\bar{\nu} < {}^\circ\left(\frac{1}{n}\sum_{a\in A} f(a)\right) = {}^\circ\left(\frac{1}{n}\sum_{a\in A} e(a)\right) = \int_A {}^\circ e d\bar{\nu}, \tag{5.26}$$

so ${}^\circ f$ does not correspond to an allocation of the associated Loeb measure economy. In Example 5.5.5, we present an example due to Manelli of a hyperfinite economy χ with a (non S-integrable) core allocation f such that $\frac{1}{n}\sum_{a\in A} {}^*\phi_B(f(a), p, e(a)) \not\simeq 0$. However, core equivalence holds in the associated continuum economy χ, in the sense that $g \in \mathcal{C}({}^\circ\chi)$ implies $g \in \mathcal{Q}({}^\circ\chi)$. Indeed, Proposition 5.5.6 shows that, in the absence of monotonicity assumptions, any S-integrable core allocation f is close to an element of the core of ${}^\circ\chi$. In other words, the integrability condition in the definition of the continuum core is revealed by the hyperfinite formulation to be a strong endogenous assumption.

4. In Example 5.5.8, we present Manelli's example of an economy χ with endowment e and core allocation f, both of which are S-integrable, such that

$$\frac{1}{n}\sum_{a\in A} {}^*\phi_B(f(a), p, e(a)) \simeq 0 \tag{5.27}$$

for some $p \in {}^*\Delta$ but there is no $p \in {}^*\Delta$ such that $\frac{1}{n}\sum_{a\in A} {}^*\phi(f(a), p, \succ_a, e(a)) \simeq 0$. Core equivalence holds in the associated continuum economy χ, in the sense that $g \in \mathcal{C}(^\circ\chi)$ implies $g \in \mathcal{Q}(^\circ\chi)$. Indeed, $^\circ f \in \mathcal{C}(^\circ\chi)$, so $^\circ f \in \mathcal{Q}(^\circ\chi)$. In the example, the commodity bundles which show p is not an approximate supporting price for f are infinite; they thus pose no barrier to the verification of the support condition in the continuum economy.

5. The condition

$${}^*\phi(f(a), p, \succ_a, e(a)) \simeq 0 \tag{5.28}$$

in the hyperfinite formulation implies the condition

$$\phi(^\circ f(a), {}^\circ p, {}^\circ \succ_a, {}^\circ e(a)) = 0 \tag{5.29}$$

in the Loeb continuum economy which in turn implies

$$^\circ f(a) \in Q(^\circ p, {}^\circ\succ_a, {}^\circ e(a)). \tag{5.30}$$

In the presence of strong convexity, equation 5.28 implies that

$$f(a) \simeq {}^*Q(p, \succ_a, e(a)); \tag{5.31}$$

without strong convexity, equation 5.31 may fail, as shown by Example 5.5.9. The formulas 5.28 and 5.31 are nearly internal; using the Transfer Principle, we show in Theorem 5.5.10 that strong convexity of preferences implies a stronger form of convergence for sequence of finite economies. However, strong convexity is not needed to deduce formula 5.30 (which corresponds to the conclusion of Aumann's Equivalence Theorem) from 5.29. Thus, in the continuum economy, convexity plays no role in the theorem. Since formula 5.30 is far from internal, it is not amenable to application of the Transfer Principle. Thus, the conclusion of Aumann's Theorem does not reflect the behavior of sequences of finite economies, in the sense that it does not capture the implications of convexity for the form of convergence.

Example 5.5.4 (Tenant Farmers) In this example, we construct a hyperfinite economy in which the endowments are not S-integrable. Core convergence of the associated sequence of finite economies follows from Theorem 5.5.10; however, the sequence does not satisfy the hypotheses of Hildenbrand (1974) or Trockel (1976).

1. We consider a hyperfinite economy $\chi : A \to {}^*(P \times \mathbf{R}^k_+)$, where $A = \{1, \ldots, n^2\}$ for some $n \in {}^*\mathbf{N} \setminus \mathbf{N}$. For all $a \in A$, the preference of a is given by a utility function $u(x, y) = 2\sqrt{2}x^{1/2} + y$. The endowment is given by

$$e(a) = \begin{cases} (n+1, 1) & \text{if } a = 1, \ldots n \\ (1, 1) & \text{if } a = n+1, \ldots, n^2 \end{cases} \tag{5.32}$$

Think of the first commodity as land, while the second commodity is food. The holdings of land are heavily concentrated among the agents $1, \ldots, n$, a small fraction of the total population. Land is useful as an input to the production of food; however, the marginal product of land diminishes rapidly as the size of the plot worked by a given individual increases.

2. There is a unique Walrasian equilibrium, with $p = \left(\frac{1}{2}, \frac{1}{2}\right)$ and allocation

$$f(a) = \begin{cases} (2, n) & \text{if } a = 1, \ldots, n \\ (2, 0) & \text{if } a = n+1, \ldots, n^2. \end{cases} \tag{5.33}$$

Thus, the "tenant farmers" $n+1, \ldots, n^2$ purchase the right to use land with their endowment of food; they then feed themselves from the food they are able to produce on their rented plot of land.

3. By part (4) of Theorem 5.5.2, $g \in ({}^*\mathcal{C})(\chi) \Rightarrow g(a) \simeq (2, 0)$ for $\bar{v}$-almost all $a \in A$, so that almost all of the tenant farmers receive allocations infinitely close to their Walrasian consumption. A slight refinement of Theorem 5.5.2 in Anderson (1981) shows that

$$^\circ\left(\frac{1}{|A|}\sum_{a=n+1}^{n^2} g(a)\right) = (2, 0) \tag{5.34}$$

and

$$^\circ\left(\frac{1}{|A|}\sum_{a=1}^{n} g(a)\right) = (0, 1). \tag{5.35}$$

Thus, the per capita consumption allocated to the two classes (landowners and tenant farmers) is infinitely close to the Walrasian consumptions of those classes.

4. In the associated sequence of finite economies, if $g_n \in \mathcal{C}(\chi_n)$, one concludes by transfer that

$$\left(\frac{1}{|A_n|} \sum_{a=n+1}^{n^2} g_n(a) \right) \to (2,0) \tag{5.36}$$

and

$$\left(\frac{1}{|A_n|} \sum_{a=1}^{n} g_n(a) \right) \to (0,1). \tag{5.37}$$

5. If one forms the associated continuum economy ${}^\circ\chi$ via the Loeb measure construction, one gets

$$\int_A {}^\circ e(a) d\bar{\nu} = (1,1) \not\simeq (2,1) = \frac{1}{|A|} \sum_{a \in A} e(a). \tag{5.38}$$

In other words, the measure-theoretic economy ${}^\circ\chi$ has less aggregate endowment than the hyperfinite economy χ. In ${}^\circ\chi$, the unique Walrasian equilibrium has price $\left(\frac{\sqrt{2}}{1+\sqrt{2}}, \frac{1}{1+\sqrt{2}}\right)$ and consumption $(1,1)$ almost surely. Thus, the continuum economy does not capture the behavior of the sequence χ_n of finite economies. Trockel (1976) proposed a solution involving rescaling the weight assigned to the agents in the sequence of finite economies. However, the example violates Trockel's hypotheses, since the preferences do not converge under Trockel's rescaling to a strongly monotone preference as he requires. We conclude that the assumption that endowments are integrable in the continuum model represents a serious restriction on the ability of the continuum to capture the behavior of large finite economies.

Example 5.5.5 [2] **(Manelli)**

[2] Examples 5.5.5, 5.5.8, and 5.5.9 were originally given in the context of a sequence of finite exchange economies.

1. We consider a hyperfinite exchange economy $\chi : A \to {}^*(\mathcal{P}_c \times \mathbf{R}^2_+)$. $A = \{1, \ldots, n+2\}$ with $n \in {}^*\mathbf{N} \setminus \mathbf{N}$. The endowment map is $e(1) = e(2) = 0$, $e(a) = (1,1)$ $(a = 3, \ldots, n+2)$. Let V denote the cone $\{0\} \cup \{x \in \mathbf{R}^2_{++} : 0.5 < \frac{x^1}{x^2} < 2\}$. Consider the allocation

$$f(1) = (n, 0),\ f(2) = (0, \tfrac{n}{2}),$$
$$f(a) = (0, \tfrac{1}{2})\ (a = 3, \ldots, n). \tag{5.39}$$

The preferences have the property that

$$x \succ_a f(a) \iff x - f(a) \in {}^*V. \tag{5.40}$$

It is not hard to see that there are internal complete, transitive preferences that satisfy equation 5.40. In addition, we can choose $\succ_a$ so that ${}^\circ\succ_a$ is locally non-satiated for each $a \in A$.

2. It is not hard to verify that $f \in {}^*\mathcal{C}(\chi)$. However, f is not approximable by a core allocation of ${}^\circ\chi$. Indeed,

$$\int_A {}^\circ f d\bar{\nu} = (0, \frac{1}{2}) \neq (1,1) = \int_A {}^\circ e d\bar{\nu}, \tag{5.41}$$

so ${}^\circ f$ is not even an allocation of ${}^\circ\chi$.

3. Given $p \in {}^*\Delta_+$,

$$\frac{1}{n+2} \sum_{a \in A} \phi_B(f(a), p, e(a))$$

$$= \frac{n|p^1| + \frac{n}{2}|p^2| + n|p^1 + \frac{p^2}{2}|}{n+2} \geq \frac{n}{2(n+2)} \not\simeq 0. \tag{5.42}$$

4. ${}^\circ\chi$ is an Aumann continuum economy with locally nonsatiated preferences. As Hildenbrand notes on page 845 of Chapter 18 of this Handbook, a careful examination of the original proof of Aumann's Equivalence Theorem shows that $\mathcal{C}({}^\circ\chi) \subset \mathcal{Q}({}^\circ\chi)$. In particular,

$$g \in \mathcal{C}({}^\circ\chi) \Rightarrow \exists p \in \Delta \int_A \phi(g(a), p, {}^\circ\succ_a, {}^\circ e(a)) d\bar{\nu} = 0. \tag{5.43}$$

   Comparing equations 5.42 and 5.43, one sees that the decentralization properties of ${}^*\mathcal{C}(\chi)$ are totally different from those of $\mathcal{C}({}^\circ\chi)$. By the Transfer Principle, one can construct a sequence of finite economies whose cores have the decentralization properties exhibited by ${}^*\mathcal{C}(\chi)$ rather than those exhibited by the Aumann continuum economy $\mathcal{C}({}^\circ\chi)$.

5. In Proposition 5.5.6, we show that if h is S-integrable and $h \in {}^*\mathcal{C}(\chi)$, then ${}^\circ h \in \mathcal{C}({}^\circ\chi)$; hence,

$$h \in \mathcal{C}({}^\circ\chi) \Rightarrow \exists p \in \Delta \int_A \phi(h(a), p, {}^\circ\succ_a, {}^\circ e(a)) d\bar{\nu} = 0 \tag{5.44}$$

   by item (4). Consequently, the properties of the internal core are significantly different from those of the set of S-integrable core allocations. By the Transfer Principle, one can construct a sequence of finite economies whose core allocations have the decentralization properties exhibited by ${}^*\mathcal{C}(\chi)$. Consequently, the restriction to integrable allocations inherent in the definition of the core in the Aumann economy is thus a strong endogenous assumption which prevents the Aumann economy from capturing the properties of certain sequences of finite economies.

Proposition 5.5.6 (Brown, Robinson, Rashid) *Suppose* $\chi : A \to {}^*(P_c \times \mathbf{R}^k_+)$ *is a hyperfinite exchange economy. If* e

and f *are S-integrable, and* $f \in {}^*\mathcal{C}(\chi)$, *then* ${}^\circ f \in \mathcal{C}({}^\circ\chi)$.

Proof: $\int_A {}^\circ f d\bar{\nu} = {}^\circ(\frac{1}{n}\sum_{a\in A} f(a)) = {}^\circ(\frac{1}{n}\sum_{a\in A} e(a)) = \int_A {}^\circ e d\bar{\nu}$ by Theorem 4.4.6. Thus, ${}^\circ f$ is an allocation of ${}^\circ\chi$.

Suppose ${}^\circ f \notin \mathcal{C}({}^\circ\chi)$. Then there exists $S \in \mathcal{A}$ with $\bar{\nu}(S) > 0$ and an integrable function $g : S \to \mathbf{R}^k_+$ such that $\int_S g d\bar{\nu} = \int_S e d\bar{\nu}$ and $g(a) {}^\circ\!\succ_a {}^\circ f(a)$ for $\bar{\nu}$-almost all $a \in S$. By Theorem 4.1.6, there exists $T' \in \mathcal{A}$ such that $\bar{\nu}((S \setminus T') \cup (T' \setminus S)) = 0$. Define $g(a) = 0$ for $a \in T' \setminus S$. By Theorem 4.4.6, there is an S-integrable function $G : T' \to {}^*\mathbf{R}^k_+$ such that $G(a) \simeq g(a)$ for $\bar{\nu}$-almost all $a \in T'$. Let $J = \{j \in \{1, \ldots, k\} : \int_S e^j d\bar{\nu} = 0\}$. We can choose G such that $G(a)^j = 0$ for all $a \in T'$, $j \in J$. Let $T = \{a \in T' : G(a) \succ_a f(a)\}$. $T \in \mathcal{A}$ by the Internal Definition Principle; moreover, $\nu(T' \setminus T) \simeq 0$. Given $m \in \mathbf{N}$, let $T_m = \{a \in T : y \in {}^*\mathbf{R}^k_+, |y - G(a)| < \frac{1}{m} \Rightarrow y \succ_a f(a)\}$. Then $T_m \in \mathcal{A}$ by the Internal Definition Principle and $\bar{\nu}(\cup_{m\in\mathbf{N}} T_m) = \bar{\nu}(T)$ by Propositions 3.1.5 and 5.1.3, and the fact that ${}^\circ g(a) {}^\circ\!\succ_a {}^\circ f(a)$ for $\bar{\nu}$-almost all $a \in S$. Since G is S-integrable, there exists $m \in \mathbf{N}$ such that

$$\frac{1}{n}\sum_{a\in T_m} G(a)^j \geq \frac{\frac{1}{n}\sum_{a\in T} G(a)^j}{2} \tag{5.45}$$

for $j \in \{1, \ldots, k\} \setminus J$. Let $H(a) = G(a)$ if $a \in T \setminus T_m$. For $a \in T_m$, define

$$H(a)^j = \frac{\sum_{a\in T} e(a)^j}{|T_m|} \text{ if } j \in J \tag{5.46}$$

and

$$H(a)^j = \left(1 - \frac{\sum_{b\in T} G(b)^j - e(b)^j}{\sum_{b\in T_m} G(b)^j}\right) G(a)^j \text{ if } j \notin J. \tag{5.47}$$

Then $H(a) = G(a) \succ_a f(a)$ for $a \in T \setminus T_m$. For $a \in T_m$, $H(a) \in {}^*\mathbf{R}^k_+$, and $|H(a) - G(a)| \simeq 0$, so $H(a) \succ_a f(a)$. Thus,

$H(a) \succ_a f(a)$ for all $a \in T$. An easy calculation shows that $\sum_{a\in T} H(a) = \sum_{a\in T} e(a)$, so $f \notin {}^*\mathcal{C}(\chi)$, a contradiction which completes the proof.■

Example 5.5.7 In this example, we show that the converse to Proposition 5.5.6 does not hold. Specifically, we construct an S-integrable allocation f such that $^\circ f \in \mathcal{C}(\chi)$ but $f \notin {}^*\mathcal{C}(\chi)$. In a sense, this example is merely a failure of lower hemicontinuity on the part of the core, a well-known phenomenon. Its importance lies in showing that the topology on $\mathcal{P}_c$ is inappropriate for the study of economies where large consumptions could matter. We consider a hyperfinite exchange economy $\chi : A \to {}^*(\mathcal{P}_c \times \mathbf{R}^2_+)$. $A = \{1, \ldots, n+1\}$ with $n \in {}^*\mathbf{N} \setminus \mathbf{N}$. The endowment map is $e(a) = (1,1)$ for all $a \in A$. Let $p = (1 - \frac{1}{n}, \frac{1}{n})$. The preferences have the property that

$$x \succ_1 (1,1) \Longleftrightarrow \left[[x >> (1,1)] \vee [x >> (0, \frac{n}{2})]\right];$$

$$x \succ_a (1,1) \Longleftrightarrow p \cdot x > p \cdot (1,1) \; (a = 2, \ldots, n+1). \quad (5.48)$$

Consider the allocation $f = e$. f is Pareto dominated in χ by the allocation

$$\begin{aligned} g(1) &= \left(\tfrac{1}{n}, \tfrac{3n+3}{4}\right); \\ g(a) &= \left(1 + \tfrac{1}{n} - \tfrac{1}{n^2}, \tfrac{n+1}{4n}\right) \; (a = 2, \ldots, n+1). \end{aligned} \quad (5.49)$$

Note however that $^\circ f$ (which equals f) is a Walrasian allocation of $^\circ\chi$ with Walrasian price $(1,0)$. One cannot block $^\circ f$ by $^\circ g$ precisely because g is not S-integrable. Accordingly, the restriction to integrable blocking allocations inherent in the definition of the core in the Aumann continuum economy is a significant endogenous assumption.

Example 5.5.8 (Manelli)

1. We consider a hyperfinite exchange economy $\chi : A \to {}^*(\mathcal{P}_c \times \mathbf{R}^2_+)$. $A = \{1, \ldots, 2n\}$ with $n \in {}^*\mathbf{N} \setminus \mathbf{N}$. The endowment map is $e(a) = (1,1)$ for all $a \in A$. Let V denote the cone $\{0\} \cup \{x \in \mathbf{R}^2_{++} : 0.5 < \frac{x^1}{x^2} < 2\}$. Consider the allocation

$$\begin{array}{c} f(a) = (0, \frac{1}{2}),\ (a = 1, \ldots, n); \\ f(a) = (2, \frac{3}{2}),\ (a = n+1, \ldots, 2n). \end{array} \tag{5.50}$$

The preferences have the property that

$$x \succ_a f(a) \iff x - f(a) \in {}^*V\ (a = 2, \ldots, 2n);$$

$$x \succ_1 f(1) \iff [[x - f(1) \in {}^*V] \vee [x >> (n, 0)]] \tag{5.51}$$

It is not hard to see that there are internal complete, transitive preferences that satisfy equation 5.51. In addition, we can choose $\succ_a$ so that ${}^\circ\!\succ_a$ is locally non-satiated for each $a \in A$.

2. It is not hard to verify that $f \in {}^*\mathcal{C}(\chi)$. Moreover, e and f are S-integrable, so ${}^\circ f \in \mathcal{C}({}^\circ\chi)$ by Proposition 5.5.6. As in item 4 of Example 5.5.5, there exists $p \in \Delta$ such that

$$\int_A \phi({}^\circ f(a), p, {}^\circ\!\succ_a, {}^\circ e(a))\,d\bar{\nu} = 0. \tag{5.52}$$

Indeed, it is easy to see that $p = \pm\left(\frac{1}{3}, -\frac{2}{3}\right)$. Consequently,

$$\frac{1}{|A|}\sum_{a \in A} {}^*\phi_B(f(a), p, e(a)) \simeq \int_A \phi_B({}^\circ f(a), p, {}^\circ e(a)) = 0 \tag{5.53}$$

by Theorem 4.4.6.[3] However, with $p = \pm\left(\frac{1}{3}, -\frac{2}{3}\right)$,

$$\frac{1}{|A|}\sum_{a\in A} {}^*\phi_S(f(a), p, \succ_a) = -\infty. \tag{5.54}$$

Comparing equations 5.54 and 5.52, one sees that the decentralization properties of f are quite different from those of ${}^\circ f$. By the Transfer Principle, one can construct a sequence of finite economies whose cores have the decentralization properties exhibited by f rather than those exhibited by ${}^\circ f$.

Example 5.5.9 (Anderson, Mas-Colell) We consider a hyperfinite exchange economy $\chi : A \to {}^*(\mathcal{P}_c \times \mathbf{R}^2_+)$. $A = \{1, \ldots, n\}$ with $n \in {}^*\mathbf{N} \setminus \mathbf{N}$. Fix a transcendental number $\xi \in [0, 1]$. The endowment map is $e(a) = (1 + \xi^a)(1, 1)$ for all $a \in A$. Let $\delta = \min\{|\sum_{a\in A} h_a(1 + \xi^a)| : h \text{ internal}, h_a \in \{-1, 0, 1\}, h_a \text{ not all } 0\}$. Since ξ is transcendental, $\delta \in {}^*\mathbf{R}_{++}$. One can construct a homothetic preference $\succ \in {}^*\mathcal{P}_c$ such that $(\frac{1}{2}, \frac{3}{2}) \succ (1, 1)$ and $(\frac{3}{2}, \frac{1}{2}) \succ (1, 1)$, but such that $f = e \in {}^*\mathcal{C}(\chi)$; the idea is to make the region around $(\frac{1}{2}, \frac{3}{2})$ and $(\frac{3}{2}, \frac{1}{2})$ which is preferred to $(1, 1)$ very small.[4] For any price $q \in {}^*\Delta$, ${}^\circ|f(a) - {}^*D(q, \succ_a, e(a))| \geq \frac{1}{\sqrt{2}}$ for all $a \in A$. However, ${}^\circ f \in \mathcal{W}({}^\circ\chi)$, in fact ${}^\circ f \in D((\frac{1}{2}, \frac{1}{2}), {}^\circ{\succ_a}, {}^\circ e(a))$ for all $a \in A$. As a consequence, the Aummann continuum economy fails to distinguish between the equivalence conditions in equation 5.28 (which says that the demand gap of the core allocation is small) and 5.31 (which says that the core allocation is close to the demand set). In particular, convexity plays no role in Aumann's equivalence theorem, while it significantly alters the form of the equivalence theorem for hyperfinite

[3] It is also easy to verify equation 5.53 by direct reference to the hyperfinite economy χ.

[4] See Anderson and Mas-Colell (1988) for details.

economies; by the Transfer Principle, convexity significantly alters the form of core convergence for sequences of large finite economies.

Theorem 5.5.10 Brown, Robinson, Khan, Rashid, Anderson *Let $\chi_n : A_n \to (P_c \times \mathbf{R}^k_+)$ be a sequence of finite exchange economies.*

1. *If*
   (a) $|A_n| \to \infty$;
   (b) *for each $n \in \mathbf{N}$ and $a \in A_n$, $\succ_a$*
      i. *is monotonic;*
      ii. *satisfies free disposal;*
   (c) i. $\overline{\lim}\frac{1}{n}\sum_{a\in A} e(a) << \infty$;
      ii. $\underline{\lim}\frac{1}{n}\sum_{a\in A} e(a) >> 0$; *and*
   (d) $\max_{a\in A_n} \frac{|e(a)|}{|A_n|} \to 0$;

   then for every sequence $f_n \in \mathcal{C}(\chi_n)$, there exists a sequence $p_n \in \Delta_+$ such that

$$\frac{1}{|A_n|}\sum_{a\in A_n} \phi(f_n(a), p_n, \succ_a, e(a)) \to 0. \tag{5.55}$$

2. *If the assumptions in (1) hold and in addition there is an compact set K of strongly monotonic preferences and $\delta \in \mathbf{R}_{++}$ such that for each commodity i and each $n \in \mathbf{N}$,*

$$\frac{|\{a \in A_n :\succ_a \in K,\ e(a)^i \geq \delta\}|}{|A_n|} \geq \delta, \tag{5.56}$$

   then there is a compact set $D \subset \Delta_{++}$ and $n_0 \in \mathbf{N}$ such that $p_n \in D$ for all $n \geq n_0$.

3. *If the assumptions in (1) and (2) hold and in addition the endowment sequence* $\{e_n : n \in \mathbf{N}\}$ *is uniformly integrable, then the sequence* $\{f_n : n \in \mathbf{N}\}$ *is uniformly integrable.*

4. *If the assumptions in (1) hold and in addition*

   (a) *for all* $\gamma \in \mathbf{R}_{++}$*, there is an compact set* K *of strongly convex preferences such that for all* $n \in \mathbf{N}$*,*

   $$\frac{|\{a \in A_n : \succ_a \in K\}|}{|A_n|} > 1 - \gamma; \tag{5.57}$$

   (b) *there is a* $\delta \in \mathbf{R}_{++}$ *such that, for each commodity* i*,*

   $$\frac{|\{a \in A_n : e(a)^i \geq \delta\}|}{|A_n|} \geq \delta; \tag{5.58}$$

   (c) $\succ_a$ *is irreflexive, convex, and strongly convex for all* $n \in \mathbf{N}$ *and all* $a \in A_n$*;*

   then for each $\epsilon \in \mathbf{R}_{++}$*,*

   $$\frac{|\{a \in A_n : |f_n(a) - D(p_n, \succ_a, e(a))| > \epsilon\}|}{|A_n|} \to 0. \tag{5.59}$$

5. *If the assumptions in (1) and (4) hold and, in addition,* e *is S-integrable, then there exists a sequence* $\epsilon_n \to 0$ *and* $g_n \in \mathcal{W}_{\epsilon_n}(\chi_n)$ *such that*

   $$\frac{1}{|A_n|} \sum_{a \in A_n} |f_n(a) - g_n(a)| \to 0. \tag{5.60}$$

Proof:

1. This follows immediately from Anderson (1978); see also Dierker (1975). The proof given in Anderson (1978) was originally discovered by translating nonstandard proofs of part (1) of Theorem 5.5.2 and a weaker version of part (1) of Theorem 5.5.10. Note that if $n \in {}^*\mathbf{N} \setminus \mathbf{N}$, then χ_n satisfies the hypotheses of part (1) of Theorem 5.5.2.

2. Suppose the additional assumption in (2) holds. By Transfer, for all $n \in {}^*\mathbf{N}$, $\nu(\{a \in A_n : \succ_a \in {}^*K, e(a)^i \geq \delta\}) \geq \delta$. If $\succ_a \in {}^*K$, then ${}^\circ\!\succ_a \in K$ by Theorem 3.3.2, so ${}^\circ\!\succ_a$ is strongly monotonic. Hence, for $n \in {}^*\mathbf{N} \setminus \mathbf{N}$, χ_n satisfies the assumptions of part (2) of Theorem 5.5.2. Hence, ${}^\circ p_n \in \Delta_{++}$. Hence, for $n \in {}^*\mathbf{N} \setminus \mathbf{N}$, ${}^\circ p_n \in \Delta_{++}$. Let $M = \{n \in \mathbf{N} : p_n \notin \Delta_{++}\}$. If M is infinite, then there exists $n \in {}^*M \cap ({}^*\mathbf{N} \setminus \mathbf{N})$, a contradiction. Hence M is finite; let $n_0 = (\max M)+1$. Let $D = \{{}^\circ p_n : n \in {}^*\mathbf{N}, n \geq n_0\}$. D is compact by Proposition 3.3.7, $D \subset \Delta_{++}$, and $p_n \in D$ for all $n \geq 0$, $n \in \mathbf{N}$.

3. Suppose that the sequence e_n is uniformly integrable. Then for $n \in {}^*\mathbf{N} \setminus \mathbf{N}$, e_n is S-integrable by Proposition 4.4.8. By part (3) of Theorem 5.5.2, f_n is S-integrable for $n \in {}^*\mathbf{N}$. Then the sequence $\{f_n : n \in \mathbf{N}\}$ is uniformly integrable by Proposition 4.4.8.

4. Fix $\epsilon \in \mathbf{R}_{++}$. It is easy to see that the assumptions in (4) imply that the assumptions of part (4) of Theorem 5.5.2 hold for $n \in {}^*\mathbf{N} \setminus \mathbf{N}$. Thus, for $n \in {}^*\mathbf{N} \setminus \mathbf{N}$,

$$\nu_n(\{a \in A_n : |f_n(a) - {}^*D(p_n, \succ_a, e(a))| > \epsilon\}) \simeq 0. \tag{5.61}$$

By Proposition 3.1.9, for $n \in \mathbf{N}$,

$$\nu_n(\{a \in A_n : f_n(a) - D(p_n, \succ_a, e(a)) > \epsilon\}) \to 0. \tag{5.62}$$

5. For $n \in \mathbf{N}$, choose p_n and $g_n \in D(p_n, \succ_a, e(a))$ to minimize $\frac{1}{|A_n|} \sum_{a \in A_n} |f_n(a) - g_n(a)|$. If $n \in {}^*\mathbf{N} \setminus \mathbf{N}$, then χ_n satisfies the hypotheses of part (5) of Theorem 5.5.2, so

$$\frac{1}{|A_n|} \sum_{a \in A_n} |f_n(a) - g_n(a)| \simeq 0. \tag{5.63}$$

By Proposition 3.1.10,

$$\epsilon_n = \frac{1}{|A_n|} \sum_{a \in A_n} |f_n(a) - g_n(a)| \to 0. \tag{5.64}$$

Then $g_n \in \mathcal{W}_{\epsilon_n}(\chi_n)$, which completes the proof.

■

5.6 Approximate Equilibria

This section will give a discussion of Khan (1975), Khan and Rashid (1982), and Anderson, Khan and Rashid (1982).

5.7 Pareto Optima

This section will give a discussion of Khan and Rashid (1975) and Anderson (1988).

5.8 Bargaining Set

This section will give a discussion of Geanakoplos (1978).

5.9 Value

This section will contain a discussion of Brown and Loeb (1976).

5.10 "Strong" Core Theorems

This section will contain a discussion of Anderson (1985) and Hoover (1989).

Chapter 6

Continuum of Random Variables

6.1 The Problem

In modelling a variety of economic situations, it is desirable to have a continuuum of independent identically distributed random variables, and to be able to assert that, with probability one, the distribution of outcomes of those random variables equals the theoretical distribution; in other words, there is individual uncertainty but no aggregate uncertainty. Some applications include Lucas and Prescott (1974), Diamond and Dybvig (1983), Bewley (1986), and Faust (1988); see Feldman and Gilles (1985) for other references.

There is no difficulty in defining a continuum of independent, identically distributed random variables. Suppose $(\Omega_0, \mathcal{C}_0, \rho_0)$ is a probability space, and $X : \Omega_0 \to \mathbf{R}$ a random variable with distribution function F. Let $(\Omega, \mathcal{B}, \rho) = \prod_{t\in[0,1]}(\Omega_0, \mathcal{B}_0, \rho_0)$, and define $X_t(\omega) = X(\omega_t)$. Then the family $\{X_t : t \in [0,1]\}$ is a continuum of independent random variables with distribution F.

The problem arises in the attempt to formulate the statement that there is no aggregate uncertainty. Suppose $([0,1],\mathcal{B},\mu)$ is the Lebesgue measure space. Given $\omega \in \Omega$, the empirical distribution function should be defined as $F_\omega(r) = \mu(\{t \in [0,1] : X_t(\omega) \leq r\}$. Unfortunately, $\{t \in [0,1] : X_t(\omega) \leq r\}$ need not be measurable, so the empirical distribution function need not be defined.

Judd (1985) considered a slightly different construction of $(\Omega,\mathcal{B},\mu)$ due to Kolmogorov. In it, he shows that $\{\omega : F_\omega$ is defined$\}$ is a non-measurable set with outer measure 1 and inner measure 0. Thus, one can find an extension μ' of the Kolmogorov measure μ such that F_ω is defined for μ'-almost all ω. However, $\{\omega : F_\omega = F\}$ is not measurable with respect to μ'; in fact, it has μ' outer measure 1 and μ' inner measure 0. Thus, one can find an extension μ'' of μ' with the property that $\mu''(\{\omega : F_\omega = F\}) = 1$. However, the extensions to μ' and μ'' are arbitrary, leaving the status of economic predictions from such models unclear.

A variety of standard constructions have been proposed to alleviate the problem (Feldman and Gilles(1985), Uhlig (1988), and Green (1989)). Much earlier, Keisler gave a broad generalization of the Law of Large Numbers for hyperfinite collections of random variables on Loeb measure spaces (Theorem 4.11 of Keisler (1977)). Since Loeb measure spaces are standard probability spaces in the usual sense, this provides a solution of the continuum of random variables problem. In section 6.2, we provide a simplified version of Keisler's result. In section 6.3, we describe a nontatonnement price adjustment model due to Keisler (1979, 1986, 1990, 1992, 1996); in Keisler's model, individual uncertainty over trading times in a hyperfinite exchange economy results in no aggregate uncertainty.

6.2 Loeb Space Construction

Loeb probability spaces are standard probability spaces in the usual sense, but they have many special properties. In the following construction, the internal algebra is guaranteed to be rich enough to ensure that the measurability problems outlined in Section 6.1 never arise. The construction also satisfies an additional uniformity condition highlighted in Green (1989), since the conclusion holds on every subinterval of the set of traders (conclusion 2b of Theorem 6.2.2, below).

Construction 6.2.1 Let $(A, \mathcal{A}, \nu)$ be as in the construction of Lebesgue measure (Construction 4.2.1). Suppose $Y : A \to {}^*\mathbf{R}$ is ν-measurable, and $^\circ|Y(a)| < \infty$ for $\bar{\nu}$-almost all $a \in A$. Define $\Omega = \prod_{a \in A} A$, $\mathcal{R} = ({}^*\mathcal{P})(\Omega)$, $\rho(B) = \frac{|B|}{|\Omega|}$ for $B \in \mathcal{R}$, $Y_a(\omega) = Y(\omega_a)$, and $X_a(\omega) = {}^\circ Y(\omega_a)$.

Theorem 6.2.2 (Keisler) *Consider Construction 6.2.1. Let F be the distribution function for $^\circ Y$, i.e. $F(r) = \bar{\nu}(\{a \in A : {}^\circ Y(a) \leq r\})$.*

1. *X_a is $\bar{\rho}$-measurable, and has distribution function F, for all $a \in A$;*
2. *for $\bar{\rho}$-almost all $\omega \in \Omega$, for all $r \in \mathbf{R}$, for all $s, t \in [0, 1]$ satisfying $s < t$*
   - *(a) $\{a \in A \cap {}^*[s, t] : X_a \leq r\} \in \bar{\mathcal{R}}$ and*
   - *(b) $\bar{\nu}(\{a \in A \cap {}^*[s, t] : X_a \leq r\}) = (t - s)F(r)$;*

Proof:

1. X_a is $\bar{\rho}$-measurable by Theorem 4.4.2. For all $a \in A$, the distribution function of X_a equals F, the distribution function of $^\circ Y$.

2. If $r \in \mathbf{R}$, let $C_r = \{a \in A : {}^\circ Y(a) \leq r\}$. Since $C_r = \cap_{m=1}^{\infty}\{a \in A : Y(a) \leq r + \frac{1}{m}\}$, ${}^\circ\nu(\{a \in A : Y(a) \leq r + \frac{1}{m}\}) \to \bar{\nu}(C_r)$ as $m \to \infty$. Therefore, there exists $m \in {}^*\mathbf{N} \setminus \mathbf{N}$ such that $\nu(\{a \in A : Y(a) \leq r + \frac{1}{m}\}) \simeq \bar{\nu}(C_r)$, by Proposition 3.1.9; let $D_r = \{a \in A : Y(a) \leq r + \frac{1}{m}\}$.

   Let $W = \{(r,s,t) \in \mathbf{R} \times [0,1] \times [0,1] : s < t\}$. By Theorem 1.13.4, we may find an internal set $T \subset {}^*W$ such that $W \subset T$ and $|T| < n^{1/4}$, where $n = |A|$. Let $T_1 = \{r \in {}^*\mathbf{R} : \exists s,t\ (r,s,t) \in T\}$. Given a finite set $V \subset \mathbf{R}$, let G_V denote the set of internal functions $g : T_1 \to \mathcal{A}$ such that $r \in \mathbf{R} \Rightarrow g(r) = D_r$. G_V is nonempty for all V by the Internal Definition Principle; by saturation, $\cap_{V \in \mathcal{FP}(\mathbf{R})} G_V \neq \emptyset$. Choose $g \in \cap_{V \in \mathcal{FP}(\mathbf{R})} G_V$ and define $D_r = g(r)$ for all $r \in T_1$. For $(r,s,t) \in T$, let k be the greatest element of ${}^*\mathbf{N}$ less than or equal to $n(t-s)$. Let $A_{st} = A \cap {}^*[s,t]$. $n\nu(Y^{-1}(D_r) \cap A_{st})$ has a *-binomial distribution $B(k, \nu(D_r))$, so it has mean $k\nu(D_r)$ and standard deviation $\sqrt{k\nu(D_r)\,(1-\nu(D_r))} < \sqrt{k}$ by Feller (1957) and the Transfer Principle. Thus, $\nu(Y^{-1}(D_r) \cap A_{st})$ has mean $\frac{k\nu(D_r)}{n}$ and standard deviation less than $\frac{\sqrt{k}}{n} \leq \frac{1}{\sqrt{n}}$. Therefore,

$$\rho\left(\left\{\omega : \left|\nu(Y^{-1}(D_r) \cap A_{st}) - \frac{k\nu(D_r)}{n}\right| > n^{-\frac{1}{4}}\right\}\right) \leq \frac{1}{\sqrt{n}} \tag{6.1}$$

   by Chebycheff's Inequality (Feller (1957)). Therefore

$$\rho\left(\left\{\omega : \exists (r,s,t) \in T,\ \left|\nu(Y^{-1}(D_r) \cap A_{st}) - \frac{k\nu(D_r)}{n}\right| > n^{-\frac{1}{4}}\right\}\right) \leq n^{1/4}\frac{1}{\sqrt{n}} \simeq 0. \tag{6.2}$$

Therefore

$$\bar{\rho}(\{\omega : \forall(r,s,t) \in W \quad \nu(Y^{-1}(D_r) \cap A_{st}) = (t-s)F(r)\}) = 1, \tag{6.3}$$

so

$$\bar{\rho}(\{\omega : \forall(r,s,t) \in W \quad \bar{\nu}(\{a \in A_{st} : {}^\circ Y(a) \leq r\}) \geq (t-s)F(r)\}) = 1. \tag{6.4}$$

Similarly,

$$\bar{\rho}(\{\omega : \forall(r,s,t) \in W \quad \bar{\nu}(\{a \in A_{st} : {}^\circ Y(a) \leq r\}) \leq (t-s)F(r)\}) = 1, \tag{6.5}$$

so

$$\bar{\rho}(\{\omega : \forall(r,s,t) \in W, \quad \bar{\nu}(\{a \in A_{st} : {}^\circ Y(a) \leq r\}) = (t-s)F(r)\}) = 1. \tag{6.6}$$

■

Remark 6.2.3 Let G be an arbitrary distribution function. There exists a random variable Z defined on the Lebesgue measure space $[0,1]$ with distribution function F. Define $Z' : A \to \mathbf{R}$ by $Z'(a) = Z({}^\circ a)$. Z' has distribution function G by Theorem 4.2.2. There exists a ν-measurable function $Y : A \to {}^*\mathbf{R}$ such that ${}^\circ Y(a) = Z'(a)$ almost surely by Theorem 4.4.2; the distribution function of ${}^\circ Y$ is G. Thus, Construction 6.2.1 allows us to produce a continuum of independent random variables with any desired distribution.

6.3 Price Adjustment Model

In the tatonnement story of the determination of equilibrium prices, it is assumed that a fictitious Walrasian auctioneer announces a price vector, determines the market excess demand at that price, and then adjusts the price in a way which hopefully leads eventually to equilibrium. No trade is allowed until the equilibrium price is reached. This story has two critical flaws:

1. If no trade is allowed at the non-equilibrium prices called out by the auctioneer, why should individual agents bother to communicate their excess demands to the auctioneer? If they do not convey their excess demands, how does the auctioneer determine what the social excess demand is? The more we require the auctioneer to know, the less the tattonement story fills the role of providing foundations for a theory of *decentralization* by prices.

2. Convergence to the equilibrium price requires a countable number of steps. If there is a technological lower bound on the length of time needed for the auctioneer to elicit the excess demand information, equilibrium cannot be reached in finite time. Thus, no trade occurs in finite time.

Thus, it is highly desirable to replace the tatonnement story with a model which allows trade out of equilibrium, and in which the information required to adjust prices is kept to a minimum.

Keisler has developed such a model using a hyperfinite exchange economy in which agents are chosen to trade randomly. remainder of this section, we sketch Keisler's result, listing the principal assumptions and conclusions in a special case. For a complete statement, see Keisler (1979,1986,1990,1992,1996).[1]

[1] In the monograph, we intend to expand this section to include a formal statement of Keisler's result in the special case considered here.

The set of agents is $A = \{1, \ldots, n\}$ and the time line is $T = \left\{ \frac{j}{n} : j \in {}^*\mathbf{N}, j \leq n^2 \right\}$ for $n \in {}^*\mathbf{N} \setminus \mathbf{N}$. There are $k \in \mathbf{N}$ commodities. There is a central market. At each time $t \in T$, one agent is chosen at random to go to the market and trade. Thus, the underlying probability space is $(\Omega, \mathcal{B}, \nu)$, as described below.

1. $\Omega = A^T$. Thus, an element $\omega \in \Omega$ is an internal function from T to A. If $\omega(t) = a$, then agent a is chosen to go to market at time t.

2. $\mathcal{B}$ is the set of all internal subsets of Ω.

3. $\nu(B) = \frac{|B|}{|A|}$ for $B \in \mathcal{B}$.

$D(p, I, a)$ denotes the demand of agent a with income I and price vector p. The prevailing price in the market is set initially at an arbitrary price $p(\frac{1}{n})$. The market has an initial inventory $I(\frac{1}{n}) = (n\epsilon, \ldots, n\epsilon)$. Each agent begins with an endowment $f(a, 0)$. The prevailing price at time t, denoted $p(t)$, the market inventory at time t, denoted $I(t)$, and the commodity bundle of agent a at time t, denoted $f(a, t)$ are determined inductively by the formula

$$\begin{aligned}
f(a,t) \quad &= \begin{cases} f(a, t - \frac{1}{n}) & \text{if } \omega(t) \neq a \\ D(p(t), p(t) \cdot f(a, t - \frac{1}{n}), a) & \text{if } t > 0 \\ & \ \& \ \omega(t) = a \end{cases} \\
p\left(t + \tfrac{1}{n}\right) \quad &= p(t) + \lambda \left[f(\omega(t), t) - f(\omega(t), t - \tfrac{1}{n}) \right] \\
I\left(t + \tfrac{1}{n}\right) \quad &= I(t) + \left[f(\omega(t), t) - f(\omega(t), t - \tfrac{1}{n}) \right].
\end{aligned} \tag{6.7}$$

We hope to replace Keisler's parametrization assumption (discussed briefly below) with a compactness condition.

In other words, in each time period, the agent who trades purchases his/her demand given the prevailing price and his/her holding from previous trades, the net trade of these agents is taken from the market inventory, and the price is adjusted proportionately to the net trade of this agent.

The parameters λ and n are chosen so that $\lambda = n^{-c}$ for some $c \in (0, 1)$, so

$$^\circ(\lambda n) = \infty, \ \lambda \log(\lambda n) \simeq 0 \tag{6.8}$$

(in particular $\lambda \simeq 0$). The inventory parameter $\epsilon \simeq 0$, but ϵ is not too small.

The demand functions of the agents are assumed to be parametrizable in a certain fashion. This parametrization assumption appears to be a form of compactness condition on the demand functions. An economy with a finite (in the standard sense) number of types of agents with C^1 demands satisfying a global Lipschitz condition will satisfy the parametrization assumption. It is also assumed that the initial endowment $f(a, 0)$ is uniformly bounded by some $M \in \mathbf{N}$ and that $p(0)$ is finite.

The evolution of the economy is a random process, depending on the realization of the random variable ω which determines which agents trade at each time. We can associate a deterministic price adjustment process defined by the differential equation

$$\begin{aligned} q(0) &= p(0) \\ q'(t) &= {}^\circ\left(\tfrac{1}{n}\textstyle\sum_{a\in A}[D(p(t), p(t)\cdot f(a,0), a) - f(a,0)]\right). \end{aligned} \tag{6.9}$$

We assume that the solution of equation 6.9 is exponentially stable, with limit $\mathbf{p}$, a Walrasian equilibrium price, for every initial value in a neighborhood of $p(0)$.

Keisler shows that for $\bar{\nu}$-almost all $\omega \in \Omega$, the following properties hold.

1. $p(t) \simeq {}^*q(\lambda n t)$ for all $t \in T$ with ${}^\circ t < \infty$. Note that since $q(t) \to \mathbf{p}$ as $t \to \infty$ and λn is infinite, $p(t) \simeq \mathbf{p}$ for finite $t \in T$ satisfying ${}^\circ(\lambda n t) = \infty$. Thus,

   (a) the path followed by the price is, up to an infinitesimal, deterministic; and

   (b) the price becomes infinitely close to the Walrasian equilibrium price $\mathbf{p}$ in infinitesimal time and stays infinitely close to $\mathbf{p}$ for all finite times.

2. $I(t) \gg 0$ for all $t \in T$. Thus, an initial market inventory which is infinitesimal compared to the number of agents suffices to ensure that the trades desired by the agents are feasible when the agents come to market.

3. For almost all agents a, there exists $t(a) \in T$ with ${}^\circ t(a) < \infty$ such that $f(a,t) \simeq D(\mathbf{p}, \mathbf{p} \cdot f(a,0), a)$ for all $t \geq t(a)$. Thus, almost all agents trade at a price infinitely close to the Walrasian price $\mathbf{p}$, and they consume their demands at $\mathbf{p}$. An infinitesimal proportion of the trade takes place at prices outside the monad of the Walrasian price $\mathbf{p}$.

Chapter 7

Noncooperative Game Theory

Chapter 8

Stochastic Processes

Chapter 9

Translating Nonstandard Proofs

Because hyperfinite economies possess both the continuous properties of measure-theoretic economies (via the Loeb measure construction) and the discrete properties of large finite economies (via the Transfer Principle), they provide a tool for converting measure-theoretic proofs into elementary ones. The strategy for doing this involves taking a measure-based argument, and interpreting it for Loeb measure economies; the interpretation typically involves the use of formulas with iterated applications of external constructs. One can then proceed on a step-by-step basis to replace the external constructs with internal ones; each time one does this, the conclusion of the theorem is typically strengthened. In most cases, the process terminates with one or more external constructs still present, and no tractable internal arguments to replace them. However, it is occasionally possible to replace all the external constructs; if one succeeds in doing this, the internal proof is (with *'s deleted) a valid standard proof which is elementary in the sense that measure theory is not used.

An extended discussion of translation techniques is given in Rashid (1987). Given the limitations of space, we will limit ourselves to listing a few examples.

1. The elementary proof of a core convergence result in Anderson (1978) was obtained by applying the translation process to a nonstandard version of the Kannai-Bewley-Grodal-Hildenbrand approach to core limit theorems using weak convergence reported in Hildenbrand (1974). Much of the groundwork for the translation was laid in Brown and Robinson (1974,1975) who first developed nonstandard exchange economies, and in Khan (1974b) and Rashid (1979). A critical phase in the translation was carried out by Khan and Rashid (1976), who showed that one can dispense with the assumption that almost all agents in the hyperfinite economy have preferences which are nearstandard in the space of monotone preferences.

2. Anderson, Khan and Rashid (1982) presents an elementary proof of the existence of approximate Walrasian equilibria (in the sense that per capita market excess demand is small) in which the bound on the excess demand is independent of compactness conditions on preferences such as uniform monotonicity. The proof is a translation of the nonstandard proof in Khan and Rashid (1982). As in item 1, a key to the successful completion of the translation was the discovery that one preferences in the hyperfinite economy need not be nearstandard in the space of monotone preferences.

3. Anderson (1985,1988) proved that "strong" versions of core convergence theorems and the second welfare theorem hold with probability one in sequences of economies obtained by sampling agents' characteristics from a

probability distribution, even with nonconvex preferences; here, "strong" means that agent's consumptions are close to their demand sets. The proofs are highly external, and so would appear poor candidates for translation. However, they required checking certain conditions for standard prices only; since the set of standard prices can be embedded in a hyperfinite set by Theorem 1.13.4, this suggested strongly that the key was to consider only finitely many prices at a time. Hoover (1989) recently succeeded in giving a standard proof by carrying out the translation. Hoover's proof is elementary in the sense that it uses little measure theory beyond the bare bones necessary to define sequences of economies obtained by sampling from a probability distribution.

Chapter 10

Further Reading

There are a number of other applications of nonstandard analysis in economics which space did not permit us to discuss in Anderson (1990). We hope to cover some of them in the monograph, but for now we limit ourselves to the following listing of references:

1. Richter (1971) and Blume, Brandenburger and Dekel (1991a,1991b) on the representation of preferences;

2. Lewis (1977), Brown and Lewis (1981), and Stroyan (1983) on infinite time horizon models;

3. Geanakoplos and Brown (1982) on overlapping generations models;

4. Muench and Walker (1981) and Emmons (1984) on public goods economies; and

5. Simon and Stinchcombe (1989) on equilibrium refinements in noncooperative games.

There are a number of approachable books giving an introduction to nonstandard analysis. We particularly recommend Hurd and Loeb (1985), which gives a thorough nonstandard development of the principal elements of real analysis. Keisler (1976), an instructor's manual to accompany a calculus text based on infinitesimals, is very useful for those who find mathematical logic intimidating. An entirely different approach to nonstandard analysis is given in Nelson (1977).

Rashid (1987) provides a broader survey of the applications of nonstandard analysis to the large economies literature, and gives an extended discussion of techniques for translating nonstandard proofs into elementary standard proofs.

There is an extensive literature on stochastic processes, including Brownian motion and stochastic integration, based on the Loeb measure. Since stochastic processes play an important role in finance, this is a potentially fertile area for future applications of nonstandard analysis to economics. See Anderson (1976), Keisler (1984), and Albeverio, Fenstad, Høegh-Krohn and Lindstrøm (1986).

Appendix A

Proof of the Existence of Nonstandard Extensions

This appendix will provide a complete proof of the existence of nonstandard extensions as defined in Chapter 2. æ

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