

Bus Ad 239B–Spring 2003

Lecture Notes, Part III

19 Replicating Contingent Claims

With this section, we begin Chapter 5 of Nielsen, the Martingale Valuation Principle. This is the heart of the book, and represents the current state of the art of the theory on pricing contingent claims. Regardless of whether the underlying securities price process is assumed to be an Itô process, the standard method of pricing contingent claims, including options, is to construct a state price process for the securities price process and compute the Martingale Valuation.

Definition 19.1 A *contingent claim* is a random variable Y , interpreted as a claim which pays off $Y(\omega)$ for every $\omega \in \Omega$. A self-financing trading strategy $\bar{\Delta}$ replicates a contingent claim Y at time T in the price system \bar{S} if

$$P\left(\left\{\omega \in \Omega : \bar{\Delta}(\omega, T)\bar{S}(\omega, T) = Y(\omega)\right\}\right) = 1$$

A contingent claim Y is *marketed* at time T with respect to \bar{S} and Π if it is replicated at time T in the price system \bar{S} by some self-financing trading strategy $\bar{\Delta}$ which is admissible for \bar{S} and Π .

Remark 19.2 A necessary condition for Y to be replicated at T is that Y be \mathcal{F}_T -measurable. A necessary condition for Y to be marketed at time T is that $\Pi(T)Y \in L^1(\Omega)$, i.e. $E(|\Pi(T)Y|) < \infty$; to see this, note that since $\bar{\Delta}$ is admissible, $\Pi\bar{\Delta}\bar{S}$ is a martingale, and hence $\Pi(t)\bar{\Delta}(t)\bar{S}(t) \in L^1(\Omega)$ for all t , including T .

Definition 19.3 If Y is any contingent claim such that $\Pi(T)Y \in L^1(\Omega)$, the *martingale value process of Y* is the process

$$V(Y; \Pi)(t) = \frac{1}{\Pi(t)}E(\Pi(T)Y|\mathcal{F}_t)$$

for $0 \leq t \leq T$.

Proposition 19.4 *If Y is marketed at time T with respect to \bar{S} and Π by the self-financing admissible trading strategy $\bar{\Delta}$, then $\bar{\Delta}\bar{S}(t) = V(Y; \Pi)(t)$. In particular, any two admissible self-financing trading strategies with respect to Π and \bar{S} that replicate Y at time T must have the same value process.¹*

Proof: Since $\bar{\Delta}$ is admissible, $\Pi\bar{\Delta}\bar{S}$ is a martingale, hence

$$\Pi(t)\bar{\Delta}(t)\bar{S}(t) = E(\Pi(T)\bar{\Delta}(T)\bar{S}(T)|\mathcal{F}_t) = E(\Pi(T)Y|\mathcal{F}_t)$$

so

$$\bar{\Delta}(t)\bar{S}(t) = \frac{1}{\Pi(t)}E(\Pi(T)Y|\mathcal{F}_t) = V(Y; \Pi)(t)$$

■

Remark 19.5 The martingale value process is defined even if Y is not marketed or replicated at time T . Indeed, Y need not even be \mathcal{F}_T -measurable.

Proposition 19.6 *Suppose that either*

1. *Y is replicated at time T by an admissible self-financing trading strategy $\bar{\Delta}$ with respect to Π and \bar{S} ; or*
2. *$\Pi(T)Y \in L^1(\Omega)$ and $\{\mathcal{F}_t : t \in \mathcal{T}\}$ is the filtration generated by the Wiener process W .*

Then $V(Y; \Pi)(t)$ is an Itô process.

Proof: If Y is replicated by an admissible self-financing trading strategy $\bar{\Delta}$, then $V(Y; \Pi)(t) = \bar{\Delta}(t)\bar{S}(t)$ (this is the self-financing condition), which is an Itô process. If $\Pi(T)Y \in L^1(\Omega)$ and $\{\mathcal{F}_t\}$ is the filtration generated by the Wiener process W , then the martingale $E(\Pi(T)Y|\mathcal{F}_t)$ is an Itô process by the Martingale Representation Theorem, so $V(Y; \Pi)(t) = E(\Pi(T)Y|\mathcal{F}_t)(t)/\Pi(t)$ is an Itô process (see the section on changing units of account). ■

¹The trading strategy need not be unique. The trading strategy will be unique if $K = N + 1$ and $\text{rank } \bar{\sigma} = N + 1$ or there is a money-market account, $K = N$ and $\text{rank } \bar{\sigma} = N$.

Proposition 19.7 (Proposition 5.1 in Nielsen) *Let Y be a contingent claim which is replicated by two self-financing trading strategies $\bar{\Delta}$ and $\bar{\Theta}$ such that $\bar{\Delta}$ is admissible for \bar{S} and Π and $\Pi\bar{\Theta}\bar{S}$ is bounded below. Then $\Pi\bar{\Delta}\bar{S}$ is bounded below and, with probability one,*

$$\bar{\Delta}\bar{S}(t) \leq \bar{\Theta}(t)\bar{S}(t)$$

for all t .

Proof: See Nielsen. ■

Remark 19.8 In my view, this proposition is an unsuccessful attempt to resolve the lack of uniqueness inherent in the martingale valuation principle unless markets are dynamically complete. Why is the lowest price at which the claim can be replicated the natural candidate for the price that will prevail for the claim in the market? Why not the highest price? More fundamentally, if there are two replicating strategies with different prices on a set of positive measure, there will be an arbitrage; go long on the cheaper strategy, short on the long strategy, place the difference in a money-market account, then at time T close out the long and short positions on the replicating strategy, and pocket the amount invested in the money-market account, plus interest. In other words, in a real market with a money-market account, there can't be two different prices for strategies which replicate the same claim; the theory should provide a way to rule out one of the strategies, or to force their prices to be equal. Proposition 19.4 does this provided there is a unique state price process; any two admissible self-financing trading strategies with respect to Π and \bar{S} must have the same value process. However, if $\Pi \neq \Pi'$ are state price processes, then in general $V(Y; \Pi) \neq V(Y; \Pi')$, and the theory does not provide a unique price.

20 Delta Hedging

The problem of hedging a holding is essentially the problem of replicating the instantaneously risky part of a claim; see Nielsen.

21 Making a Trading Strategy Self-Financing

This section is preliminary to proving the Complete Markets Theorem. The strategy for replicating a contingent claim Y will be to calculate its martingale value process V , find $\bar{\gamma}$ whose dispersion equals the dispersion of V , then trade in the money-market to make $\bar{\gamma}$ into a self-financing trading strategy $\bar{\Delta}$.

Proposition 21.1 (Proposition 5.2 in Nielsen) *Let \bar{b} be a money-market account with value process M . Then $\alpha\bar{b} \in \mathcal{L}(\bar{S}/M)$ and*

$$\mathcal{G}(\alpha\bar{b}; \bar{S}/M) = \alpha(0)$$

(i.e. there is no gain or loss from trading) for every 1-dimensional trading strategy α .

Proof: This is more or less obvious. Prices are normalized so the money-market value process is constant, hence there are no gains or losses from holding it. Read the formal proof in Nielsen. ■

Proposition 21.2 (Proposition 5.3 in Nielsen) *Let \bar{b} be a money-market account with value process M . Suppose $\bar{\gamma}$ is a trading strategy in $\mathcal{L}(\bar{S}/M)$ that is not necessarily self-financing. Let*

$$\bar{\Delta} = \bar{\gamma} + \mathcal{D}(\bar{\gamma}; \bar{S}/M)\bar{b}$$

Then $\bar{\Delta}$ is self-financing, has initial value $\bar{\gamma}(0)\bar{S}(0)$ and the dispersion of the value process $\bar{\Delta}\bar{S}$ is $\bar{\Delta}\bar{\sigma} = \bar{\gamma}\bar{\sigma}$.

Proof: Again, this is more or less obvious. Recall that \mathcal{D} is the cumulative net withdrawal process, so $\bar{\Delta}$ is self-financing because it puts all recognized capital gains and losses generated by $\bar{\gamma}$ into the Money-Market Account. $\mathcal{D}(\bar{\gamma}; \bar{S}/M)$ counts up any gains or losses generated by $\bar{\gamma}$ when prices are normalized so $M = 1$. Since $\bar{b}\bar{S} = M = 1$, the accumulated gain buys exactly $\mathcal{D}(\bar{\gamma}; \bar{S}/M)\bar{b}$ units of the money-market account. By Proposition 21.1, the holdings in the money-market account do not generate any gains or losses that could upset the self-financing constraint. It is obvious that $\bar{\Delta}$ has the same dispersion as $\bar{\gamma}$. See Nielsen for details. ■

Up until now, we have shown how to take a (not necessarily self-financing) trading strategy and make it self-financing, while keeping the same initial

value and not changing the dispersion. It is natural to ask whether the strategy is unique. In general, the answer is no, but the value process is uniquely defined, given a particular securities price process \bar{S} and state-price process Π .

Proposition 21.3 (Proposition 5.4 in Nielsen) *Let Π be a state price process for \bar{S} . Let V be an Itô process and let $\bar{\Delta}$ be a self-financing trading strategy. Then V is the value process of $\bar{\Delta}$ if and only if*

1. ΠV has zero drift
2. $\bar{\Delta}(0)\bar{S}(0) = V(0)$
3. the dispersion coefficient of V equals $\bar{\Delta}\bar{\sigma}$ almost everywhere.

Proof: If $V = \bar{\Delta}\bar{S}$, then (2) is immediate and (1) follows because $\bar{\Delta}$ is self-financing, from the argument on the top of page 131 of Nielsen (Proposition 15.3 of these Lecture Notes). (3) follows from the uniqueness of Itô coefficients.

Conversely, suppose we have (1)-(3). We will show that $\frac{V}{M}$ and $\frac{\bar{\Delta}\bar{S}}{M}$ have the same Itô coefficients, hence $\frac{V}{M} = \frac{\bar{\Delta}\bar{S}}{M}$, hence $V = \bar{\Delta}\bar{S}$. Since V is an Itô process,

$$dV = a_V dt + b_V dW$$

for some $a_V \in \mathcal{L}^1$ and $b_V \in \mathcal{L}^2$. The self-financing constraint implies that

$$d(\bar{\Delta}\bar{S}) = \bar{\Delta}\bar{\mu} dt + \bar{\Delta}\bar{\sigma} dW$$

Recall (see page 132 of Nielsen, dividing by M sets $r = 0$) that

$$\Pi = \Pi(0)M(0)\frac{\eta[0, \Leftrightarrow\lambda]}{M}$$

so

$$\eta[0, \Leftrightarrow\lambda]V/M = \frac{\Pi V}{\Pi(0)M(0)}$$

has zero drift by assumption. From Itô's Formula for Quotients (page 69 of Nielsen), since the dispersion of M is zero,

$$d\left(\frac{V}{M}\right) = \frac{1}{m} [a_V \Leftrightarrow V r M] dt + \frac{b_V}{M} dW$$

so the drift of $\eta[0, \Leftrightarrow \lambda]V/M$ is

$$\eta[0, \Leftrightarrow \lambda] \left[\frac{a_V \Leftrightarrow V r M}{M} \Leftrightarrow \frac{b_V \lambda^T}{M} \right]$$

$\eta[0, \Leftrightarrow \lambda] > 0$, so the drift of $\eta[0, \Leftrightarrow \lambda]V/M$ is zero if and only if

$$\frac{a_V}{M} \Leftrightarrow V r = \frac{b_V \lambda^T}{M}$$

Since the drift of V/M is $\frac{a_V - V r M}{M}$, the drift of V/M equals $\frac{b_V \lambda^T}{M}$.

Since the process

$$\eta[0, \Leftrightarrow \lambda] \bar{\Delta} \bar{S} / M = \frac{\Pi \bar{\Delta} \bar{S}}{\Pi(0)M(0)}$$

has zero drift (since $\bar{\Delta}$ is self-financing), an analogous calculation (using the fact that $\bar{\Delta} \bar{\sigma}$ is the dispersion of $\bar{\Delta} \bar{S}$) shows that the drift of $\bar{\Delta} \bar{S} / M$ is

$$\bar{\Delta} \bar{\sigma} \lambda^T / M$$

By (3), $b_V = \bar{\Delta} \bar{\sigma}$, so

$$\frac{b_V \lambda^T}{M} = \frac{\bar{\Delta} \bar{\sigma} \lambda^T}{M}$$

so $\frac{V}{M}$ and $\frac{\bar{\Delta} \bar{S}}{M}$ have the same drift. The dispersion of $\frac{V}{M}$ is $\frac{\bar{\Delta} \bar{\sigma}}{M}$, which equals the dispersion of $\frac{\bar{\Delta} \bar{S}}{M}$. By (2),

$$\bar{\Delta}(0) \bar{S}(0) = V(0)$$

Therefore, V/M and $\bar{\Delta} \bar{S} / M$ have the same Itô coefficients, and hence are equal, so V and $\bar{\Delta} \bar{S}$ are equal. ■

22 The Complete Markets Theorem

Definition 22.1 We say *markets are dynamically complete with respect to* \bar{S} and Π if every contingent claim Y which is measurable with respect to \mathcal{F}_T and satisfied $\Pi(T)Y \in L^1(\Omega)$ is marketed at time T with respect to \bar{S} and Π , i.e. Y can be replicated by an admissible, self-financing trading strategy.

Remark 22.2 Dynamic completeness depends, of course, on \bar{S} ; you can't talk about what claims are replicable until you know the trading and capital gain opportunities, which obviously depend on \bar{S} . Changing the state price process Π changes the set of contingent claims that $\Pi(T)Y \in L^1(\Omega)$, and it changes the set of admissible trading strategies. Surprisingly, the Complete Markets Theorem shows that dynamic completeness depends only on \bar{S} and not on Π . This is tempered by the fact that if Y is marketed at time T with respect to Π and Π' , there is no obvious reason why the martingale value processes $\Pi(Y; \Pi)$ and $\Pi(Y; \Pi')$ should agree. Note also that markets are necessarily dynamically *incomplete* if $N < K$.

Theorem 22.3 (Complete Markets Theorem, Theorem 5.6 in Nielsen)

Assume that $\mathcal{F} = \mathcal{F}^W$, i.e. the filtration is the filtration generated by the Wiener process. Assume there is a money-market account, and let Π be a state price process for \bar{S} . Then markets are dynamically complete with respect to \bar{S} and Π if and only if $\bar{\sigma}$ has rank K almost everywhere.

Before we prove the Complete Markets Theorem, we prove the following proposition:

Proposition 22.4 (Proposition 5.5 in Nielsen) *Assume there is a money-market account. Let V be an Itô process such that ΠV has zero drift.*

1. *If $\bar{\sigma}$ has rank K almost everywhere, then V is the value process of a self-financing trading strategy.*
2. *If, in addition, $N = K$, the trading strategy is unique.*

Proof: Since V is an Itô process,

$$dV = a_V dt + b_V dW$$

with $a_V \in \mathcal{L}^1$ and $b_V \in \mathcal{L}^2$. Let $\bar{b} \in \mathcal{L}(\bar{S})$ be a money-market account, $M = \bar{b}\bar{S}$. We need to construct a self-financing trading strategy whose value process equals V . First, we match the volatility. Let

$$\bar{\gamma} = b_V (\bar{\sigma}^T \bar{\sigma})^{-1} \bar{\sigma}^T$$

$\bar{\sigma}^T \bar{\sigma}$ is invertible by the argument given in these notes in the proof of Proposition 4.4 of Nielsen.) We need to show that $\bar{\gamma} \in \mathcal{L}(\bar{S}/M)$.

$$\frac{\bar{\gamma} \bar{\sigma}}{M} = \frac{b_V \left(\bar{\sigma}^T \bar{\sigma} \right)^{-1} \bar{\sigma}^T \bar{\sigma}}{M} = \frac{b_V}{M} \in \mathcal{L}^2$$

since M is a positive Itô process, hence almost surely uniformly bounded away from zero on each finite time interval $[0, T]$. Since M is a money-market account, by item 4 in Remark 18.5 in these Lecture Notes, the drift of \bar{S}/M is

$$\begin{aligned} \frac{\bar{\gamma}(\bar{\mu} \Leftrightarrow r \bar{S})}{M} &= \frac{\bar{\gamma} \bar{\sigma} \lambda^T}{M} \\ &= \frac{b_V \lambda^T}{M} \in \mathcal{L}^1 \end{aligned}$$

since $b_V, \lambda^T \in \mathcal{L}^2$. $\bar{\gamma} + \mathcal{D}(\bar{\gamma}; \bar{S}/M) \bar{b}$ is self-financing and has the same initial value and dispersion as $\bar{\gamma}$, by Proposition 5.3 in Nielsen. Let

$$\bar{\Delta} = \bar{\gamma} + \mathcal{D}(\bar{\gamma}; \bar{S}/M) \bar{b} + \frac{V(0) \Leftrightarrow \bar{\gamma}(0) \bar{S}(0) \bar{b}}{M(0)}$$

so

$$\begin{aligned} &\bar{\Delta}(0) \bar{S}(0) \\ &= \bar{\gamma}(0) \bar{S}(0) + \left(\mathcal{D} \left(\bar{\gamma}; \frac{\bar{S}}{M} \right) (0) \right) \bar{b}(0) \bar{S}(0) + \frac{V(0) \Leftrightarrow \bar{\gamma}(0) \bar{S}(0) \bar{b}}{M(0)} \bar{b}(0) \bar{S}(0) \\ &= \bar{\gamma}(0) \bar{S}(0) + \frac{V(0) \Leftrightarrow \bar{\gamma}(0) \bar{S}(0) \bar{b}}{M(0)} M(0) \quad \left(\text{since } \mathcal{D} \left(\bar{\gamma}; \frac{\bar{S}}{M} \right) (0) = 0 \right) \\ &= V(0) \\ \bar{\Delta} \bar{\sigma} &= \bar{\gamma} \bar{\sigma} \\ &= b_V \end{aligned}$$

so V is the value process of $\bar{\Delta}$, by Proposition 5.4 of Nielsen; this completes the proof of (1).

(2) is a simple exercise in linear algebra. Let Δ, Θ be two self-financing trading strategies in $\mathcal{L}(\bar{S})$, with $\bar{\Delta} \bar{S} = \bar{\Theta} \bar{S}$. Since $\bar{b} \bar{S} = M > 0$, and $\bar{b} \bar{\sigma} = 0$, \bar{S} is not in the span of the columns of $\bar{\sigma}$, so the rank of the matrix $(\bar{S}, \bar{\sigma})$

(formed by adjoining an additional column vector, equal to \bar{S} , on the right side of $\bar{\sigma}$) is $K + 1 = N + 1$. Since $\bar{\Delta}\bar{S} = \bar{\Theta}\bar{S}$, their dispersions must also agree by the uniqueness of Itô coefficients, so

$$\begin{aligned}(\bar{\Delta} \Leftrightarrow \bar{\Theta})\bar{S} &= 0 \\(\bar{\Delta} \Leftrightarrow \bar{\Theta})\bar{\sigma} &= 0 \\(\bar{\Delta} \Leftrightarrow \bar{\Theta})(\bar{S}, \bar{\sigma}) &= 0\end{aligned}$$

which implies that $\bar{\Delta} \Leftrightarrow \bar{\Theta} = 0$. ■

Proof of the Complete Markets Theorem: First, suppose that $\text{rank } \bar{\sigma} = K$ almost everywhere. Let Y be a contingent claim which is \mathcal{F}_t -measurable and $\Pi(T)Y \in L^1(\Omega)$. Let $X(t) = E(\Pi(T)Y|\mathcal{F}_t)$; X is a martingale. Since $\mathcal{F} = \mathcal{F}^W$, the Martingale Representation Theorem implies that X is an Itô process, and in particular has zero drift. Define $V = X/\Pi$. Π is positive, hence almost surely uniformly bounded away from zero on $[0, T]$ for each T , so V is an Itô Process. Since $\Pi V = X$, ΠV has zero drift. By Proposition 5.5 in Nielsen, V is the value process of a self-financing trading strategy $\bar{\Delta}$. Since ΠV is a martingale, $\Pi\bar{\Delta}\bar{S} = \Pi V$ is a martingale, so $\bar{\Delta}$ is admissible.

$$\begin{aligned}\Pi(T)V(T) &= X(T) \\ &= E(\Pi(T)Y|\mathcal{F}_T) \\ &= \Pi(T)Y \\ \text{so } Y &= V(T) = \bar{\Delta}(T)\bar{S}(T)\end{aligned}$$

so Y is marketed at time T .

Conversely, suppose markets are dynamically complete with respect to \bar{S} and Π . It is obvious that $\bar{\sigma}$ must have rank K almost everywhere; if not, you can't replicate the components of the Wiener process. See Nielsen for details. ■

23 How to Replicate

The strategy is as follows:

1. Find a state price process and the martingale value process V of a claim.

2. Hope that V depends only on t and $\bar{S}(t)$. Nielsen says this is “usually” true. I think a fair statement is there are few examples in the literature in which people have successfully calculated the replicating trading strategy if the value process depends on things other than t and $\bar{S}(t)$. In economies with more than one agent, equilibrium prices will definitely not be functions of t and $\bar{S}(t)$ alone. With more than one agent, the equilibrium value of claims depends on the wealths of individuals, which at time t will depend on the whole history of prices $\bar{S}(s), s \in [0, t]$.
3. Choose $\bar{\Delta}$ so that it has the same derivatives as V with respect to the security prices \bar{S} ; then $\bar{\Delta}$ prescribes that the number of shares in security n should be the partial derivative of V with respect to \bar{S}_n .

The following proposition is a little hard to digest. The following comments should help in understanding it, and appreciating its limitations.

1. The proposition shows that, if one calculates a trading strategy by the method just outlined, it will in fact be a replicating strategy. This looks circular, because we assume in item (2) that V is the value process of some self-financing trading strategy on $[0, T]$. Often, we will know in advance that (2) is satisfied because markets are dynamically complete. Even if we don't know that markets are dynamically complete, we can use the recipe given by Item (6) to compute $\bar{\Delta}$, then try to show that (2) is satisfied by $\bar{\Delta}$.
2. Φ is assumed to be defined and differentiable on $\mathcal{O} \times (0, T)$ rather than on $\mathcal{O} \times [0, T]$. The distinction is important; for example, if the claim is a standard call option with exercise price X at time T , then $\Phi(S, T)$ has a kink at $S = X$.
3. The construction of Φ requires solving a partial differential equation (PDE). The existence of solutions of PDEs is not well developed. The literature contains papers in which authors assert the existence of solutions of PDEs without citing a theorem that covers the case at hand; such assertions should be treated with skepticism.
4. Recall that S is the vector of prices of the risky securities.

Proposition 23.1 (Proposition 5.7 in Nielsen) *Suppose*

0. there is a state prices process Π for \bar{S} ;
1. the zeroth security is a money-market account with value process M ;
2. V is the value process of some self-financing trading strategy on $[0, T]$;
3. $\mathcal{O} \subseteq \mathbf{R}^N$ is an open set such that

$$P\left(\left\{\omega : \forall_{t \in [0, T]} S(t) \in \mathcal{O}\right\}\right) = 1$$

4. $\Phi : \mathcal{O} \times (0, T) \rightarrow \mathbf{R}$ is C^2 with respect to $S \in \mathcal{O}$ and C^1 with respect to $t \in (0, T)$;
5. $P\left(\left\{\omega : \forall_{t \in [0, T]} V(\omega, t) = \Phi(S(\omega, t), t)\right\}\right) = 1$;
6. $\bar{\Delta} = (\Delta_0, \Delta)$ is a process such that

- $\Delta(t) = \Phi_S(S(t), t)$ for $t \in (0, T)$ i.e. $\Delta_n(t) = \frac{\partial \Phi(S(t), t)}{\partial S_n}$;
- $\Delta(0)$ is \mathcal{F}_0 -measurable and $\Delta(T)$ is \mathcal{F}_T -measurable;
- $\Delta_0(t) = \frac{V(t) - \Delta(t)S(t)}{M(t)}$ for all $t \in [0, T]$

Then $\bar{\Delta}$ is a self-financing trading strategy in $\mathcal{L}(\bar{S})$ and $V(t) = \bar{\Delta}(t)\bar{S}(t)$ for all $t \in [0, T]$. If in addition

7. V is the value process of some admissible self-financing trading strategy then $\bar{\Delta}$ is admissible.

Proof: The only hard part of this is showing that $\bar{\Delta} \in \mathcal{L}^1(\bar{S})$. As defined, $\bar{\Delta}$ is adapted. $\bar{\Delta}$ is measurable, since it is measurable on $\Omega \times \{0\}$ and $\Omega \times \{T\}$ by assumption; and it is measurable in $(\omega, t) \in \Omega \times (0, t)$ because it is continuous in $(S(t), t)$ and $S(t)$ is measurable in (ω, t) . By assumption, V is the value process of some self-financing $\bar{\theta} \in \mathcal{L}(S)$, i.e. $V = \bar{\theta}\bar{S}$, so V is an Itô Process.

$$dV = d(\bar{\theta}\bar{S}) = \bar{\theta}\bar{\mu} dt + \bar{\theta}\bar{\sigma} dW$$

by the self-financing constraint.

$$\bar{\Delta}(t)\bar{S}(t) = \Delta_0(t)M(t) + \Delta(t)S(t) = V(t) \Leftrightarrow \Delta(t)S(t) + \Delta(t)S(t) = V = \bar{\theta}\bar{S}$$

In particular, V is the value process of $\bar{\Delta}$ on $[0, T]$.

$\Pi = \eta[\Leftrightarrow r, \Leftrightarrow \lambda]$ for some $r \in \mathcal{L}^1$ and $\lambda \in \mathcal{L}^2$; moreover, $\bar{\mu} = r\bar{S} + \bar{\sigma}\lambda^T$. By Itô's Lemma, the dispersion of $V(t) = \Phi(S(t), t)$ is $\Phi_S(S(t), t)\bar{\sigma}$. For $t \in (0, T)$, since the first row of $\bar{\sigma}$ is zero,

$$\begin{aligned} \bar{\Delta}(t)\bar{\sigma}(t) &= (0, \Delta(t))\bar{\sigma}(t) \\ &= (0, \Phi_S(S(t), t))\bar{\sigma}(t) \\ &= \bar{\theta}(t)\bar{\sigma}(t) \in \mathcal{L}^2 \\ \bar{\Delta}\bar{\mu} &= r\bar{\Delta}\bar{S} + \bar{\Delta}\bar{\sigma}\lambda^T \\ &= rV + \bar{\Delta}\bar{\sigma}\lambda^T \\ &= r\bar{\theta}\bar{S} + \bar{\theta}\bar{\sigma}\lambda^T \\ &= \bar{\theta}\bar{\mu} \in \mathcal{L}^1 \end{aligned}$$

Thus, $\bar{\Delta} \in \mathcal{L}(\bar{s})$. Since $\bar{\theta}\bar{\sigma} = \bar{\Delta}\bar{\sigma}$ and $\bar{\theta}\bar{\mu} = \bar{\Delta}\bar{\mu}$, $\bar{\theta}$ and $\bar{\Delta}$ have the same cumulative gains process; since $\bar{\theta}$ is self-financing, so is $\bar{\Delta}$.

If V is the value process of an admissible self-financing trading strategy, then ΠV is a martingale, so $\Pi\bar{\Delta}\bar{S}$ is a martingale, so $\bar{\Delta}$ is admissible. ■

24 Example on Cash-or-Nothing Options

It would be nice to do an example carefully at this point, but I think the exposition in Section 5.6 of Nielsen is reasonably complete, and I think the cash-or-nothing option (which pays \$1 if $S(T) \geq X$, zero otherwise) is a weird option. In Chapter 6, Nielsen computes the price of a standard call option by piecing together two exotic options (the Cash-or-Nothing Option being one of them). Rather than doing this, I prefer to compute the price of the standard call option directly. Therefore, I will leave this Example for you to read.

25 The State Price Process as Primitive

So far, we have followed the following strategy:

1. Begin with a securities price process \bar{S}
2. Compute a state price process Π

3. Value all claims from Π alone; \bar{S} is not needed for this
4. If a replicating portfolio is desired, compute this from Π and \bar{S} .

In this section, we explore an alternative process:

1. Begin with any positive Itô process Π . We will think of Π as a state price process, but notice that since we don't have a securities price process \bar{S} specified, we can't require that $\Pi\bar{S}$ have zero drift; *any* positive Itô process Π will do.
2. Given a claim Y which is \mathcal{F}_T -measurable and such that $\Pi(T)Y \in L^1(\Omega)$, Y has a unique martingale value process $V(Y; \Pi)$ given Π .
3. Choose a set of claims Y_0, \dots, Y_N satisfying the conditions in the previous paragraph, and *define* $\bar{S}_n = V(Y_n; \Pi)$. This defines a securities price process \bar{S} for which Π is a state price process.

The process just described is closely related to equilibrium methods. Suppose the securities pay exogenously specified dividends, only at time T ; by arbitrage, $\bar{S}_n(T)$ must be the vector of dividends. A state price process is constructed as the marginal utility of consumption in period T . Then the equilibrium value of the securities will, by the first order conditions, be given by the martingale value process

$$\bar{S}_n(t) = \frac{E\left(\bar{S}_n(T)\Pi(T)|\mathcal{F}_t\right)}{\Pi(t)}$$

The next proposition proceeds in a slightly different way. We start with the state price process Π . This allows us to price all claims such that Y is \mathcal{F}_T -measurable and $\Pi(T)Y \in L^1(\Omega)$. In order to replicate claims, we need to define basic securities. Rather than defining these as the martingale value processes of specific chosen claims, we construct securities with a given invertible relative dispersion matrix $\hat{\sigma}$. Then the Complete Markets Theorem assures us that all claims that are priced by Π can be replicated using the constructed basic securities.

Proposition 25.1 *Given any positive Itô process $\Pi = \Pi(0)\eta[\Leftrightarrow r, \Leftrightarrow \lambda]$ and any invertible $K \times K$ process $\hat{\sigma} \in \mathcal{L}^2$, there exists a K -dimensional vector S*

of positive price processes with relative dispersion matrix $\hat{\sigma}$ such that if M is the money market account $M(0)\eta[r, 0]$, then Π is a state price process for

$$\bar{S} = \begin{pmatrix} M \\ S \end{pmatrix}$$

Proof: $\Pi = \Pi(0)\eta[\Leftrightarrow r, \Leftrightarrow \lambda]$ for some $r \in \mathcal{L}^1$ and some $1 \times K$ -dimensional $\lambda \in \mathcal{L}^2$. Let $N = K$. We'll construct $S = (S_1, \dots, S_N)^T$. Set

$$\hat{\iota} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}_{N \times 1}, \quad \hat{\mu} = r\hat{\iota} + \hat{\sigma}\lambda^T,$$

$$\iota = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}_{(N+1) \times 1}, \quad \bar{\mu} = \begin{pmatrix} r \\ \hat{\mu} \end{pmatrix}, \quad \bar{\sigma} = \begin{pmatrix} 0 & \cdots & 0 \\ & \hat{\sigma} & \end{pmatrix}$$

$\hat{\sigma}, \lambda \in \mathcal{L}^2, r \in \mathcal{L}^1$, so $\bar{\mu} \in \mathcal{L}^1$. Choose initial prices $M(0) > 0, S_n(0) > 0 (n = 1, \dots, N = K)$ and let

$$M = M(0)\eta[r, 0], \quad S_n = S_n(0)\eta[\hat{\mu}_n, \hat{\sigma}_n], \quad S = \begin{pmatrix} S_1 \\ \vdots \\ S_N \end{pmatrix}, \quad \bar{S} = \begin{pmatrix} M \\ S \end{pmatrix}$$

where $\hat{\sigma}_n$ is the n^{th} row of $\hat{\sigma}$. Notice that

$$\begin{aligned} dM &= Mr \, dt \\ dS_n &= S_n \hat{\mu}_n \, dt + S_n \hat{\sigma}_n \, dW \\ \bar{\mu} &= r\iota + \bar{\sigma}\lambda^T \end{aligned}$$

Thus, Π is a state price process for \bar{S} (recall from Problem Set 7 the defining relation for a state price process in terms of a geometric process). ■

26 Risk-Adjusted Probabilities

In this section, we show how to incorporate the risk-adjustment into the probabilities, so the only explicit adjustment is the time-discounting. The

advantage of this is that the computation of the value of claims is often easier; the disadvantage is that it is less general as we must assume that λ satisfies the Novikov condition (or, more generally, that $\eta[0, \Leftrightarrow\lambda]$ is a martingale), and this is often difficult to verify. Our goal is to find conditions under which the risk-adjusted probability measure Q (with Radon-Nikodym derivative $\eta[0, \Leftrightarrow\lambda](T)$) makes $\frac{\bar{S}}{M}$ into a martingale.

Let Π be a state price process. Recall the following two facts:

- $V = \bar{\Delta}\bar{S}$, $\bar{\Delta}$ admissible, self-financing
 $\Rightarrow \Pi V$ is a martingale
 $\Rightarrow V(t) = \frac{1}{\Pi(t)} E(\Pi(T)V(T)|\mathcal{F}_t)$
- $\Pi = \Pi(0)M(0)\frac{\eta[0, \Leftrightarrow\lambda]}{M}$ (see page 132 of Nielsen)

The second fact shows it is possible to split Π into the risk-adjustment $\eta[0, \Leftrightarrow\lambda]$ and the time-discounting M .

Definition 26.1 Let $\mathcal{T} = [0, T]$. If $E(\eta[0, \Leftrightarrow\lambda](T)) = 1$, the *risk-adjusted probability measure* Q (on the horizon T) is the probability measure on (Ω, \mathcal{F}) whose density with respect to P is $\eta[0, \Leftrightarrow\lambda](T)$. $\eta[0, \Leftrightarrow\lambda]$ is called the *density process* of the risk-adjusted probability measure, or the *likelihood process*.

Remark 26.2 Remember that $E(\eta[0, \Leftrightarrow\lambda](T)) = 1$ if and only if $\eta[0, \Leftrightarrow\lambda]$ is a martingale; we need this to assure that the density $\eta[0, \Leftrightarrow\lambda](T)$ defines a *probability* measure Q . It may be painful to check this in specific examples. Nielsen says that this is not a particularly objectionable assumption because it amounts to saying that buying and holding the money-market account is an admissible trading strategy. I find this argument hard to evaluate. Certainly, if we have a state price process for which buying and holding the money-market account is *not* an admissible strategy, it will be hard to replicate interesting claims with admissible trading strategies. But it seems to me Nielsen's argument is backwards. One would really like to characterize the set of price processes that have a state price process $\Pi = \eta[\Leftrightarrow r, \Leftrightarrow\lambda]$ for which $\eta[0, \Leftrightarrow\lambda]$ is a martingale; in my view, it is the size and nature of this set of securities price processes that should determine how reasonable the assumption is.

Given a claim Y at time T , we can discount back to time t by taking

$$X(t) = \frac{M(t)}{M(T)}Y$$

Notice that X is \mathcal{F}_T -measurable, but usually not \mathcal{F}_t -measurable.

Proposition 26.3 *Let X be any random variable. Then*

$$X \in L^1(\Omega, Q) \Leftrightarrow X\eta[0, \Leftrightarrow\lambda](T) \in L^1(\Omega, P)$$

In this case,

$$E_Q(X|\mathcal{F}_t) = E_P\left(\frac{X\eta[0, \Leftrightarrow\lambda](T)}{\eta[0, \Leftrightarrow\lambda](t)} \middle| \mathcal{F}_t\right)$$

Proof:

$$\begin{aligned} X \in L^1(\Omega, Q) &\Leftrightarrow E_Q(|X|) < \infty \\ &\Leftrightarrow \int_{\Omega} |X|\eta[0, \Leftrightarrow\lambda](T) dP < \infty \\ &\Leftrightarrow X\eta[0, \Leftrightarrow\lambda](T) \in L^1(\Omega, P) \end{aligned}$$

Suppose $X \in L^1(\Omega, Q)$ and $B \in \mathcal{F}_t$. Then

$$\begin{aligned} \int_B E_Q(X|\mathcal{F}_t) dQ &= \int_B X dQ \\ &= \int_B X\eta[0, \Leftrightarrow\lambda](T) dP \\ &= \int_B E_P(X\eta[0, \Leftrightarrow\lambda](T)|\mathcal{F}_t) dP \\ &= \int_B \frac{E_P(X\eta[0, \Leftrightarrow\lambda](T)|\mathcal{F}_t)}{\eta[0, \Leftrightarrow\lambda](t)} \eta[0, \Leftrightarrow\lambda](t) dP \\ &= \int_B E_P\left(\frac{X\eta[0, \Leftrightarrow\lambda](T)}{\eta[0, \Leftrightarrow\lambda](t)} \middle| \mathcal{F}_t\right) \eta[0, \Leftrightarrow\lambda](t) dP \\ &\quad \text{since } \eta[0, \Leftrightarrow\lambda](t) \text{ is } \mathcal{F}_t\text{-measurable (Nielsen, Prop. B.26.4)} \\ &= \int_B E_P\left(\frac{X\eta[0, \Leftrightarrow\lambda](T)}{\eta[0, \Leftrightarrow\lambda](t)} \middle| \mathcal{F}_t\right) \eta[0, \Leftrightarrow\lambda](T) dP \\ &\quad \text{since } E(X\eta[0, \Leftrightarrow\lambda](T)|\mathcal{F}_t) \text{ is } \mathcal{F}_t\text{-measurable} \\ &\quad \text{and } \eta[0, \Leftrightarrow\lambda] \text{ is a martingale} \end{aligned}$$

$$= \int_B E_P \left(\frac{X\eta[0, \Leftrightarrow\lambda](T)}{\eta[0, \Leftrightarrow\lambda](t)} \middle| \mathcal{F}_t \right) dQ$$

so

$$E_Q(X|\mathcal{F}_t) = E_P \left(\frac{X\eta[0, \Leftrightarrow\lambda](T)}{\eta[0, \Leftrightarrow\lambda](t)} \middle| \mathcal{F}_t \right)$$

■

Proposition 26.4 *If $V = \bar{\Delta}\bar{S}$ is the value process of a self-financing trading strategy with $\bar{\Delta}(T)\bar{S}(T) = Y$, then $\bar{\Delta}$ is admissible if and only if V/M is a martingale with respect to Q , in which case*

$$\frac{\bar{\Delta}(t)\bar{S}(t)}{M(t)} = E_Q \left[\frac{Y}{M(T)} \middle| \mathcal{F}_t \right]$$

Proof: $\bar{\Delta}$ is admissible if and only if $\Pi V = \Pi(0)M(0)\frac{\eta[0, -\lambda](t)}{M}V$ is a P -martingale if and only if $\frac{\eta[0, -\lambda](T)}{M}V$ is a P -martingale if and only if V/M is a Q -martingale. ■

Definition 26.5 Suppose that Q (with Radon-Nikodym derivative $\eta[0, \Leftrightarrow\lambda]$) is an equivalent martingale measure. Then $\Pi = \eta[\Leftrightarrow r, \Leftrightarrow\lambda]$ is a state price process for \bar{S} , so that $\eta[0, \Leftrightarrow\lambda]$ is a state price process for $\frac{\bar{S}}{M}$. We introduce the notation

$$V(Y/M(T); Q)(t) = V(Y/M(T); \eta[0, \Leftrightarrow\lambda])(t)$$

Notice that

$$\begin{aligned} V(Y/M(T); Q)(t) &= V(Y/M(T); \eta[0, \Leftrightarrow\lambda])(t) \\ &= \frac{1}{\eta[0, \Leftrightarrow\lambda](t)} E \left[\eta[0, \Leftrightarrow\lambda](T) \frac{Y}{M(T)} \middle| \mathcal{F}_t \right] \\ &= \frac{\eta[0, \Leftrightarrow\lambda](t)}{\eta[0, \Leftrightarrow\lambda](t)} E \left(\frac{Y}{M(T)} \middle| \mathcal{F}_t \right) \quad (\text{since } \eta[0, \Leftrightarrow\lambda] \text{ is a martingale}) \\ &= E_Q \left[\frac{Y}{M(T)} \middle| \mathcal{F}_t \right] \end{aligned}$$

so $V(Y/M(T); Q) = V(Y; \Pi)/M$.

Since P and Q are equivalent measures, they have the same null sets, so (Ω, \mathcal{F}, Q) is complete, $\{\mathcal{F}_t : t \in \mathcal{T}\}$ is augmented with respect to Q , and \mathcal{L}^1 and \mathcal{L}^2 are the same for P as for Q .

Definition 26.6 Recall

$$W^\lambda(t) = \int_0^t \lambda^T ds + W(t)$$

If $\eta[0, \Leftrightarrow \lambda]$ is a martingale, W^λ is called the *risk-adjusted Wiener process* (note that by Girsanov's Theorem, it is a standard Wiener process with respect to Q).

Proposition 26.7 *If X is an Itô process with*

$$dX = a dt + b dW$$

then the drift of $\eta[0, \Leftrightarrow \lambda]X$ is zero if and only if $a = b\lambda^T$ if and only if $dX = b dW^\lambda$.

Proof: The drift of $\eta[0, \Leftrightarrow \lambda]X$ is zero if and only if $\eta[0, \Leftrightarrow \lambda]$ is a state price process for X if and only if $(\bar{\mu} \Leftrightarrow r\bar{S}) = \bar{\sigma}\lambda^T$ (see page 132 of Nielsen) if and only if $a = b\lambda^T$, since $r = 0$. Since

$$\begin{aligned} dX &= a dt + b dW \\ &= a dt + b(dW^\lambda \Leftrightarrow \lambda^T dt) \\ &= (a \Leftrightarrow b\lambda^T) dt + b dW^\lambda \end{aligned}$$

$a = b\lambda^T$ if and only if $dX = b dW^\lambda$. ■

In particular, if S is normalized (recall this means there is a self-financing trading strategy \bar{b} such that $\bar{b}\bar{S} = 1$, so $r = 0$),

$$\begin{aligned} \bar{\mu} &= \bar{\sigma}\lambda^T \\ d\bar{S} &= \bar{\sigma}\lambda^t dt + \bar{\sigma} dW \\ &= \bar{\sigma} dW^\lambda \end{aligned}$$

If \bar{S} is not necessarily normalized,

$$\bar{\mu} \Leftrightarrow r\bar{S} = \bar{\sigma}\lambda^T$$

$$\begin{aligned}
d\left(\frac{\bar{S}}{M}\right) &= \frac{1}{M} d\bar{S} \Leftrightarrow \bar{S} \frac{r}{M} dt \\
&= \frac{1}{M} (\bar{\mu} \Leftrightarrow r \bar{S}) dt + \frac{1}{M} \bar{\sigma} dW \\
&= \frac{1}{M} \bar{\sigma} \lambda^T dt + \frac{1}{M} \bar{\sigma} dW \\
&= \frac{1}{M} \bar{\sigma} dW^\lambda
\end{aligned}$$

This proves the following proposition:

Proposition 26.8 *If $\bar{\Delta}$ is a self-financing trading strategy in $\mathcal{L}(\bar{S}/M)$, then*

$$d\left(\frac{\bar{\Delta}\bar{S}}{M}\right) = \bar{\Delta} d\left(\frac{\bar{S}}{M}\right) = \frac{1}{M} \bar{\Delta} \bar{\sigma} dW^\lambda$$

Remark 26.9 In my view, **Propositions 5.8** and **5.9** of Nielsen are additional unsatisfactory attempts to deal with the nonuniqueness of state price processes. You may read these on your own.

Example 26.10 [Example 5.10 in Nielsen] Let's calculate the risk-adjusted probability measure Q in the Black-Scholes Model.

$$\begin{aligned}
K &= 1 \\
M &= M(0)\eta[r, 0] = M(0)e^{rt} \\
S &= S(0)\eta[\mu, \sigma] = S(0)e^{(\mu - \sigma^2/2)t + \sigma W} \\
\mu \Leftrightarrow r &= \lambda\sigma \text{ so } \lambda = \frac{\mu \Leftrightarrow r}{\sigma} \\
\frac{dQ}{dP} &= \eta[0, \Leftrightarrow\lambda](T) = e^{(-\lambda^2/2)T - \lambda W(T)}
\end{aligned}$$

Let Y be any claim such that Y is \mathcal{F}_t -measurable.

$$\begin{aligned}
\frac{Y}{M(T)} \in L^1(\Omega, Q) &\Leftrightarrow \Pi(T)Y \in L^1(\Omega, P) \\
V\left(\frac{Y}{M(T)}; Q\right)(t) &= E_Q\left[\frac{Y}{M(T)} \middle| \mathcal{F}_t\right] \\
&= \frac{1}{M(T)} E_Q[Y | \mathcal{F}_t]
\end{aligned}$$

The last equality is true because $M(T)$ is independent of ω in the Black-Scholes model, and hence \mathcal{F}_t -measurable. Notice that $V(Y/M(T); Q)$ is a martingale because it does not reflect time-discounting. In terms of units of account, the value of the claim is

$$\begin{aligned} & M(t)V(Y/M(T); Q)(t) \\ &= \frac{M(t)}{M(T)} E_Q[Y|\mathcal{F}_t] \\ &= e^{-r(T-t)} E_Q[Y|\mathcal{F}_t] \end{aligned}$$

Read through the calculation of the value of the Cash-or-Nothing Option, and note that it is easier to do it through the Risk-Adjusted Probability Q than through the state price process Π .

27 The Black-Scholes Model

The model is laid out in the previous example, so we will not repeat the model here. Note that markets are dynamically complete by Theorem 5.6 in Nielsen.

In the original Black-Scholes Model, μ , σ and r are all constants. We will follow Nielsen in including a little more generality: we assume σ and r are constants, but

$$\mu = \sigma\lambda + r$$

for some $\lambda \in \mathcal{L}^2$ such that $E_P\eta[0, \Leftrightarrow\lambda](T) = 1$.

By Girsanov's Theorem, if Q has density $\eta[0, \Leftrightarrow\lambda](T)$ with respect to P , then

$$W^\lambda(t) = \int_0^t \lambda ds + W(t)$$

is a standard Wiener process with respect to Q . In terms of dW^λ , the differential of $S(t)$ and $S(t)/M(T)$ are the same as before; they are not affected by the fact that μ is not constant.

28 Valuing The Standard Call Option

We now have all the ingredients to compute the value process of the standard call option in the Black-Scholes Model. First, we need a preliminary lemma:

Lemma 28.1 *If N is the cumulative distribution function of the standard normal distribution, then*

$$\int_a^b e^{\alpha y} dN(y) = e^{\alpha^2/2} (N(b \Leftrightarrow \alpha) \Leftrightarrow N(a \Leftrightarrow \alpha))$$

Proof:

$$\begin{aligned} \int_a^b e^{\alpha y} dN(y) &= \frac{1}{\sqrt{2\pi}} \int_a^b e^{\alpha y} e^{-y^2/2} dy \\ &= \frac{1}{\sqrt{2\pi}} \int_a^b e^{(2\alpha y - y^2)/2} dy \\ &= \frac{e^{\alpha^2/2}}{\sqrt{2\pi}} \int_a^b e^{-(y-\alpha)^2/2} dy \\ &= \frac{e^{\alpha^2/2}}{\sqrt{2\pi}} \int_{a-\alpha}^{b-\alpha} e^{-y^2/2} dy \\ &= e^{\alpha^2/2} (N(b \Leftrightarrow \alpha) \Leftrightarrow N(a \Leftrightarrow \alpha)) \end{aligned}$$

■

Theorem 28.2 *In the Black-Scholes Model, consider a call option on the stock, with exercise price X at date T , where the stock price at time t is S . Then the martingale value process of the call option is*

$$C = SN(d_1) \Leftrightarrow e^{-r(T-t)} XN(d_2)$$

where

$$\begin{aligned} d_1 &= \frac{\ln(S/X) + \left(r + \frac{\sigma^2}{2}\right)(T \Leftrightarrow t)}{\sigma\sqrt{T \Leftrightarrow t}} \\ d_2 &= \frac{\ln(S/X) + \left(r \Leftrightarrow \frac{\sigma^2}{2}\right)(T \Leftrightarrow t)}{\sigma\sqrt{T \Leftrightarrow t}} \\ &= d_1 \Leftrightarrow \sigma\sqrt{T \Leftrightarrow t} \end{aligned}$$

Proof:

$$S(t) = S(0)e^{\int_0^t (\mu(s) - \sigma^2/2) ds + \sigma W}$$

$$\begin{aligned}
&= S(0)e^{(r-\sigma^2/2)t}e^{(\mu(s)-r)t+\sigma W} \\
&= S(0)e^{(r-\sigma^2/2)t}e^{\sigma(\lambda t+W)} \\
&= S(0)e^{(r-\sigma^2/2)t}e^{\sigma W^\lambda} \\
&= S(0)e^{\gamma t+\sigma W^\lambda}
\end{aligned}$$

where $\gamma = r \Leftrightarrow \frac{\sigma^2}{2}$ and W^λ is a standard Wiener process with respect to the risk-adjusted probability measure Q . Therefore, the conditional distribution of $S(T)$ conditional on the information at time t , is the distribution of

$$S(t)e^{\gamma(T-t)+\sigma\sqrt{T-ty}}$$

where y is standard normal. The martingale value process is given by

$$\begin{aligned}
\Phi(S, t) &= e^{-r(T-t)}E_Q(\max\{S(T) \Leftrightarrow X, 0\}|S(t) = S) \\
&= e^{-r(T-t)}\int_{-\infty}^{\infty}\max\{0, Se^{\gamma(T-t)+\sigma\sqrt{T-ty}} \Leftrightarrow X\}dN(y) \\
&= e^{-r(T-t)}\int_{\frac{\ln(X/S)-\gamma(T-t)}{\sigma\sqrt{T-t}}}^{\infty}Se^{\gamma(T-t)+\sigma\sqrt{T-ty}}dN(y) \\
&\quad \Leftrightarrow e^{-r(T-t)}\int_{\frac{\ln(X/S)-\gamma(T-t)}{\sigma\sqrt{T-t}}}^{\infty}XdN(y) \\
&= Se^{(\gamma-r)(T-t)}\int_{\frac{\ln(X/S)-\gamma(T-t)}{\sigma\sqrt{T-t}}}^{\infty}e^{\sigma\sqrt{T-ty}}dN(y) \\
&\quad \Leftrightarrow Xe^{-r(T-t)}\int_{\frac{\ln(X/S)-\gamma(T-t)}{\sigma\sqrt{T-t}}}^{\infty}dN(y) \\
&= Se^{(\gamma-r+\sigma^2/2)(T-t)}\left(1 \Leftrightarrow N\left(\frac{\ln(X/S) \Leftrightarrow \gamma(T \Leftrightarrow t)}{\sigma\sqrt{T \Leftrightarrow t}} \Leftrightarrow \sigma\sqrt{T \Leftrightarrow t}\right)\right) \\
&\quad \Leftrightarrow Xe^{-r(T-t)}\left(1 \Leftrightarrow N\left(\frac{\ln(X/S) \Leftrightarrow \gamma(T \Leftrightarrow t)}{\sigma\sqrt{T \Leftrightarrow t}}\right)\right) \\
&= SN\left(\frac{\ln(S/X) + \left(r + \frac{\sigma^2}{2}\right)(T \Leftrightarrow t)}{\sigma\sqrt{T \Leftrightarrow t}}\right) \\
&\quad \Leftrightarrow Xe^{-r(T-t)}N\left(\frac{\ln(S/X) + \left(r \Leftrightarrow \frac{\sigma^2}{2}\right)(T \Leftrightarrow t)}{\sigma\sqrt{T \Leftrightarrow t}}\right) \\
&\quad \text{(since } \gamma \Leftrightarrow r + \sigma^2/2 = 0 \text{ and } 1 \Leftrightarrow N(a) = N(\Leftrightarrow a)) \\
&= SN(d_1) \Leftrightarrow e^{-r(T-t)}XN(d_2)
\end{aligned}$$

■

29 Replicating the Standard Call Option

$$C = SN(d_1) \Leftrightarrow e^{-r(T-t)} XN(d_2)$$

where

C is the value of the call option

S is the current price of the stock

X is the exercise price of the option at date T

N is the standard normal cumulative distribution function

$$d_1 = \frac{\ln(S/X) + \left(r + \frac{\sigma^2}{2}\right)(T \Leftrightarrow t)}{\sigma\sqrt{T \Leftrightarrow t}}$$

$$\begin{aligned} d_2 &= \frac{\ln(S/X) + \left(r \Leftrightarrow \frac{\sigma^2}{2}\right)(T \Leftrightarrow t)}{\sigma\sqrt{T \Leftrightarrow t}} \\ &= d_1 \Leftrightarrow \sigma\sqrt{T \Leftrightarrow t} \end{aligned}$$

Let's calculate the replicating strategy.

$$\begin{aligned} \bar{\Delta} &= \Phi_S(S, t) \\ &= N(d_1) + SN'(d_1) \frac{\partial d_1}{\partial S} \Leftrightarrow e^{-r(T-t)} XN'(d_2) \frac{\partial d_2}{\partial S} \\ &= N(d_1) + \left(SN'(d_1) \Leftrightarrow e^{-r(T-t)} XN'(d_2)\right) \frac{\partial d_1}{\partial S} \end{aligned}$$

since $d_1 \Leftrightarrow d_2$ does not depend on S . We will show that

$$SN'(d_1) \Leftrightarrow e^{-r(T-t)} XN'(d_2) = 0$$

and hence $\bar{\Delta} = N(d_1)$. To this end, compute

$$\begin{aligned} &\ln\left(\frac{SN'(d_1)}{e^{-r(T-t)} XN'(d_1 \Leftrightarrow \sigma\sqrt{T \Leftrightarrow t})}\right) \\ &= \ln S + \ln N'(d_1) + r(T \Leftrightarrow t) \Leftrightarrow \ln X \Leftrightarrow \ln N'(d_1 \Leftrightarrow \sigma\sqrt{T \Leftrightarrow t}) \\ &= \ln S \Leftrightarrow \ln \sqrt{2\pi} \Leftrightarrow \frac{d_1^2}{2} + r(T \Leftrightarrow t) \Leftrightarrow \ln X + \ln \sqrt{2\pi} + \frac{(d_1 \Leftrightarrow \sigma\sqrt{T \Leftrightarrow t})^2}{2} \end{aligned}$$

$$\begin{aligned}
&= \ln S \Leftrightarrow \frac{d_1^2}{2} + r(T \Leftrightarrow t) \Leftrightarrow \ln X + \frac{(d_1)^2}{2} \Leftrightarrow d_1 \sigma \sqrt{T \Leftrightarrow t} + \frac{\sigma^2(T \Leftrightarrow t)}{2} \\
&= \ln S \Leftrightarrow \ln X + \left(r + \frac{\sigma^2}{2} \right) (T \Leftrightarrow t) \Leftrightarrow d_1 \sigma \sqrt{T \Leftrightarrow t} \\
&= d_1 \sigma \sqrt{T \Leftrightarrow t} \Leftrightarrow d_1 \sigma \sqrt{T \Leftrightarrow t} \\
&= 0
\end{aligned}$$

Therefore,

$$SN'(d_1) \Leftrightarrow e^{-r(T-t)} XN'(d_1 \Leftrightarrow \sigma \sqrt{T \Leftrightarrow t}) = 0$$

so

$$\bar{\Delta} = N(d_1)$$

Now, we apply Nielsen's much-maligned Proposition 5.7. Note that

- There is a state price process $\Pi(t) = \eta[\Leftrightarrow r, \Leftrightarrow \lambda](t)$.
- There is a money-market account.
- $C(T) = \max\{S(T) \Leftrightarrow X, 0\}$ is \mathcal{F}_T -measurable. Assume $E(\Pi(T)S(T)) < \infty$.² By the Complete Markets Theorem, the value of the call option is the value process of an admissible self-financing trading strategy.

²As Nielsen points out in other places, this assumption says that buying-and-holding the stock $S(T)$ is an admissible trading strategy. Since

$$0 \leq E(\Pi(T)C(T)) \leq E(\Pi(T)S(T))$$

our assumption implies that $E(\Pi(T)C(T)) < \infty$. Note that

$$\begin{aligned}
E(\Pi(T)S(T)) &= E\left(e^{\int_0^T (-r - \lambda^2/2) dt - \sigma W + \int_0^T (\mu - \sigma^2/2) dt + \sigma W}\right) \\
&= E\left(e^{\int_0^T (\mu - r - (\lambda^2 + \sigma^2)/2) dt}\right) \\
&= E\left(e^{\int_0^T (\lambda \sigma - (\lambda^2 + \sigma^2)/2) dt}\right) \\
&= E\left(e^{\int_0^T (-\lambda - \sigma)^2/2 dt}\right)
\end{aligned}$$

Since $\lambda = \frac{\mu - r}{\sigma}$, it suffices to know that μ is bounded. In particular, if μ is constant as in the original Black-Scholes model, then $E(\Pi(T)C(T)) < \infty$. More generally, if we strengthen the assumption $E(\eta[0, -\lambda](T)) = 1$ to the Novikov condition (Nielsen page 77),

- If we let $\bar{\Delta} = (\Delta_0, \Delta)$, where Δ_0 is as specified in Proposition 5.7, then $\bar{\Delta}$ is an admissible, self-financing trading strategy replicating C .

30 The Value Function—Executive Summary

In this and the next section, we will cover the material in Section 6.2 in Nielsen, except for the material on the Black-Scholes PDE. As Nielsen notes, the Black-Scholes PDE played a critical role in the development of the theory, but has been largely superseded in the theory by the martingale valuation method. Nielsen asserts the Black-Scholes PDE is useful in computation, presumably because it can be solved numerically; I have no reason to doubt this, but I know absolutely nothing about numerical solutions of PDEs, so I would not have anything useful to say on that subject.

In this section, Nielsen sets up a formula for evaluating a wide class of contingent claims in the Black-Scholes model. The notation is quite cumbersome. Hence, we provide a summary in this section, and the details in the next section.

Definition 30.1 A contingent claim Y is *path-independent* if

$$Y = g(S(T))$$

for some function $g : (0, \infty) \rightarrow \mathbf{R}$. In other words, Y depends only on the terminal value of S , rather than on the whole history of the price process.

We need some conditions on g to ensure that the integrals we need to evaluate the martingale value of the claim make sense. We know that the tails of the distribution of $S(T)$ are small, but if g grows sufficiently rapidly as $S(T) \rightarrow 0$ or $S(T) \rightarrow \infty$, then $g(S(T))$ might not be integrable.

Assumption 30.2 *We assume*

then $-\lambda - \sigma$ will also satisfy the Novikov condition, which says that

$$E \left(e^{\int_0^T (-\lambda - \sigma)^2 / 2 dt} \right) < \infty$$

1. g is locally integrable, i.e. for any bounded interval $[c, d]$, $\int_{[c,d]} |g| d\lambda < \infty$, where here λ denotes Lebesgue measure.
2. g satisfies the following polynomial growth condition:

$$\exists_{\beta > 0} \exists_{y_0 \geq 0} \forall_{y \in (y_0, \infty)} |g(y)| \leq 1 + y^\beta$$

Remark 30.3 Obviously, standard options like puts and call satisfy the polynomial growth condition.

Theorem 30.4 Suppose g satisfies Assumption 30.2. The martingale value process V of the claim $g(S(T))$ is

$$\Phi(S, t) = e^{-r(T-t)} v \left(\frac{\ln(S/S(0)) \Leftrightarrow \gamma t}{\sigma}, T \Leftrightarrow t \right)$$

where

$$v(x, T \Leftrightarrow t) = E_Q \left(g \left(S_0 e^{\gamma t + \sigma W^\lambda(T)} \right) \middle| W^\lambda(t) = x \right)$$

and

$$\gamma = r \Leftrightarrow \frac{\sigma^2}{2}$$

31 The Value Process—The Gory Details

In this section, we go through the cumbersome notational exercise needed to derive Theorem 30.4. The stock price process in the Black-Scholes model is lognormal. It is very convenient to express the value of a contingent claim in terms of integrals with respect to a normal distribution. The following notation facilitates this.

Let

$$\gamma = r \Leftrightarrow \frac{\sigma^2}{2}$$

For each $t \in [0, T]$, define

$$h[t] : \mathbf{R} \rightarrow (0, \infty)$$

by

$$h[t](x) = S(0) e^{\gamma t + \sigma x}$$

Notice that $h[t]$ is strictly increasing in $x \in \mathbf{R}$ (the stock is risky in the Black-Scholes Model, which entails $\sigma \neq 0$, and by convention, we take $\sigma > 0$, since reversing the sign of σ doesn't change the Model). Thus, $h[t]$ is a bijection from \mathbf{R} to $(0, \infty)$. Thus, there is an inverse function

$$h[t]^{-1} : (0, \infty) \rightarrow \mathbf{R}$$

defined by

$$h[t]^{-1}(S) = \frac{1}{\sigma} (\ln S \Leftrightarrow \ln S(0) \Leftrightarrow \gamma t)$$

The point of all this is as follows. Since

$$\begin{aligned} \frac{dS}{S} &= \mu dt + \sigma dW \\ &= (\mu \Leftrightarrow \sigma \lambda) dt + \sigma dW^\lambda \\ &= r dt + \sigma dW^\lambda \\ S(t) &= S(0)e^{(r-\sigma^2/2)t + \sigma W^\lambda(t)} \\ &= S(0)e^{\gamma t + \sigma W^\lambda(t)} \\ S(\omega, t) &= h[t](W^\lambda(\omega, t)) \\ W^\lambda(\omega, t) &= h[t]^{-1}(S(\omega, t)) \end{aligned}$$

Define

$$f[g] : \mathbf{R} \rightarrow \mathbf{R}$$

by

$$f[g] = g \circ h[T]$$

i.e.,

$$\begin{aligned} f[g](x) &= g(S(0)e^{\gamma T + \sigma x}) \\ f[g](W^\lambda(T)) &= g(h[T](W^\lambda(\omega, T))) \\ &= g(S(\omega, T)) \end{aligned}$$

In order to know that the value of the claim in terms of the stock price will be integrable and well-behaved, we need to have a growth condition on g . There is no problem with moderate values of $\ln S$; the potential problems arise for $\ln S$ large and positive (S large and positive) and $\ln S$ large and negative (S close to zero). Since the tails of the normal distribution fall off very quickly, the growth condition on g is not very stringent.

Assumption 31.1 We assume

1. g is locally integrable, i.e. for any bounded interval $[c, d]$, $\int_{[c, d]} |g| d\lambda < \infty$, where here λ denotes Lebesgue measure.
2. $f[g] \in G_0(a)$ for some $a > 0$ satisfying $T < 1/2a$, where

$$G_0(\alpha) = \left\{ f : \mathbf{R} \rightarrow \mathbf{R}, \exists_{\kappa, C} |f(y)| \leq C e^{\alpha y^2} \text{ for almost all } y \in \mathbf{R} \text{ with } |y| \geq \kappa \right\}$$

Proposition 31.2 (Proposition 6.1 in Nielsen) A sufficient condition for $f[g]$ to be in $G_0(\alpha)$ for all $\alpha > 0$ is that g satisfies the following polynomial growth condition:

$$\exists_{\beta > 0} \exists_{y_0 \geq 0} \forall_{y \in (y_0, \infty)} |g(y)| \leq 1 + y^\beta$$

Proof: We find

$$\begin{aligned} |f[g](x)| &= |g(h[T](x))| \\ &\leq 1 + (h[T](x))^\beta \\ &= 1 + \left(S(0) e^{\gamma T + \sigma x} \right)^\beta \\ &= 1 + S(0)^\beta e^{\beta \gamma T + \beta \sigma x} \\ &\leq e^{\alpha x^2} \end{aligned}$$

for x sufficiently large. ■

Remark 31.3 Obviously, standard options like puts and call satisfy the polynomial growth condition.

Our goal is to find the value function of a claim $g(S(T))$, express it in terms of the Wiener process W^λ , then transform it by $h[t]^{-1}$. To this end, let

$$p : \mathbf{R} \times (0, \infty) \rightarrow (0, \infty)$$

be the density of the Normal with mean 0 and variance τ

$$p(x, \tau) = \frac{e^{-x^2/2\tau}}{\sqrt{2\pi\tau}}$$

Let

$$v : \mathbf{R} \times [0, T) \rightarrow \mathbf{R}$$

be defined by

$$\begin{aligned} v(x, \tau) &= \int_{-\infty}^{\infty} f[g](y)p(y \Leftrightarrow x, \tau) dy \\ &= E(f[g](x + W(\tau))) \\ &\quad \text{because } x + W(\tau) \text{ has density } p(y \Leftrightarrow x, \tau) \\ &= E_Q(f[g](x + W^\lambda(\tau))) \\ &\quad \text{because the expectation depends only the distribution, and} \\ &\quad W^\lambda \text{ is a standard Wiener process with respect to } Q \end{aligned}$$

Assumption 31.1 implies that the expectation in the definition of v exists (Nielsen Proposition C.2) and v is infinitely differentiable (Nielsen Proposition C.3).

$$\begin{aligned} v(x, T \Leftrightarrow t) &= E_Q f[g](x + W^\lambda(T) \Leftrightarrow W^\lambda(t)) \\ &\quad \text{since } W^\lambda(T) \Leftrightarrow W^\lambda(t) \text{ is Normal, mean 0, variance } T \Leftrightarrow t \\ &= E_Q(f[g](W^\lambda(T)) | W^\lambda(t) = x) \\ v(W^\lambda(t), T \Leftrightarrow t) &= E_Q(f[g](W^\lambda(T)) | \mathcal{F}_t) \\ &= E_Q[g(S(T)) | \mathcal{F}_t] \\ &= e^{r(T-t)} V(t) \end{aligned}$$

Thus, v expresses the martingale value V of the claim $g(S(T))$ as a function of $(W^\lambda(t), T \Leftrightarrow t)$ and as a future rather than a present value.

Let

$$\begin{aligned} \Phi(S, t) &= e^{-r(T-t)} v(h[t]^{-1}(S), T \Leftrightarrow t) \\ &= e^{-r(T-t)} v\left(\frac{\ln(S/S(0)) \Leftrightarrow \gamma t}{\sigma}, T \Leftrightarrow t\right) \end{aligned}$$

Φ is C^∞ since v is.

Proposition 31.4 $\Phi(S(t), t) = V(t)$, the martingale value process of the claim $g(S(T))$.

Proof:

$$\begin{aligned}\Phi(S(t), t) &= e^{-r(T-t)}v\left(h[t]^{-1}(S(t)), T \Leftrightarrow t\right) \\ &= e^{-r(T-t)}v\left(W^\lambda(t), T \Leftrightarrow t\right) \\ &= e^{-r(T-t)}E_Q[g(S(T))|\mathcal{F}_t] \\ &= V(t)\end{aligned}$$

as claimed. ■

Remark 31.5 Notice that μ doesn't enter into the formula. Hence, in the Black-Scholes model, the martingale value process of every claim that can be priced is independent of μ , except to the extent that μ determines S . Thus, this puzzling feature of the Black-Scholes model applies in great generality to pricing claims, not just the standard options like puts and calls. A partial explanation is that the stock price S encodes information about μ , and it happens that S is a sufficient statistic for the effect of μ on the martingale value of the claim. But this really begs the question of *why* S is a sufficient statistic; it falls out from the mathematics, but there does not seem to be a clear intuition.

We have now worked through the martingale valuation method in considerable detail. As we have seen, it allows us to value quite general claims in the Black-Scholes Model. More importantly, it can be used to price standard options when the price process is *not* geometric Brownian motion. We developed the machinery when the price process is an Itô process; it can be extended to non-Itô price processes as well.

32 The Fundamental Theorem of Finance

In discrete models, the Fundamental Theorem of Finance asserts that if a pricing process \bar{S} is arbitrage-free, and includes a money-market account with value process M , there exists an equivalent probability measure Q such that $\frac{\bar{S}}{M}$ is a Q -martingale.

In this section, we explore the extent to which the Fundamental Theorem of Finance extends to continuous-time models. We assume throughout that \bar{S} is an Itô Process with

$$d\bar{S} = \bar{\mu} dt + \bar{\sigma} dW$$

and \bar{S} includes a money-market account with value process M and interest rate process $r \in \mathcal{L}^1$.

1. First, we have to decide what we mean by absence of arbitrage. We saw that even the Black-Scholes model admits arbitrage if we allow doubling strategies. The focus of the martingale approach is on admissible trading strategies, but these are defined in terms of a specific state price process; since part of the point is to find conditions on which the absence of arbitrage implies the existence of a state price process, we cannot define absence of arbitrage in terms of admissible trading strategies. We could restrict attention to trading strategies $\bar{\Delta} \in \mathcal{H}^2(\bar{S})$, i.e.

$$\bar{\Delta}\bar{\mu} \in \mathcal{H}^1, \quad \bar{\Delta}\bar{\sigma} \in \mathcal{H}^2$$

However, it's not clear that the absence of arbitrage with respect to strategies in $\mathcal{H}(\bar{S})$ gives enough information; for example, if $b \notin \mathcal{H}^2$, then buy-and-hold strategies do not lie in $\mathcal{H}(\bar{S})$. Here is a potential candidate:

Candidate Theorem 32.1 *Assume $\bar{S} \in \mathcal{H}(W)$, i.e. $a \in \mathcal{H}^1$ and $b \in \mathcal{H}^2$. If \bar{S} admits no arbitrage strategy in $\mathcal{H}(\bar{S})$, there is an equivalent measure Q such that \bar{S} is a Q -martingale.*

Is this true? Here are some potential problems. P and Q are equivalent if and only if $\mathcal{L}_P(W) = \mathcal{L}_Q(W)$. However, the equivalence of P and Q is not enough to ensure that $\mathcal{H}_P(W) = \mathcal{H}_Q(W)$. If we say that P and Q are boundedly equivalent provided that P and Q are equivalent and there exist $0 < m < M < \infty$ such that $m \leq \frac{dQ}{dP} \leq M$, then $\mathcal{H}_P(W) = \mathcal{H}_Q(W)$ if and only if P and Q are boundedly equivalent. However, even in the Black-Scholes model, P and Q are not boundedly equivalent. To see this, note that

$$\frac{dQ}{dP} = \eta[0, \Leftrightarrow \lambda](T) = e^{\lambda W(T)}$$

Since $W(T)$ has full support on $(-\infty, \infty)$, $\frac{dQ}{dP}$ has full support on $(0, \infty)$. Thus, it might be the case that $\bar{S} \in \mathcal{H}_P(W)$, but $\bar{S} \notin \mathcal{H}_Q(W)$, that \bar{S} has zero drift with respect to Q , but \bar{S} is not a Q -martingale. Thus, there seem to be significant barriers to proving the Candidate Theorem.

2. The only theorem I have found on the literature that shows that absence of arbitrage implies anything connected to state price processes or martingale measures is the following:

Theorem 32.2 (Karatzas and Shreve [4], Theorem 4.2) ³ *Suppose that \bar{S} admits no arbitrage in trading strategies such that the discounted cumulative gains process is uniformly bounded below by a constant. Then there is an adapted measurable process λ such that*

$$\bar{\mu} \Leftrightarrow r\bar{S} = \sigma\lambda^T$$

Recall that this does not show that there is a state price process; for that, we need to know that $\lambda \in \mathcal{L}^2$. Thus, it does not appear to be the case that the absence of arbitrage implies the existence of a state price process.

3. Perhaps we should weaken the demands on the equivalent measure Q . We could make the following definition:

Definition 32.3 Q is an equivalent martingale measure (EMM) for \bar{S} if Q is equivalent to P and $\frac{\bar{S}}{M}$ is a Q -martingale. Q is a risk-adjusted measure (RAM) for \bar{S} if Q is equivalent to P and $\frac{\bar{S}}{M}$ has zero drift with respect to Q , i.e. there exists W_Q , a Wiener process with respect to Q , such that

$$d\left(\frac{\bar{S}}{M}\right) = 0 dt + \tilde{\sigma}dW_Q$$

for some $\tilde{\sigma}$.⁴

4. Does existence of an EMM or an RAM imply the existence of a state price process?

Proposition 32.4 *Suppose Q is an RAM and there exists $\lambda \in \mathcal{L}^2$ such that*

$$E(\eta[0, \Leftrightarrow\lambda](T)) = 1 \text{ and } \frac{dQ}{dP} = \eta[0, \Leftrightarrow\lambda](T)$$

³This theorem appears on page 12!

⁴Our only recipe for constructing Q assumes we have $\lambda \in \mathcal{L}^2$ such that $\frac{dQ}{dP} = \eta[0, -\lambda](T)$. However, we don't know that that is the *only* way an EMM or RAM could be constructed. Hence, we can't just use W^λ for W_Q .

Then there is a state price process with respect to P .

Proof: Note that since $\frac{\bar{S}}{M}$ is an Itô Process with respect to P and W , it is an Itô Process with respect to Q and W^λ . Since it has zero drift with respect to *some* Q -Wiener Process W^Q , it must have zero drift with respect to W^λ ; but this implies that $\bar{S}\eta[\Leftrightarrow r, \Leftrightarrow \lambda]$ has zero drift with respect to P , so $\eta[\Leftrightarrow r, \Leftrightarrow \lambda]$ is a state price process with respect to P . ■

Proposition 32.4 says that the existence of a RAM is morally stronger than the existence of a state price process, but it does not show that it is technically stronger. Our only recipe for constructing a RAM or EMM is to find a state price process, hope that $E(\eta[0, \Leftrightarrow \lambda](T)) = 1$, and then set $\frac{dQ}{dP} = \eta[0, \Leftrightarrow \lambda](T)$. But that does *not* rule out the possibility that there might be an EMM or RAM which does not come from this construction. The following proposition gives one condition under which an EMM or RAM must come from this construction:

Proposition 32.5 *Suppose Q is a probability measure absolutely continuous with respect to P and $\frac{dQ}{dP}$ is \mathcal{F}_T -measurable, where $\{\mathcal{F}_t\}$ is the filtration generated by the Wiener Process W . Then there exists $\lambda \in \mathcal{L}^2$ such that*

$$\eta[0, \Leftrightarrow \lambda](T) = \frac{dQ}{dP}$$

Proof: Let

$$\eta[t] = E\left(\frac{dQ}{dP} \middle| \mathcal{F}_t\right)$$

η is a martingale; since $\{\mathcal{F}_t\}$ is the filtration generated by the Wiener Process W , the Martingale Representation Theorem implies that

$$d\eta = b dW$$

for some $b \in \mathcal{L}^2$, in particular η is an Itô Process. Since $\eta > 0$, $\eta = \eta[\Leftrightarrow r, \Leftrightarrow \lambda]$ for some $r \in \mathcal{L}^1$ and $\lambda \in \mathcal{L}^2$ so

$$d\eta = \eta[\Leftrightarrow r dt + \Leftrightarrow \lambda dW] = \Leftrightarrow \eta r dt + \Leftrightarrow \eta \lambda dW$$

By the uniqueness of Itô coefficients, $r = 0$, so $\eta = \eta[0, \Leftrightarrow \lambda]$. ■

In general, it is not clear that if Q is an EMM or an RAM, that $\frac{dQ}{dP} = \eta(T)$ for an Itô Process η . The filtration $\{\mathcal{F}_t\}$ need not be the filtration generated by the Wiener Process, and it is not clear why $\frac{dQ}{dP}$ should be $\{\mathcal{F}_T\}$ -measurable.

5. Does existence of a state price process imply existence of an RAM or an EMM? The following proposition sums up what we have shown on this subject.

Proposition 32.6 *Suppose there is a state price process $\Pi = \Pi(0)\eta[\Leftrightarrow r, \Leftrightarrow \lambda]$ for \bar{S} .*

- (a) *If $E(\eta[0, \Leftrightarrow \lambda](T)) = 1$, then Q defined by $\frac{dQ}{dP} = \eta[0, \Leftrightarrow \lambda](T)$ is an RAM.*
- (b) *If all of the buy-and-hold strategies are admissible with respect to Π , then Q is an EMM.*

Proof: Part (a) is left as an exercise. Now, we turn to part (b). Let $\bar{\Delta}_n$ be the buy-and-hold strategy which buys one unit of security n at time 0 and holds it until time T . Suppose all buy-and-hold strategies are admissible with respect to $\Pi = \eta[\Leftrightarrow r, \Leftrightarrow \lambda]$. Then $M(t) = \eta[r, 0](t)$, so $\Pi(t)M(t) = \eta[0, \Leftrightarrow \lambda](t)$. Since security zero is the money-market account with value process M , the strategy $\bar{\Delta}_0$, buying and holding the money-market account, is admissible; therefore, $\eta[0, \lambda]$ is a martingale, so the measure Q with Radon-Nikodym derivative $\eta[0, \lambda]$ is a probability measure equivalent to P . Then $\bar{\Delta}_n(t)\bar{S}(t) = \bar{S}_n(t)$. By Proposition 26.4, $\frac{\bar{S}_n(t)}{M(t)}$ is a martingale with respect to Q . Since this is true for each n , $\frac{\bar{S}(t)}{M(t)}$ is a vector martingale with respect to Q . ■

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