1. Consider a one-dimensional random walk process like the random walk process $X$ discussed in class, except that every step is of the form $\pm \sigma \sqrt{n}$ rather than $\pm 1 \sqrt{n}$. What is the limit of this random walk as $n \to \infty$ (you may argue informally, as in class)? Show that the limit is a standard one-dimensional Brownian motion with a time change.

2. Let $B$ be a standard one-dimensional Brownian motion. Evaluate
$$\limsup_{t \to 0} \frac{B(\cdot, t)}{\sqrt{2t \ln |\ln t|}}$$
Hint: Use Proposition 1.6.

3. Let $X$ be the one-dimensional random walk discussed in class. We noted in class that, with respect to the partition $t_k = k/n$, the quadratic variation
$$\sum_{k=0}^{nT-1} (X(\omega, t_{k+1}) - X(\omega, t_k))^2 = T$$
for all $\omega$.

(a) Show that if $t_k = \frac{2k}{n}$ for $k = 1, \ldots, \lfloor \frac{nT}{2} \rfloor$, then
$$\sum_{k=0}^{\lfloor \frac{nT}{2} \rfloor - 1} (X(\omega, t_{k+1}) - X(\omega, t_k))^2 \to T$$
in probability as $n \to \infty$. 

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(b) Show that, if one is allowed to choose $t_k$ as a function of $\omega$ ($k = 1, \ldots, m(n, \omega)$), then one can find a choice of $t_k(\omega)$ such that

$$m(n, \omega) \sum_{k=0}^{m(n, \omega)} (X(\omega, t_{k+1}(\omega)) - X(\omega, t_k(\omega)))^2 \rightarrow \frac{3T}{2}$$

in probability as $n \rightarrow \infty$.

(c) Conjecture what assumption is needed on the $t_k(\omega)$ to ensure that

$$m(n, \omega) \sum_{k=0}^{m(n, \omega)} (X(\omega, t_{k+1}(\omega)) - X(\omega, t_k(\omega)))^2 \rightarrow T$$

in probability as $n \rightarrow \infty$.

4. Let $X$ be the one-dimensional random walk discussed in class. Find the distribution of the random variable

$$\max_{t \in [0,T]} X(\cdot, t)$$

and deduce the distribution of the random variable

$$\max_{t \in [0,T]} B(\cdot, t)$$

where $B$ is standard one-dimensional Brownian motion. Hint: Find a relationship between $\max_{t \in [0,T]} X(\cdot, t)$ and $X(\cdot, T)$. 