Economics 201B–Second Half

Lecture 2 3/11/10

Two Graphical "Proofs" of the Existence of Walrasian Equilibrium in the Edgeworth Box

Demand: $D_i(p) = \{x \in B_i(p) : \forall_{y \in B_i(p)} x \succeq_i y\}$

Walrasian Equilibrium (in the Edgeworth Box) is a pair (p, x) where

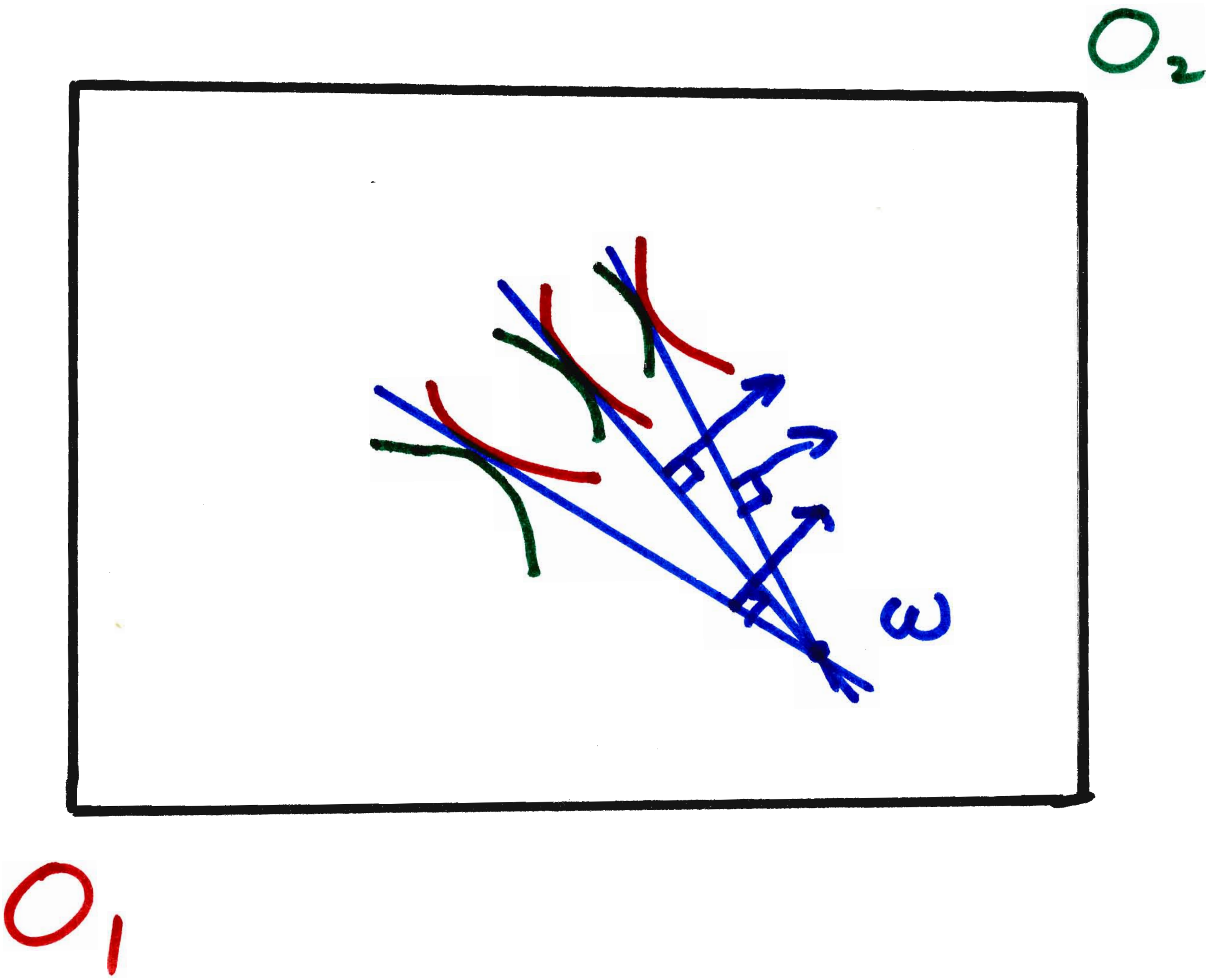
- x is an exact allocation
- $x_i \in D_i(p)$ $(i = 1, 2)$

In the following Edgeworth Box Diagram, we give a graphical representation of Walrasian Equilibrium. In fact, there are (at least) three Walrasian Equilibria in the drawing, and there is nothing apparently pathological in the preferences of the two agents. Note that if the demands of the two agents at a single price p are represented by the same point in the Edgeworth Box, it indicates that the sum of the demands equals the total supply, so we have Walrasian Equilibrium; on the other hand, if the demands of the two agents at a price p are represented by different points in the Edgeworth Box, the sum of the demands does *not* equal the total supply; p is not an equilibrium price.

Why the quotes on "Proofs"? Why the Proofs inside the quotes?

- graphical arguments prone to introduction of tacit assumptions
- these arguments can be turned into proofs; our real proof later follows the first of the two "proofs"

Price Normalization: $p \in \Delta^0 = \{p \in \mathbf{R}^2_{++} : p_1 + p_2 = 1\}$; $\Delta = \{p \in \mathbf{R}^2_{+} : p_1 + p_2 = 1\}$


O_2
ω
O_1

Notation:

- $D(p) = D_1(p) + D_2(p)$ Market Demand
- $E_i(p) = D_i(p) - \omega_i$ Excess Demand of i
- $E(p) = E_1(p) + E_2(p) = D(p) - \bar{\omega}$ Market Excess Demand
- *Offer Curve:*
  - $OC_i = \{x : \exists_{p \in \Delta^0} x \in D_i(p)\}$ This is a curve in the Edgeworth Box Diagram; OC_1 measured from O_1, OC_2 from O_2.
  - $OC = \{x : \exists_{p \in \Delta^0} x \in E(p)\}$ This is a curve in $\mathbf{R}^2$.
  - $0 \in OC \Leftrightarrow$ there is a Walrasian Equilibrium: straightforward.
  - $(OC_1 \cap OC_2) \setminus \{\omega\} \neq \emptyset \Rightarrow$ there is a Walrasian Equilibrium; we'll see why.

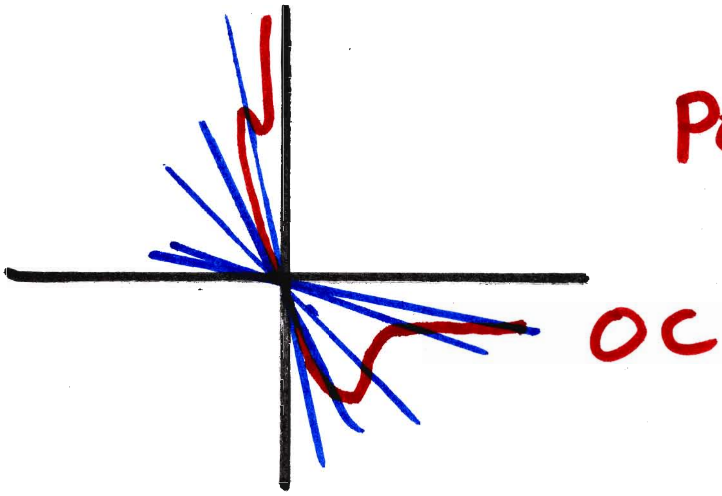
Items Common to the two "Proofs:"

- **Lemma 1** *If $p_n \in \Delta^0$ and $p_{n\ell} \to 0$ as $n \to \infty$, then $|D_i(p_n)| \to \infty$.*

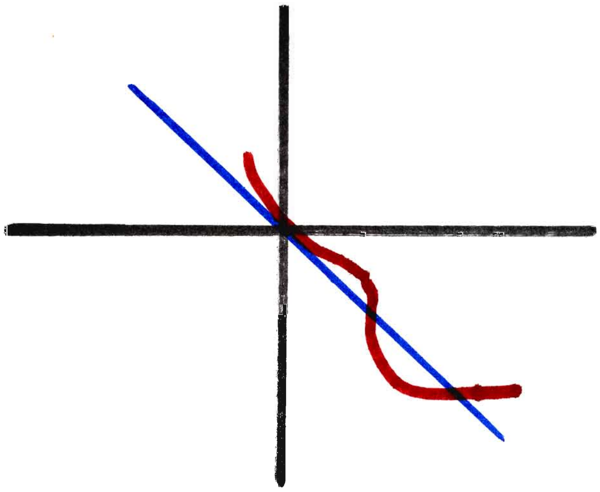
  This follows from strong monotonicity, and was likely proved in 201A. We'll prove later in a more general case.

- *Walras' Law:*
  - $p \cdot D_i(p) \leq p \cdot \omega_i$. Comes from definition, with no assumptions on preferences.
  - By strong monotonicity, can't have $p \cdot D_i(p) < p \cdot \omega_i$, so $p \cdot D_i(p) = p \cdot \omega_i$, so $p \cdot E_i(p) = 0$, so $p \cdot E(p) = 0$. In particular,

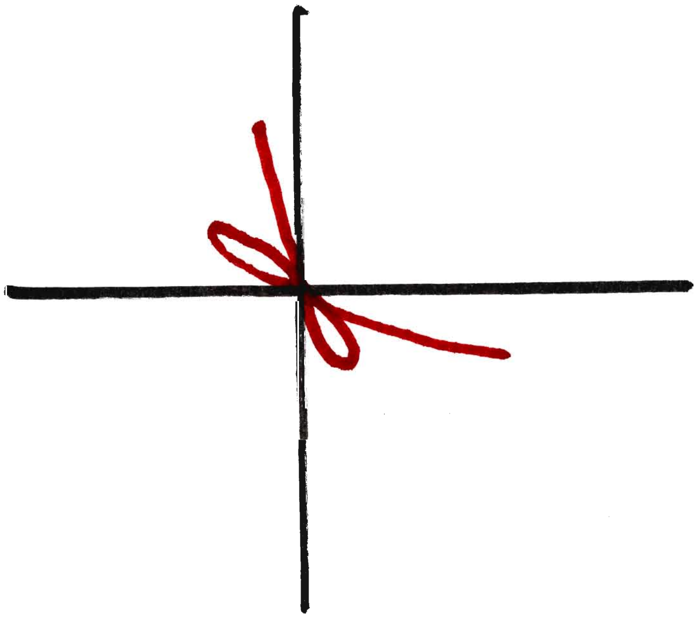
$$\begin{gathered} \nexists_{p \in \Delta^0} \ (D_i(p) < \omega_i \vee D_i(p) > \omega_i) \\ \nexists_{p \in \Delta^0} \ (E(p) < 0 \vee E(p) > 0) \end{gathered} \tag{1}$$



POSSIBLE OFFER CURVE



CAN'T BE OFFER CURVE



POSSIBLE OC: MULTIPLE EQUILIBRA

- Observe that this is where we use the fact that $D_i(p) \geq 0$, equivalently $E_i(p) \geq -\omega_i$:

$$\begin{aligned} (p_n)_2 D(p_n)_2 &\leq p_n \cdot D(p_n) \\ &= p_n \cdot \bar{\omega} \\ &\leq \max\{\bar{\omega}_1, \bar{\omega}_2\} \end{aligned}$$

If $p_{n1} \to 0$, $p_{n2} \to 1$, so for n sufficiently large, $D(p_n)_2 \leq 2\max\{\bar{\omega}_1, \bar{\omega}_2\}$, so $D(p_n)_2 \not\to \infty$. Therefore,

$$\begin{aligned} p_{n1} \to 0 \Rightarrow D(p_n)_1 \to \infty \Rightarrow E(p_n)_1 \to \infty \\ p_{n2} \to 0 \Rightarrow D(p_n)_2 \to \infty \Rightarrow E(p_n)_2 \to \infty \end{aligned} \tag{2}$$

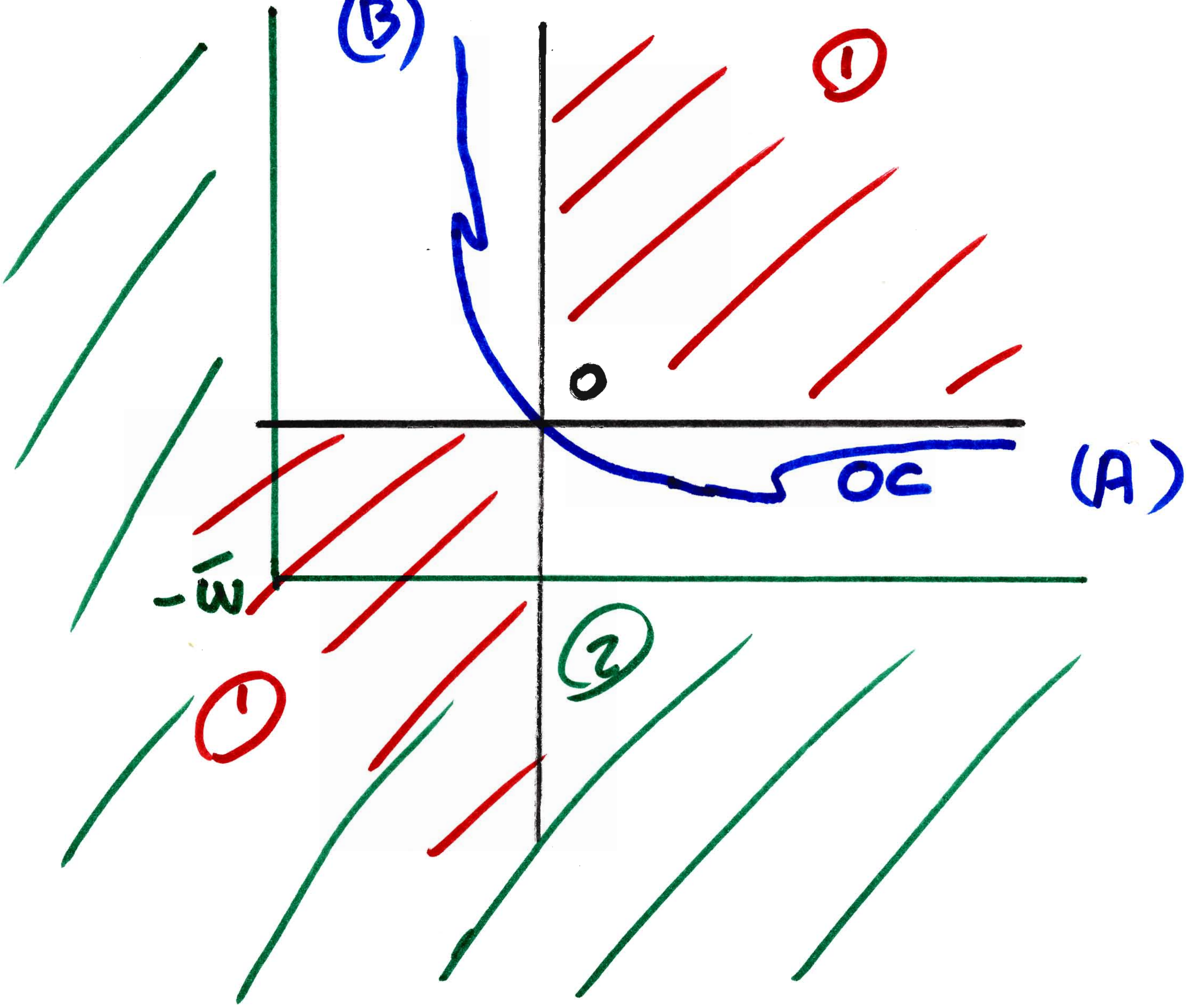
- Given $p \in \Delta^0$, $D_1(p), D_2(p)$ and $E(p)$ each consist of a single element. In other words, every ray through the origin with negative slope intersects OC in exactly one point other than zero. In the Edgeworth Box diagram, each ray through ω with negative slope intersects OC_1 and OC_2 in exactly one point, other than ω, each. Given a point $x \in OC, x \neq 0$, there is a unique $p \in \Delta^0$ such that $x \in E(p)$; p is the perpendicular to the ray through 0 and x. Given a point $x \in OC_i$, $x \neq \omega$, there is a unique $p \in \Delta^0$ such that $x \in D_i(p)$; p is the perpendicular to the ray through ω and x.

"Proof 1:" (In Consumption Space, using OC)

- $$\begin{aligned} 0 \in OC &\Leftrightarrow \exists_{p \in \Delta^0}\, E(p) = 0 \\ &\Leftrightarrow \text{Walrasian Equilibrium exists} \end{aligned}$$

  Hence, it suffices to show that $0 \in OC$

- $$E(p) = D(p) - \bar{\omega} \geq -\bar{\omega} \tag{3}$$



(B)
1
0
OC
(A)
-w̄
1
2

- In the following diagram, Equations (2) and (3) tell us that OC goes from the region (A) (when p_1 is small) to the region (B) (when p_2 is small).

- Equation (1) tells us that OC avoids the first (northeast) and third (southwest) quadrants, *so OC must pass through zero, so Walrasian Equilibrium exists!*

- However, it appears that OC may go through the origin more than once, reinforcing the earlier conclusion that Walrasian Equilibrium need not be unique.

"Proof 2" (uses OC_1 and OC_2 as in diagrams in MWG, assumes preferences are smooth)

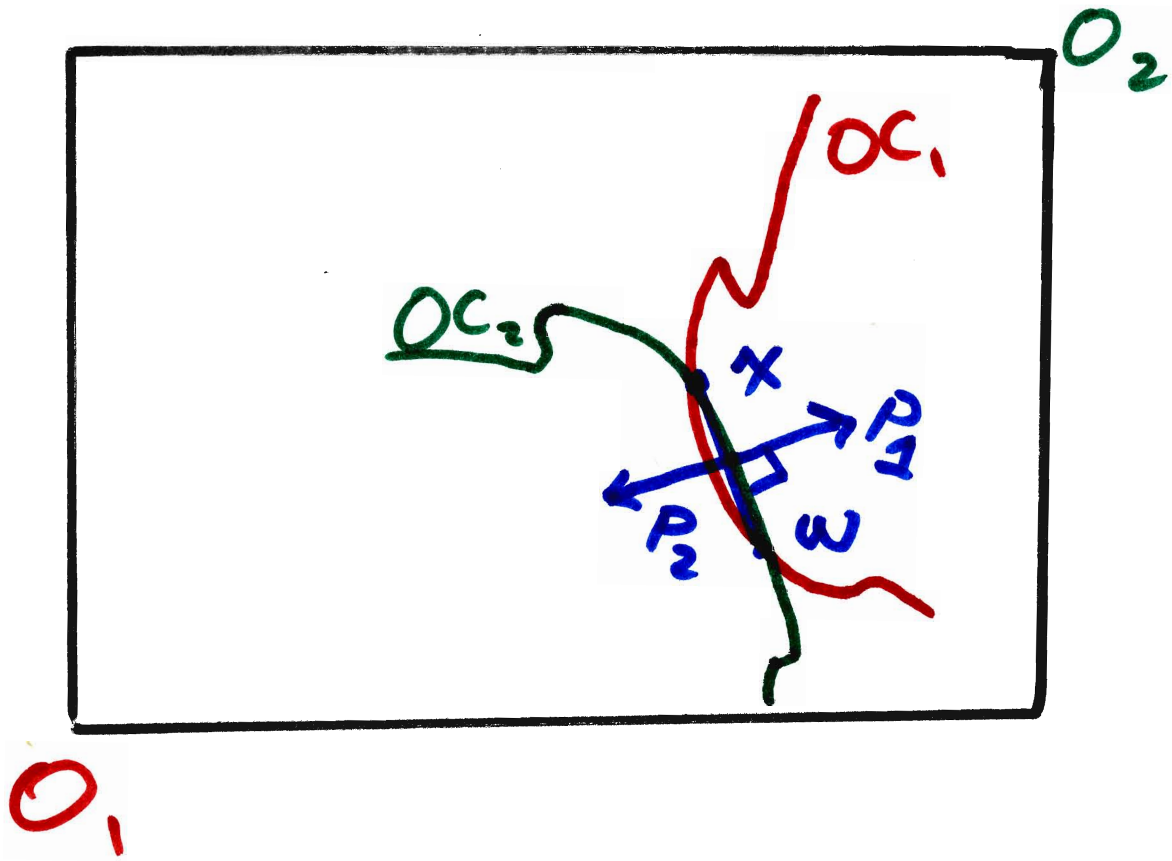
- Suppose $x \in OC_1 \cap OC_2$, $x \neq \omega$. Then $x_i = D_i(p_i)$ for some $p_i \in \Delta^0$ $(i = 1, 2)$, so $x_i \geq 0$, and hence x lies in the Edgeworth Box; although each offer curve can go outside the Edgeworth Box, any intersection of the offer curves must lie in the Edgeworth Box. There is a unique ray going through x and ω, and p_1 and p_2 are both perpendicular to it, so $p_1 = p_2$. Since x is a point in the Edgeworth Box, $x_1 + x_2 = \bar{\omega}$, so p_1 is a Walrasian Equilibrium Price. In other words, it suffices to show that $OC_1 \cap OC_2$ contains at least one $x \neq \omega$.

- $$\omega \in OC_1 \cap OC_2 \tag{4}$$

  To see this, let p_i be the "support price" to $\succeq_i$ at ω_i. In other words,

  $$y \succeq_i \omega_i \Rightarrow p_i \cdot y \geq p_i \cdot \omega_i$$

  We'll explain more carefully later why the support price exists. Then $\omega_i = D_i(p_i)$ so $\omega_i \in OC_i$, so $\omega \in OC_1 \cap OC_2$.



O_2
OC_1
OC_2
x
P_1
P_2
w
O_1

- If preferences are smooth, then

$$\begin{aligned} & p_i \cdot (D_i(p) - \omega_i) \\ &= p \cdot (D_i(p) - \omega_i) + (p_i - p) \cdot (D_i(p) - D_i(p_i)) \\ &= 0 \text{ (by Walras' Law) } + O(|p_i - p|^2) \end{aligned}$$

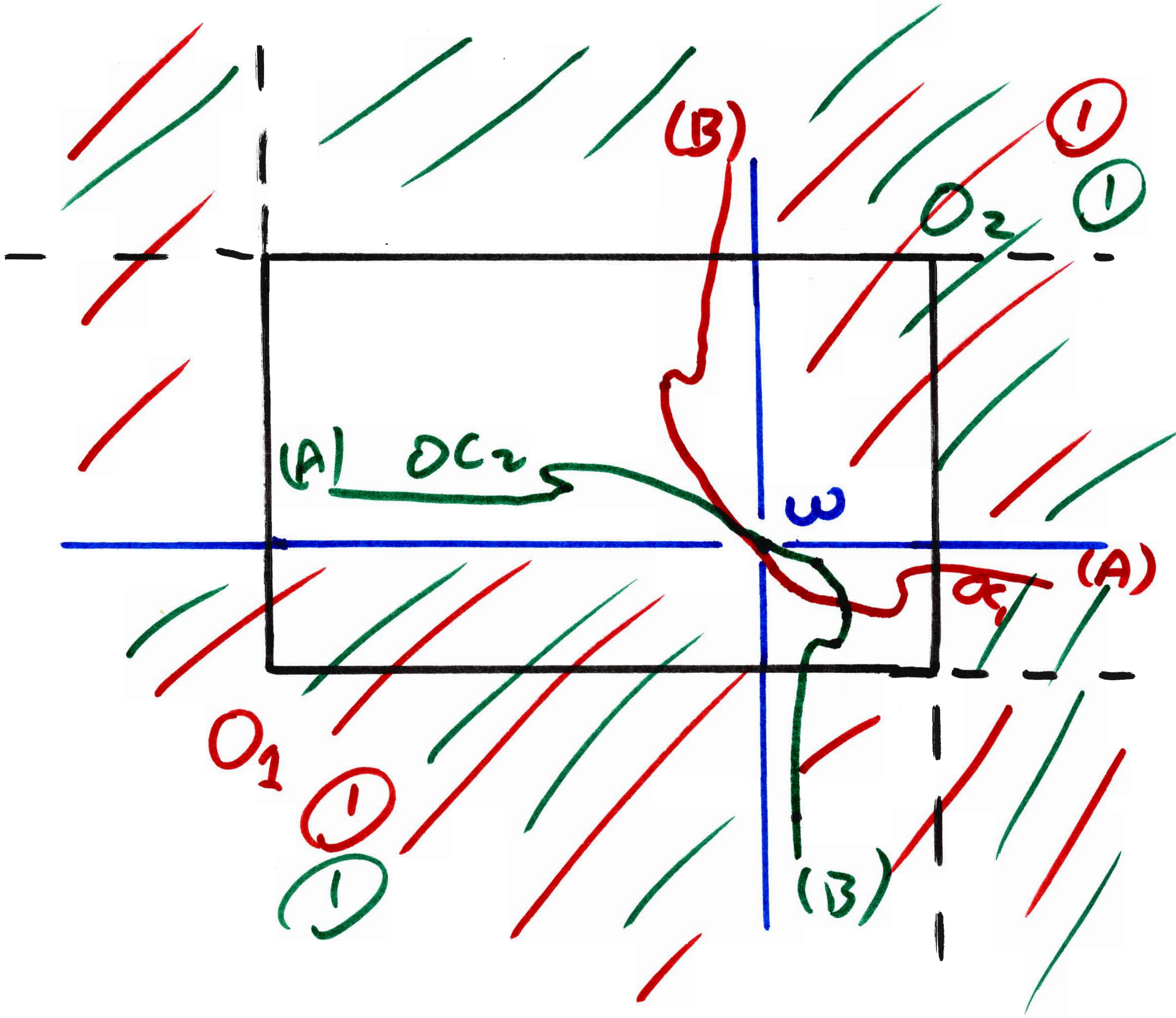
  which shows that p_i is tangent to OC_i at ω_i.

- If it turns out that $p_1 = p_2$, then this common price is a Walrasian Equilibrium Price and ω is a Walrasian Equilibrium allocation. If $p_1 \neq p_2$, then

  – OC_1 and OC_2 cross at ω.

  – By Equation (1), $OC_1 \cup OC_2$ cannot enter the quadrant northeast of ω or the quadrant southwest of ω.

  – By Equation (2), as the price of the first good moves from 0 to 1, OC_1 and OC_2 travel from (A) to (B). Notice that OC_1 at (A) lies northeast of OC_2 at (B), and OC_1 at (B) lies northeast of OC_2 at (A). Thus, OC_1 and OC_2 "must" cross an even number of times, hence they cross at some $x \neq \omega$, so Walrasian Equilibrium exists.



(B)
1
1
O2
(A)
OC2
ω
(A)
OC1
O1
1
1
(B)