Economics 201B–Second Half
Lecture 6, 4/1/10
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The Second Welfare Theorem in the Arrow Debreu Economy

Theorem 1 (Second Welfare Theorem) (Pure Exchange Case) *If x^* is Pareto Optimal in a pure exchange economy, with strongly monotone, continuous, convex preferences, there exists a price vector p^* and an income transfer T such that (p^*, x^*, T) is a Walrasian Equilibrium with Transfers.*

Outline of Proof:

- Let

$$\begin{aligned} A_i &= \{x'_i - x^*_i : x'_i \succ_i x^*_i\} \\ A &= \sum_{i=1}^{I} A_i = \{a_1 + \cdots + a_I : a_i \in A_i\} \end{aligned}$$

  Then $0 \notin A$ (if it were, we'd have a Pareto improvement).

- By Minkowski's Theorem, find $p^* \neq 0$ such that

$$\inf p^* \cdot A \geq 0$$

- Show $\left(\mathbf{R}^L_+ \setminus \{0\}\right) \subset A_i$ and hence $p^* \geq 0$.
- Show $\inf p^* \cdot A_i = 0$ for each i.
- Define T to make x^*_i affordable at p^*:

$$T_i = p^* \cdot x^*_i - p^* \cdot \omega_i$$

  Show $\Sigma_{i=1}^{I} T_i = 0$ and

$$x^*_i \in Q_i(p^*, T)$$

- Use strong monotonicity to show that $p^* \gg 0$.
- Show
$$p^* \gg 0 \Rightarrow Q_i(p^*, T) = D_i(p^*, T)$$

Now, for the details:

- Let
$$\begin{aligned} A_i &= \{x_i' - x_i^* : x_i' \succ_i x_i^*\} \\ A &= \sum_{i=1}^{I} A_i = \{a_1 + \cdots + a_I : a_i \in A_i\} \end{aligned}$$

Claim:
$$0 \notin A$$
If $0 \in A$, there exists $a_i \in A_i$ such that
$$\sum_{i=1}^{I} a_i = 0$$
Let
$$x_i' = x_i^* + a_i$$
Since $x_i' - x_i^* = a_i \in A_i$, we have
$$x_i' \succ_i x_i^*$$

$$\begin{aligned} \sum_{i=1}^{I} x_i' &= \sum_{i=1}^{I} (x_i^* + a_i) \\ &= \sum_{i=1}^{I} x_i^* + \sum_{i=1}^{I} a_i \\ &= \sum_{i=1}^{I} x_i^* \\ &= \bar{\omega} \end{aligned}$$

Therefore, x' is an exact** allocation, x' Pareto improves x^*, so x^* is not Pareto Optimal, contradiction. Therefore, $0 \notin A$.

- $$\exists_{p^* \neq 0} \ \inf p^* \cdot A \geq 0$$

  A_i is convex, so A is convex (easy exercise). By Minkowski's Theorem, there exists $p^* \neq 0$ such that

  $$0 = p^* \cdot 0 \leq \inf p^* \cdot A = \sum_{i=1}^{I} \inf p^* \cdot A_i$$

  The fact that $\inf p^* \cdot A = \Sigma_{i=1}^{I} \inf p^* \cdot A_i$ is an exercise; once you figure out what you have to prove, it is obvious.

- We claim that $p^* \geq 0$.
  Suppose not, so $p^*_\ell < 0$ for some ℓ, WLOG $p^*_1 < 0$. Let

  $$x'_i = x^*_i + \left(-\frac{1}{p^*_1}, 0, \ldots, 0\right)$$

  By strong monotonicity, $x'_i \succ_i x^*_i$, so

  $$\left(-\frac{1}{p^*_1}, 0, \ldots, 0\right) \in A_i$$

  So

  $$\begin{aligned} \inf p^* \cdot A_i &\leq p^* \cdot \left(-\frac{1}{p^*_1}, 0, \ldots, 0\right) \\ &= -1 < 0 \\ \inf p^* \cdot A &= \sum_{i=1}^{I} \inf p^* \cdot A_i \\ &\leq -I \\ &< 0 \end{aligned}$$

  a contradiction that shows $p^* \geq 0$.

- We claim that $\inf p^* \cdot A_i = 0$ for each i:
  Suppose $\varepsilon > 0$. By strong monotonicity,

  $$x^*_i + (\varepsilon, \ldots, \varepsilon) \succ_i x^*_i$$

so

$$(\varepsilon, \ldots, \varepsilon) \in A_i$$

so

$$\inf p^* \cdot A_i \leq p^* \cdot (\varepsilon, \ldots, \varepsilon)$$

Since ε is an arbitrary positive number, $\inf p^* \cdot A_i$ is less than every positive number, so

$$\inf p^* \cdot A_i \leq 0$$

Since $\Sigma_{i=1}^{I} \inf p^* \cdot A_i \geq 0$,

$$\inf p^* \cdot A_i = 0 \ (i = 1, \ldots, I)$$

- Define T to make x_i^* affordable at p^*. We claim that T is an income transfer and

$$x_i^* \in Q_i(p^*, T)$$

Let

$$T_i = p^* \cdot x_i^* - p^* \cdot \omega_i$$

$$\begin{aligned} \sum_{i=1}^{I} T_i &= \sum_{i=1}^{I} (p^* \cdot x_i^* - p^* \cdot \omega_i) \\ &= p^* \cdot \left(\sum_{i=1}^{I} x_i^* - \sum_{i=1}^{I} \omega_i \right) \\ &= p^* \cdot (\bar{\omega} - \bar{\omega}) \\ &= 0 \end{aligned}$$

so T is an income transfer.

$$\begin{aligned} p^* \cdot x_i^* &= p^* \cdot (\omega_i + (x_i^* - \omega_i)) \\ &= p^* \cdot \omega_i + p^* \cdot (x_i^* - \omega_i) \\ &= p^* \cdot \omega_i + T_i \end{aligned}$$

so

$$x_i^* \in B_i(p^*, T)$$

If $x_i' \succ_i x_i^*$, then $x_i' - x_i^* \in A_i$, so

$$\begin{aligned} p^* \cdot x_i' &= p^* \cdot (x_i^* + (x_i' - x_i^*)) \\ &= p^* \cdot x_i^* + p^* \cdot (x_i' - x_i^*) \\ &\geq p^* \cdot x_i^* + \inf p^* \cdot A_i \\ &= p^* \cdot x_i^* \\ &= p^* \cdot \omega_i + T_i \end{aligned}$$

so

$$x_i^* \in Q_i(p^*, T) * *$$

- Use strong monotonicity to show that $p^* \gg 0$.

Lemma 2 *If $\succeq_i$ is continuous and complete, and $x \succ_i y$, then there exists $\varepsilon > 0$ such that*

$$(B(x, \varepsilon) \cap X_i) \succ_i y$$

Proof: If $(B(x, \varepsilon) \cap X_i) = \{x\}$ for some $\varepsilon > 0$, i.e. x is an isolated point in X_i, then the lemma is true, since $x \succ_i y$. If x is not an isolated point in X_i, then we can find $x_n \to x$, $x_n \in X_i$, $x_n \not\succ_i y$; by completeness, we have $y \succeq_i x_n$ for each n. Since $\succeq_i$ is continuous, $y \succeq_i x$, so $x \not\succ_i y$, a contradiction which proves the lemma.■

Since $p^* \geq 0$ and $p^* \neq 0$, $p^* > 0$; since in addition $\bar{\omega} \gg 0$, $p^* \cdot \bar{\omega} > 0$, so

$$p^* \cdot \omega_i + T_i > 0 \text{ for some } i$$

If $p_\ell^* = 0$ for some ℓ (WLOG $\ell = 1$), let

$$x_i' = x_i^* + (1, 0, \ldots, 0)$$

By strong monotonicity, $x'_i \succ_i x^*_i$.

$$p^* \cdot x'_i = p^* \cdot x^*_i = p^* \cdot \omega_i + T_i > 0$$

Find ℓ (WLOG $\ell = 2$) such that

$$p^*_\ell > 0, \ x'_{2i} > 0$$

Since $x'_i \succ_i x^*_i$, let $\varepsilon > 0$ be chosen to satisfy the conclusion of the Lemma. If necessary, we may make ε smaller to ensure that $\varepsilon \leq 2x'_{2i}$. Let

$$x''_i = x'_i - (0, \varepsilon/2, 0, \ldots, 0)$$

Since $X_i = \mathbf{R}^L_+$, $x''_i \in X_i$, so by the Lemma, $x''_i \succ_i x^*_i$. But $p^* \cdot x''_i < p \cdot x'_i = p^* \cdot \omega_i + T_i$, which shows that $x^*_i \notin Q_i(p^*, T)$, a contradiction which proves that $p^* \gg 0$.

- Show

$$p^* \gg 0 \Rightarrow Q_i(p^*, T) = D_i(p^*, T)$$

  – Case 1: $p^* \cdot \omega_i + T_i = 0$. Since $p^* \gg 0$, $B_i(p^*, T) = \{0\}$, so

$$Q_i(p^*, T) = D_i(p^*, T) = \{0\}$$

  – Case 2: $p^* \cdot \omega_i + T_i > 0$
  Suppose $x \in Q_i(p^*, T)$ but $x \notin D_i(p^*, T)$. Then there exists $z \succ_i x$ such that $z \in B_i(p^*, T)$, hence $p^* \cdot z \leq p^* \cdot \omega_i + T_i$. Since $x \in Q_i(p^*, T)$, $p^* \cdot z \geq p^* \cdot \omega_i + T_i$, so

$$p^* \cdot z = p^* \cdot \omega_i + T_i > 0$$

  By Lemma 2, there exists $\varepsilon > 0$ such that

$$|z' - z| < \varepsilon, z' \in \mathbf{R}^L_+ \Rightarrow z' \succ x$$

Let

$$z' = z\left(1 - \frac{\varepsilon}{2|z|}\right)$$

Since $z \in \mathbf{R}^L_+$, $z' \in \mathbf{R}^L_+$.

$$|z' - z| = \left|\frac{\varepsilon z}{2|z|}\right| = \frac{\varepsilon}{2} < \varepsilon$$

so $z' \succ x$.

$$\begin{aligned} p^* \cdot z' &= p^* \cdot z\left(1 - \frac{\varepsilon}{2|z|}\right) \\ &= (p^* \cdot \omega_i + T_i)\left(1 - \frac{\varepsilon}{2|z|}\right) \\ &< p^* \cdot \omega_i + T_i \end{aligned}$$

which contradicts the assumption that $x \in Q_i(p^*, T)$. This shows $Q_i(p^*, T) \subset D_i(p^*, T)$; since clearly $D_i(p^*, T) \subset Q_i(p^*, T)$, $Q_i(p^*, T) = D_i(p^*, T)$.

What if preferences are not convex?

- Second Welfare Theorem may fail if preferences are nonconvex.
- Diagram gives an economy with two goods and two agents, and a Pareto optimum x^* so that so that the utility levels of x^* cannot be approximated by a Walrasian Equilibrium with Transfers.
- If p^* is the price which locally supports x^*, and T is the income transfer which makes x affordable with respect to the prices p^*, there is a unique Walrasian equilibrium with transfers (z^*, q^*, T); z^* is much more favorable to agent I and much less favorable to agent II than x^* is.

- This is the worst that can happen under standard assumptions on preferences. Given a Pareto optimum x^*, there is a Walrasian quasiequilibrium with transfers (z^*, p^*, T) such that all but L people are indifferent between x^* and z^*. Those L people are treated quite harshly (they get zero consumption). One could be less harsh and give these L people carefully chosen consumption bundles in the convex hull of their quasidemand sets, *but one would then have to forbid them from trading*, a prohibition that would in practice be difficult to enforce.