

## The Second Welfare Theorem in the Arrow Debreu Economy

**Theorem 1 (Second Welfare Theorem) (Pure Exchange Case)** *If  $x^*$  is Pareto Optimal in a pure exchange economy, with strongly monotone, continuous, convex preferences, there exists a price vector  $p^*$  and an income transfer  $T$  such that  $(p^*, x^*, T)$  is a Walrasian Equilibrium with Transfers.*

*Outline of Proof:*

- Let

$$A_i = \{x'_i - x_i^* : x'_i \succ_i x_i^*\}$$

$$A = \sum_{i=1}^I A_i = \{a_1 + \cdots + a_I : a_i \in A_i\}$$

Then  $0 \notin A$  (if it were, we'd have a Pareto improvement).

- By Minkowski's Theorem, find  $p^* \neq 0$  such that

$$\inf p^* \cdot A \geq 0$$

- Show  $(\mathbf{R}_+^L \setminus \{0\}) \subset A_i$  and hence  $p^* \geq 0$ .
- Show  $\inf p^* \cdot A_i = 0$  for each  $i$ .
- Define  $T$  to make  $x_i^*$  affordable at  $p^*$ :

$$T_i = p^* \cdot x_i^* - p^* \cdot \omega_i$$

Show  $\sum_{i=1}^I T_i = 0$  and

$$x_i^* \in Q_i(p^*, T)$$

- Use strong monotonicity to show that  $p^* \gg 0$ .
- Show

$$p^* \gg 0 \Rightarrow Q_i(p^*, T) = D_i(p^*, T)$$

Now, for the details:

- Let

$$A_i = \{x'_i - x_i^* : x'_i \succ_i x_i^*\}$$

$$A = \sum_{i=1}^I A_i = \{a_1 + \dots + a_I : a_i \in A_i\}$$

Claim:

$$0 \notin A$$

If  $0 \in A$ , there exists  $a_i \in A_i$  such that

$$\sum_{i=1}^I a_i = 0$$

Let

$$x'_i = x_i^* + a_i$$

Since  $x'_i - x_i^* = a_i \in A_i$ , we have

$$x'_i \succ_i x_i^*$$

$$\begin{aligned} \sum_{i=1}^I x'_i &= \sum_{i=1}^I (x_i^* + a_i) \\ &= \sum_{i=1}^I x_i^* + \sum_{i=1}^I a_i \\ &= \sum_{i=1}^I x_i^* \\ &= \bar{w} \end{aligned}$$

Therefore,  $x'$  is a feasible allocation,  $x'$  Pareto improves  $x^*$ , so  $x^*$  is not Pareto Optimal, contradiction.

Therefore,  $0 \notin A$ .

•

$$\exists_{p^* \neq 0} \inf p^* \cdot A \geq 0$$

$A_i$  is convex, so  $A$  is convex (easy exercise). By Minkowski's Theorem, there exists  $p^* \neq 0$  such that

$$0 = p^* \cdot 0 \leq \inf p^* \cdot A = \sum_{i=1}^I \inf p^* \cdot A_i$$

The fact that  $\inf p^* \cdot A = \sum_{i=1}^I \inf p^* \cdot A_i$  is an exercise; once you figure out what you have to prove, it is obvious.

• We claim that  $p^* \geq 0$ .

Suppose not, so  $p_\ell^* < 0$  for some  $\ell$ , WLOG  $p_1^* < 0$ . Let

$$x'_i = x_i^* + \left( -\frac{1}{p_1^*}, 0, \dots, 0 \right)$$

By strong monotonicity,  $x'_i \succ_i x_i^*$ , so

$$\left( -\frac{1}{p_1^*}, 0, \dots, 0 \right) \in A_i$$

So

$$\begin{aligned} \inf p^* \cdot A_i &\leq p^* \cdot \left( -\frac{1}{p_1^*}, 0, \dots, 0 \right) \\ &= -1 < 0 \\ \inf p^* \cdot A &= \sum_{i=1}^I \inf p^* \cdot A_i \\ &\leq -I \\ &< 0 \end{aligned}$$

a contradiction that shows  $p^* \geq 0$ .

• We claim that  $\inf p^* \cdot A_i = 0$  for each  $i$ :

Suppose  $\varepsilon > 0$ . By strong monotonicity,

$$x_i^* + (\varepsilon, \dots, \varepsilon) \succ_i x_i^*$$

so

$$(\varepsilon, \dots, \varepsilon) \in A_i$$

so

$$\inf p^* \cdot A_i \leq p^* \cdot (\varepsilon, \dots, \varepsilon)$$

Since  $\varepsilon$  is an arbitrary positive number,  $\inf p^* \cdot A_i$  is less than every positive number, so

$$\inf p^* \cdot A_i \leq 0$$

Since  $\sum_{i=1}^I \inf p^* \cdot A_i \geq 0$ ,

$$\inf p^* \cdot A_i = 0 \quad (i = 1, \dots, I)$$

- Define  $T$  to make  $x_i^*$  affordable at  $p^*$ . We claim that  $T$  is an income transfer and

$$x_i^* \in Q_i(p^*, T)$$

Let

$$T_i = p^* \cdot x_i^* - p^* \cdot \omega_i$$

$$\begin{aligned} \sum_{i=1}^I T_i &= \sum_{i=1}^I (p^* \cdot x_i^* - p^* \cdot \omega_i) \\ &= p^* \cdot \left( \sum_{i=1}^I x_i^* - \sum_{i=1}^I \omega_i \right) \\ &= p^* \cdot (\bar{\omega} - \bar{\omega}) \\ &= 0 \end{aligned}$$

so  $T$  is an income transfer.

$$\begin{aligned} p^* \cdot x_i^* &= p^* \cdot (\omega_i + (x_i^* - \omega_i)) \\ &= p^* \cdot \omega_i + p^* \cdot (x_i^* - \omega_i) \\ &= p^* \cdot \omega_i + T_i \end{aligned}$$

so

$$x_i^* \in B_i(p^*, T)$$

If  $x'_i \succ_i x_i^*$ , then  $x'_i - x_i^* \in A_i$ , so

$$\begin{aligned} p^* \cdot x'_i &= p^* \cdot (x_i^* + (x'_i - x_i^*)) \\ &= p^* \cdot x_i^* + p^* \cdot (x'_i - x_i^*) \\ &\geq p^* \cdot x_i^* + \inf p^* \cdot A_i \\ &= p^* \cdot x_i^* \\ &= p^* \cdot \omega_i + T_i \end{aligned}$$

so

$$x_i^* \in Q(p^*, T)$$

- Use strong monotonicity to show that  $p^* \gg 0$ .

**Lemma 2** *If  $\succeq_i$  is continuous and complete, and  $x \succ_i y$ , then there exists  $\varepsilon > 0$  such that*

$$(B(x, \varepsilon) \cap X_i) \succ_i y$$

**Proof:** If not, we can find  $x_n \rightarrow x$ ,  $x_n \in X_i$ ,  $x_n \not\succeq_i y$ ; by completeness, we have  $y \succeq_i x_n$  for each  $n$ .

Since  $\succeq_i$  is continuous,  $y \succeq_i x$ , so  $x \not\succeq_i y$ , a contradiction which proves the lemma. ■

Since  $p^* \geq 0$  and  $p^* \neq 0$ ,  $p^* > 0$ ; since in addition  $\bar{\omega} \gg 0$ ,  $p^* \cdot \bar{\omega} > 0$ , so

$$p^* \cdot \omega_i + T_i > 0 \text{ for some } i$$

If  $p_\ell^* = 0$  for some  $\ell$  (WLOG  $\ell = 1$ ), let

$$x'_i = x_i^* + (1, 0, \dots, 0)$$

By strong monotonicity,  $x'_i \succ_i x_i^*$ .

$$p^* \cdot x'_i = p^* \cdot x_i^* = p^* \cdot \omega_i + T_i > 0$$

Find  $\ell$  (WLOG  $\ell = 2$ ) such that

$$p_\ell^* > 0, \quad x'_{2i} > 0$$

Since  $x'_i \succ_i x_i^*$ , let  $\varepsilon > 0$  be chosen to satisfy the conclusion of the Lemma. If necessary, we may make  $\varepsilon$  smaller to ensure that  $\varepsilon \leq 2x'_{2i}$ . Let

$$x''_i = x'_i - (0, \varepsilon/2, 0, \dots, 0)$$

Since  $X_i = \mathbf{R}_+^L$ ,  $x''_i \in X_i$ , so by the Lemma,  $x''_i \succ_i x_i^*$ . But  $p^* \cdot x''_i < p^* \cdot x'_i = p^* \cdot \omega_i + T_i$ , which shows that  $x_i^* \notin Q_i(p^*, T)$ , a contradiction which proves that  $p^* \gg 0$ .

• Show

$$p^* \gg 0 \Rightarrow Q_i(p^*, T) = D_i(p^*, T)$$

– Case 1:  $p^* \cdot \omega_i + T_i = 0$ . Since  $p^* \gg 0$ ,  $B_i(p^*, T) = \{0\}$ , so

$$Q_i(p^*, T) = D_i(p^*, T) = \{0\}$$

– Case 2:  $p^* \cdot \omega_i + T_i > 0$

Suppose  $x \in Q_i(p^*, T)$  but  $x \notin D_i(p^*, T)$ . Then there exists  $z \succ_i x$  such that  $z \in B_i(p^*, T)$ ,

hence  $p^* \cdot z \leq p^* \cdot \omega_i + T_i$ . Since  $x \in Q_i(p^*, T)$ ,  $p^* \cdot z \geq p^* \cdot \omega_i + T_i$ , so

$$p^* \cdot z = p^* \cdot \omega_i + T_i > 0$$

By Lemma 2, there exists  $\varepsilon > 0$  such that

$$|z' - z| < \varepsilon, z' \in \mathbf{R}_+^L \Rightarrow z' \succ x$$

Let

$$z' = z \left( 1 - \frac{\varepsilon}{2|z|} \right)$$

Since  $z \in \mathbf{R}_+^L$ ,  $z' \in \mathbf{R}_+^L$ .

$$|z' - z| = \left| \frac{\varepsilon z}{2|z|} \right| = \frac{\varepsilon}{2} < \varepsilon$$

so  $z' \succ x$ .

$$\begin{aligned} p^* \cdot z' &= p^* \cdot z \left( 1 - \frac{\varepsilon}{2|z|} \right) \\ &= (p^* \cdot \omega_i + T_i) \left( 1 - \frac{\varepsilon}{2|z|} \right) \\ &< p^* \cdot \omega_i + T_i \end{aligned}$$

which contradicts the assumption that  $x \in Q_i(p^*, T)$ . This shows  $Q_i(p^*, T) \subset D_i(p^*, T)$ ; since clearly  $D_i(p^*, T) \subset Q_i(p^*, T)$ ,  $Q_i(p^*, T) = D_i(p^*, T)$ .

### What if preferences are not convex?

- Second Welfare Theorem may fail if preferences are nonconvex.
- Diagram gives an economy with two goods and two agents, and a Pareto optimum  $x^*$  so that so that the utility levels of  $x^*$  cannot be approximated by a Walrasian Equilibrium with Transfers.
- If  $p^*$  is the price which locally supports  $x^*$ , and  $T$  is the income transfer which makes  $x$  affordable with respect to the prices  $p^*$ , there is a unique Walrasian equilibrium with transfers  $(z^*, q^*, T)$ ;  $z^*$  is much more favorable to agent I and much less favorable to agent II than  $x^*$  is.

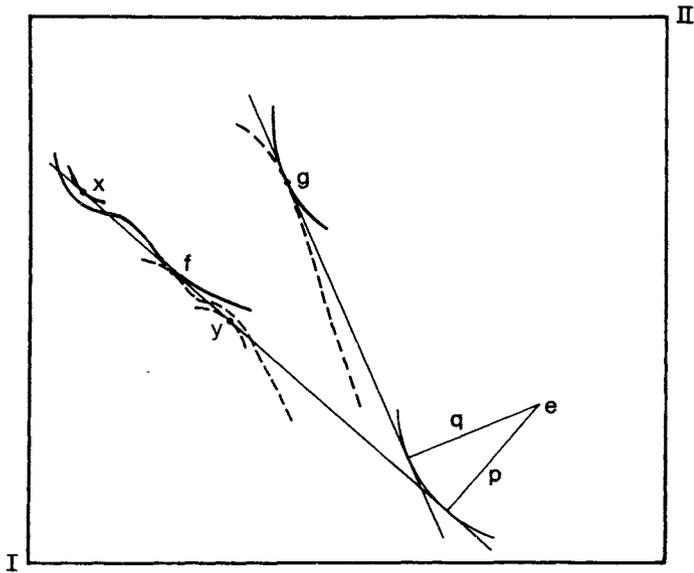


FIGURE 1.

there were an approximate Walrasian allocation  $g$ , it would have the property that  $f(a) \neq_a g(a)$  for all  $a$  (observe that  $f(a)$  is in the budget set). However, as in the convex case, allowing the government to dictate  $f$  as the initial allocation destroys the interpretation of the Second Welfare Theorem as a story of decentralized allocation.

In Theorem 3.3, we show that the government can achieve the utility levels desired for all but  $k$  agents, where  $k$  is the dimension of the commodity space. In other words, the pathology illustrated in Figure 1 disappears (at least for most agents) provided that the number of agents is large relative to the number of commodities. The proof is elementary, relying primarily on the Shapley-Folkman Theorem. We focus on a particular choice of decentralizing price  $\bar{p}$ ; this price is used by Mas-Colell in the proof of his theorem, and is closely related to the so-called gap-minimizing price studied in Anderson (1987); essentially,  $\bar{p}$  is the price which minimizes the measure by which support fails in Mas-Colell's Theorem. Given any Pareto optimum  $f$ , there is an income transfer  $t$  and a quasiequilibrium  $\tilde{f}$  with respect to  $t$  such that all but  $k$  agents are indifferent between  $f$  and  $\tilde{f}$ . If preferences are monotone and a mild assumption on the distribution of goods at  $f$  is satisfied, then we may show that  $\bar{p}$  is strictly positive, and hence  $\tilde{f}$  is a Walrasian equilibrium with respect to  $t$ . As an alternative, we can achieve an approximate equilibrium (i.e., total excess demand is bounded, independent of the number of agents)  $\hat{f}$  such that all agents are indifferent between  $f$  and  $\hat{f}$ . It is worth emphasizing that Theorem 3.3 is a universal theorem, applying to all exchange economics, rather than a generic theorem. However, there is no guarantee that  $\tilde{f}(a)$  is close to  $f(a)$  for any  $a$ . A

- This is the worst that can happen under standard assumptions on preferences. Given a Pareto optimum  $x^*$ , there is a Walrasian quasiequilibrium with transfers  $(z^*, p^*, T)$  such that all but  $L$  people are indifferent between  $x^*$  and  $z^*$ . Those  $L$  people are treated quite harshly (they get zero consumption). One could be less harsh and give these  $L$  people carefully chosen consumption bundles in the convex hull of their quasidemand sets, *but one would then have to forbid them from trading*, a prohibition that would in practice be difficult to enforce.