Economics 201B–Second Half

Lecture 8, 4/8/10

Existence of Walrasian Equilibrium (Continued)

Proposition 1 (17.C.1) Debreu-Gale-Kuhn-Nikaido Lemma *Suppose* $z : \Delta^0 \to \mathbf{R}^L$ *is a function satisfying*

1. *continuity*
2. *Walras' Law*
$$\forall_{p \in \Delta^0} \; p \cdot z(p) = 0$$
3. *bounded below:*
$$\exists_{x \in \mathbf{R}^L} \forall_{p \in \Delta^0} \; z(p) \geq x$$
4. *Boundary Condition: If* $p_n \to p$ *where* $p \in \Delta \setminus \Delta^0$, *then*
$$|z(p_n)| \to \infty$$

Then there exists $p^* \in \Delta^0$ *such that*
$$z(p^*) = 0$$

Outline of proof:

- Define a correspondence $f : \Delta^0 \to \Delta$ (so $f(p) \in 2^{\Delta}$) by
$$f(p) = \{q \in \Delta : q \cdot z(p) \geq q' \cdot z(p) \text{ for all } q' \in \Delta\}$$
f identifies the goods in highest excess demand.
- Extend the domain of f to Δ to get a compact domain, in such a way that f has closed graph. The extension is designed so that there can't be any fixed points in $\Delta \setminus \Delta^0$.
- Verify that if $p^* \in f(p^*)$, then $p^* \in \Delta^0$ and $z(p^*) = 0$.

- Check that f satisfies the hypotheses of Kakutani's Theorem.

- By Kakutani's Theorem, there exists $p^* \in \Delta$ such that $p^* \in f(p^*)$, so $p^* \in \Delta^0$ and $z(p^*) = 0$.

Details of proof:

- Define a correspondence $f : \Delta^0 \to \Delta$ (so $f(p) \in 2^\Delta$) by

$$f(p) = \{q \in \Delta : q \cdot z(p) \geq q' \cdot z(p) \text{ for all } q' \in \Delta\}$$

  f identifies the goods in highest excess demand.

$$\forall_{\ell \neq \ell_0}\ z(p)_{\ell_0} > z(p)_\ell \Rightarrow$$

$$f(p) = \{(0, \ldots, 0, \underset{\substack{\uparrow \\ \ell_0}}{1}, 0, \ldots, 0)\}$$

$$z(p)_{\ell_0} = z(p)_{\ell_1} > z(p)_\ell \text{ for all } \ell \notin \{\ell_0, \ell_1\} \Rightarrow$$

$$f(p) = \{(0, \ldots, 0, \underset{\substack{\uparrow \\ \ell_0}}{\alpha}, 0, \ldots, 0, \underset{\substack{\uparrow \\ \ell_1}}{\mathbf{1} - \alpha}, 0, \ldots, 0) : \alpha \in [0, 1]\}$$

  Notice that $f(p) \cap \Delta^0 = \emptyset$ unless $z(p)_1 = z(p)_x = \cdots = z(p)_L$, but if that happens, $z(p) = 0$ by Walras' Law. Notice also that if p_ℓ is close to 1, then the other prices are small and the boundary condition should tell us that there is some ℓ' such that $z(p)_{\ell'} > z(p)_\ell$, so $q \in f(p) \Rightarrow q_\ell = 0$; if p_ℓ is close to zero and all the other prices are far from zero, then ℓ should be the good in highest excess demand, so $q \in f(p) \Rightarrow q_\ell = 1$; this tells us heuristically that fixed points shouldn't be close to the boundary of Δ^0.

- Extend the domain of f to Δ to make it have closed graph. For $p \in \Delta \setminus \Delta^0$, let

$$\begin{aligned} f(p) &= \{q \in \Delta : p \cdot q = 0\} \\ &= \{q \in \Delta : p_\ell > 0 \Rightarrow q_\ell = 0\} \end{aligned}$$

We will verify f has closed graph on Δ in the fourth step.

- Verify that if $p^* \in f(p^*)$, then $p^* \in \Delta^0$ and $z(p^*) = 0$.

  - We claim that

$$p^* \in \Delta^0$$

If $p^* \in \Delta \setminus \Delta^0$, then

$$\begin{aligned}
& \forall_{q \in f(p^*)}\ p^* \cdot q = 0 \text{ (definition of } f) \\
\Rightarrow\ & p^* \cdot p^* = 0 \text{ (since } p^* \in f(p^*)) \\
\Rightarrow\ & p^* = 0 \\
t \ \Rightarrow\ & p^* \notin \Delta
\end{aligned}$$

contradiction. Therefore,

$$p^* \in \Delta^0$$

  - We claim that

$$p^* \in f(p^*),\, p^* \in \Delta^0 \Rightarrow z(p^*) = 0$$

We can't have $z(p^*) < 0$, for then $p^* \cdot z(p^*) < 0$, contradicting Walras' Law. Fix $\ell \in \{1, \ldots, L\}$

Let

$$\begin{aligned}
e_\ell \quad = \quad & (0, \ldots, 0, \underset{\substack{\uparrow \\ \ell}}{1}, 0, \ldots, 0)\} \\
z(p^*)_\ell \quad = \quad & e_\ell \cdot z(p^*) \\
\leq \quad & p^* \cdot z(p^*) \ (p^* \in f(p^*),\ \text{definition of } f) \\
= \quad & 0 \text{ (Walras' Law)}
\end{aligned}$$

Therefore, $z(p^*) \leq 0$ but $z(p^*) \not< 0$, so

$$z(p^*) = 0$$

- Check that f satisfies the hypotheses of Kakutani's Theorem.

  - Δ is a compact convex nonempty subset of $\mathbf{R}^L$.

  - $f : \Delta \to \Delta$ is

    * *nonempty-valued*: If $p \in \Delta^0$,

$$f(p) = \{q \in \Delta : \forall_{q' \in \Delta}\ q \cdot z(p) \geq q' \cdot z(p)\}$$

      $q \cdot z(p)$ is a continuous function of $q \in \Delta$, which is compact, so the function achieves its maximum, so $f(p) \neq \emptyset$.

      If $p \in \Delta \setminus \Delta^0$,

$$f(p) = \{q \in \Delta : q \cdot p = 0\}$$

      Since $p \in \Delta \setminus \Delta^0$, $p_\ell = 0$ for some ℓ, so if we let

$$q = (0, \ldots, 0, \underset{\substack{\uparrow \\ \ell}}{1}, 0, \ldots 0)$$

      then $q \in \Delta$ and $q \cdot p = 0$, so $f(p) \neq \emptyset$.

  - *convex-valued*: Suppose $q, \hat{q} \in f(p)$, $\alpha \in (0, 1)$. Since Δ is convex,

$$\alpha q + (1 - \alpha)\hat{q} \in \Delta$$

    If $p \in \Delta^0$, and $q' \in \Delta$,

$$\begin{aligned}
(\alpha q + (1 - \alpha)\hat{q}) \cdot z(p) &= \alpha q \cdot z(p) + (1 - \alpha)\hat{q} \cdot z(p) \\
&\geq \alpha q' \cdot z(p) + (1 - \alpha) q' \cdot z(p) \\
&\quad \text{(definition of } f;\ q, \hat{q} \in f(p)) \\
&= q' \cdot z(p)
\end{aligned}$$

    so

$$\alpha q + (1 - \alpha)\hat{q} \in f(p)$$

If $p \in \Delta \setminus \Delta^0$,

$$\begin{aligned}(\alpha q + (1-\alpha)\hat{q}) \cdot p &= \alpha q \cdot p + (1-\alpha)\hat{q} \cdot p \\ &= \alpha 0 + (1-\alpha)0 \\ &\quad \text{(definition of } f;\ q, \hat{q} \in f(p)) \\ &= 0\end{aligned}$$

so

$$\alpha q + (1-\alpha)\hat{q} \in f(p)$$

– *upper hemicontinuous*: By Theorem 3 in Lecture 7, since Δ is compact, it is enough to show that f has closed graph. Suppose $p_n \to p$, $q_n \in f(p_n)$, and $q_n \to q$. We need to show that

$$q \in f(p)$$

If $p \in \Delta^0$, then $p_n \in \Delta^0$ for n sufficiently large, so

$$f(p_n) = \{q \in \Delta : \forall_{q' \in \Delta}\ q \cdot z(p_n) \geq q' \cdot z(p_n)\}$$

z is continuous on Δ^0, so

$$z(p_n) \to z(p)$$

Suppose $q' \in \Delta$.

$$\begin{aligned}q' \cdot z(p) &= q' \cdot \lim_{n\to\infty} z(p_n) \\ &= \lim_{n\to\infty} q' \cdot z(p_n) \\ &\leq \lim_{n\to\infty} q_n \cdot z(p_n) \\ &= \lim_{n\to\infty} q_n \cdot \lim_{n\to\infty} z(p_n) \\ &= q \cdot z(p)\end{aligned}$$

so

$$q \in f(p)$$

If $p \in \Delta \setminus \Delta^0$, may have $p_n \in \Delta^0$ for some n and $p_n \in \Delta \setminus \Delta^0$ for other n. We are in one or both of the following cases; we show that in each case, $p \cdot q = 0$, and hence $q \in f(p)$.

* *Case 1*: $\{n : p_n \in \Delta^0\}$ is infinite. Then there is a subsequence p_{n_k} such that $p_{n_k} \in \Delta^0$ for all k. We need to show that $p \cdot q = 0$. Suppose $p_{\ell_0} > 0$; let $\alpha = \frac{p_{\ell_0}}{2}$. For k sufficiently large,

$$(p_{n_k})_{\ell_0} \geq \alpha$$

  $|z(p_{n_k})| \to \infty$, and $z(p_{n_k})$ is bounded below, so

$$\exists_{\ell_{n_k} \in \{1,\ldots,L\}} \; z(p_{n_k})_{\ell_{n_k}} \to \infty$$

  In the following, x is the x in the statement of the Lemma:

$$\begin{aligned}
(p_{n_k})_{\ell_0} z(p_{n_k})_{\ell_0} &= p_{n_k} \cdot z(p_{n_k}) - \sum_{\ell \neq \ell_0} (p_{n_k})_\ell z(p_{n_k})_\ell \\
&= -\sum_{\ell \neq \ell_0} (p_{n_k})_\ell z(p_{n_k})_\ell \\
&\leq \|x\|_\infty \\
z(p_{n_k})_{\ell_0} &\leq \frac{\|x\|_\infty}{\alpha}
\end{aligned}$$

  so for k sufficiently large,

$$\begin{aligned}
z(p_{n_k})_{\ell_0} < z(p_{n_k})_{\ell_{n_k}} &\Rightarrow (q_{n_k})_{\ell_0} = 0 \\
&\Rightarrow q_{\ell_0} = 0
\end{aligned}$$

  Therefore,

$$p_{\ell_0} > 0 \Rightarrow q_{\ell_0} = 0$$

  so $q \cdot p = 0$ and $q \in f(p)$.

* *Case II*: $\{n : p_n \in \Delta \setminus \Delta^0\}$ is infinite, so there is a subsequence p_{n_k} such that $p_{n_k} \in \Delta \setminus \Delta^0$ for all k. Then $q_{n_k} \cdot p_{n_k} = 0$ for all k, so

$$\begin{aligned} q \cdot p &= \left(\lim_{k\to\infty} q_{n_k}\right) \cdot \left(\lim_{k\to\infty} p_{n_k}\right) \\ &= \lim_{k\to\infty} q_{n_k} \cdot p_{n_k} \\ &= \lim_{k\to\infty} 0 \\ &= 0 \end{aligned}$$

so

$$q \in f(p)$$

- By Kakutani's Theorem, there exists $p^* \in \Delta$ such that $p^* \in f(p^*)$, so $p^* \in \Delta^0$ and $z(p^*) = 0$.■

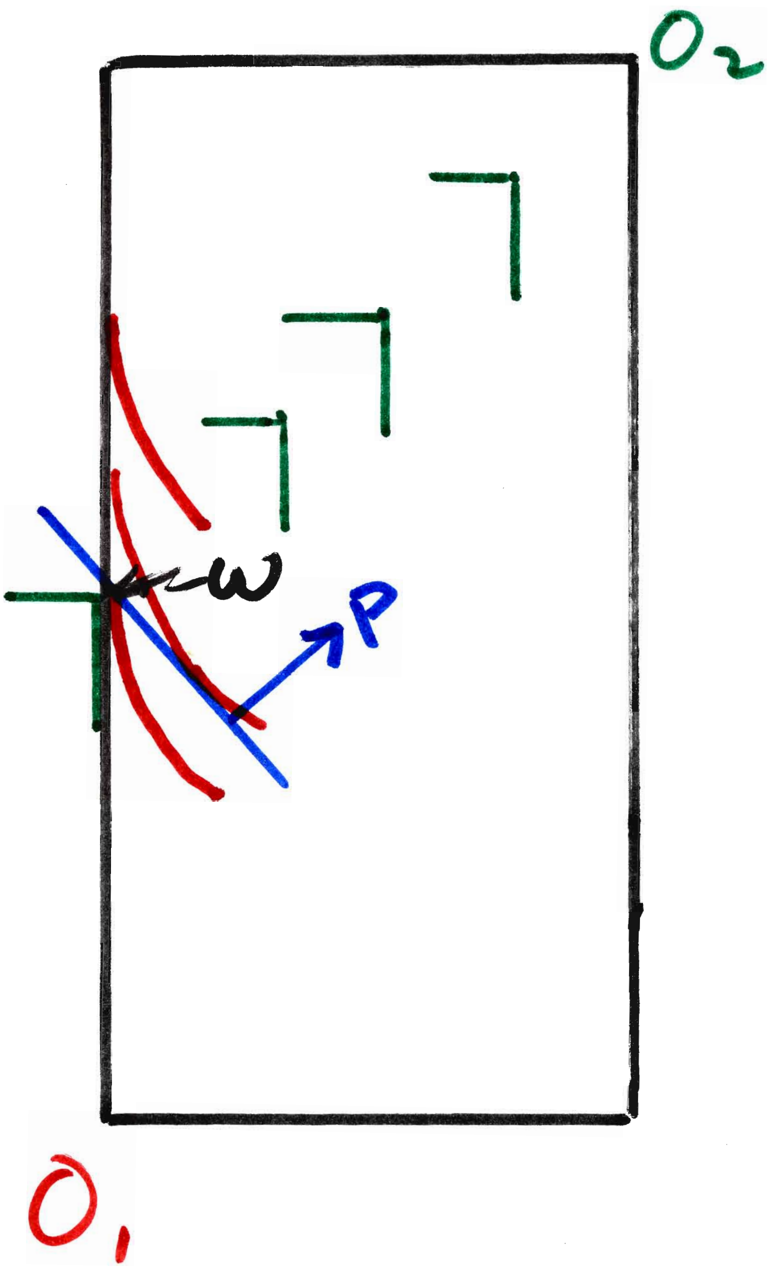
Existence of Walrasian Equilibrium (Wrap-Up)

- What happens if we weaken the strong monotonicity assumption?
  - local nonsatiation implies Walras' Law holds with equality, but is not sufficient to give Walrasian Equilibrium with $\sum_{i=1}^{I} x_i^* \leq \bar{\omega}$.
    * In Edgeworth Box Economy, let

$$\begin{aligned} u_1(x,y) &= y + \sqrt{x} \text{ (strongly monotonic)} \\ \omega_1 &= (0,1) \\ u_2(x,y) &= \min\{x,y\} \text{ (weakly monotonic)} \\ \omega_2 &= (1,1) \end{aligned}$$

For any $p \gg 0$,

$$\begin{aligned} D_2(p) = (1,1) &= \omega_2 \\ D_1(p)_1 &> \omega_{11} \end{aligned}$$



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For $p=(1,0)$ or $p=(0,1)$,

$$D_1(p)=\emptyset$$

But notice for $p=(1,0)$

$$\begin{aligned}\omega_1 &\in Q_1(p)\\ \omega_2 &\in Q_2(p)\end{aligned}$$

so $(1,0)$ is a Walrasian Quasi-Equilibrium Price.

- Even without local nonsatiation,

  $$\exists_{p^*\in\Delta,\, x_i^*\in Q_i(p^*)} \sum_{i=1}^{I} x_i^* \leq \bar{\omega}$$

  Walrasian Quasi-Equilibrium exists, some goods may be left over; local nonsatiation does not imply allocation is exact, since some prices may be zero.

- If *one* agent (WLOG agent 1) is strongly monotonic and $\omega_1 \gg 0$, then $p^* \gg 0$, so

  $$\begin{aligned}x_i^* &\in D_i(p^*)\ (i=1,\ldots,I)\\ \sum_{i=1}^{I} x_i^* &\leq \bar{\omega}\end{aligned}$$

  If, in addition, all agents exhibit local nonsatiation,

  $$\sum_{i=1}^{I} x_i^* = \bar{\omega}$$

- If $\omega_i \gg 0$ for all i,

  $$\begin{aligned}p^*\cdot\omega_i &> 0\\ x_i^* &\in D_i(p^*)\\ \sum_{i=1}^{I} x_i^* &\leq \bar{\omega}\end{aligned}$$

  Local nonsatiation need not imply allocation exact, since some prices may be zero.

- With nonconvex preferences or indivisibilities, see Lecture 12.