**Economics 201B–Second Half**

**Lecture 9, 4/13/10**

## Generic Local Uniqueness of Equilibrium

*Comparative Statics*: In what direction does the equilibrium move if the underlying parameters of the economy change?

*A Foundation for Comparative Statics*:

1. *Local Uniqueness*: For every equilibrium price $p^* \in \Delta$, there exists $\delta > 0$ such that there is no equilibrium price $q^* \in \Delta$ such that

$$q^* \neq p^*, \ |q^* - p^*| < \delta$$

2. For a sufficiently small change in the parameters of the economy, the number of equilibria is unchanged and each equilibrium moves

$$\begin{pmatrix} \text{continuously} \\ \text{differentiably} \end{pmatrix}$$

as the parameter changes.

**Remark 1** If local uniqueness fails, lattice-theoretic methods may still allow us to establish comparative statics results. We can look at an equilibrium correspondence and we may be able to say that the *set* of equilibria moves in a particular direction in response to a change in the underlying parameters (Milgrom, Shannon, others).

*Two-Good Economy*: Consider a 2-good economy, normalized prices $p \in \Delta^0$,

$$z(p) = \sum_{i=1}^{I} D_i(p) - \bar{\omega}$$

- Walras' Law with Equality implies that

$$z(p)_1 = 0 \Rightarrow z(p)_2 = 0$$

so we can capture the situation in a diagram in $\mathbf{R}^2$; let

$$\hat{z}(p_1) = z(p_1, 1 - p_1)_1$$

and plot $\hat{z}$ as a function of $p_1$.

- In Diagram I, a small shift in $\hat{z}$ results in a small shift of the equilibrium price; comparative statics are locally meaningful. Notice that

$$\hat{z}(p_1) = 0 \Rightarrow \hat{z}'(p_1) \neq 0$$

$\hat{z}$ cuts cleanly through 0, so we expect an odd number of equilibria.

- In Diagram II, there are two equilibria $p_L^*$ and $p_R^*$.
  - A small shift in $\hat{z}$ results in a small shift in $p_L^*$
  - A small upward shift in $\hat{z}$ causes $p_R^*$ to split in two; one moves left, the other moves right, so no local comparative statics.
  - A small downward shift in $\hat{z}$ cause $p_R^*$ to disappear!
  - Notice that

$$\hat{z}(p_R^*) = 0 \text{ but } \hat{z}'(p_R^*) = 0$$

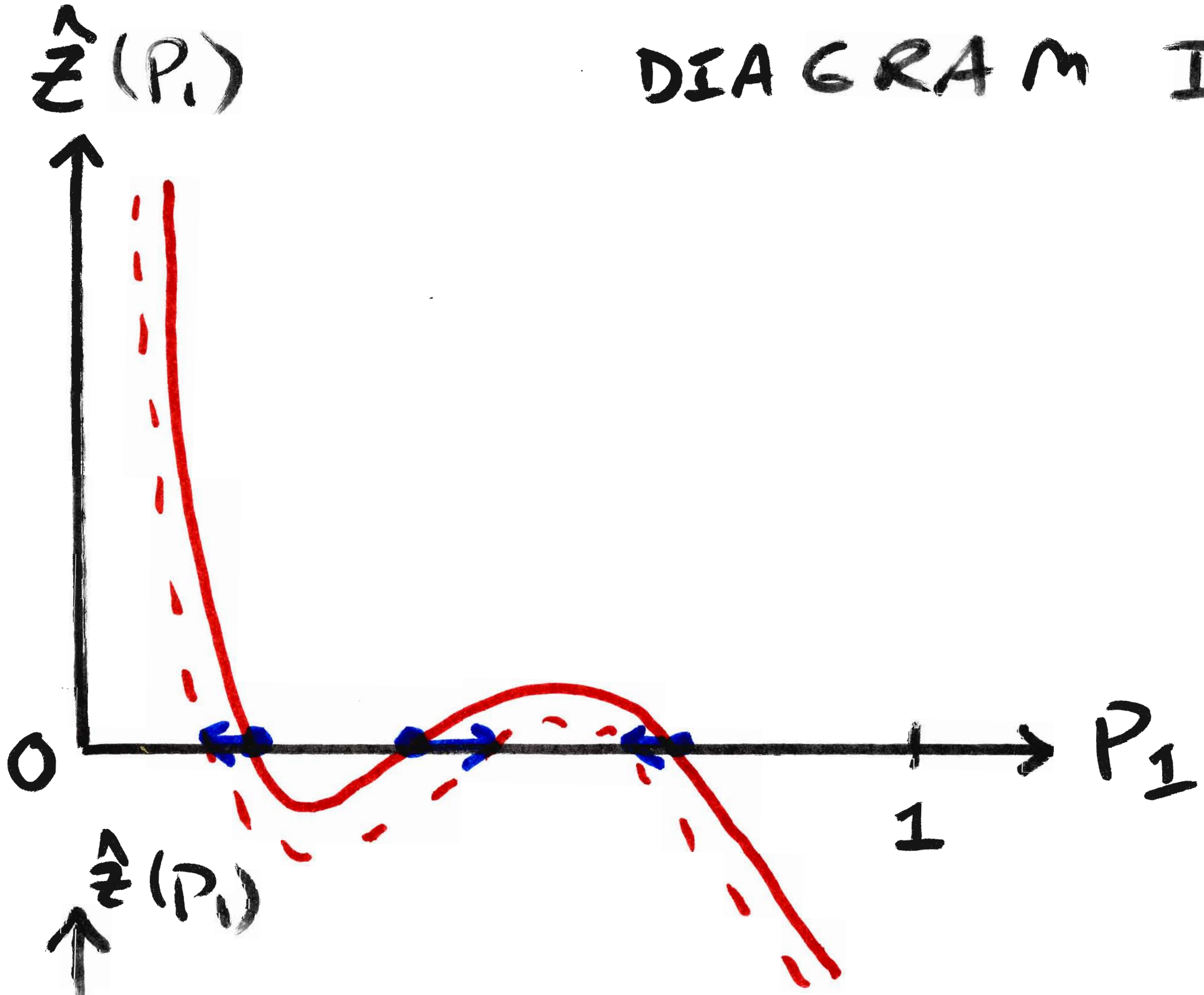
- Diagram III shows we could even have a whole interval of equilibrium prices. A small change in $\hat{z}$ results in a *discontinuous* shift in equilibrium price.

*Multi-Good Case*:

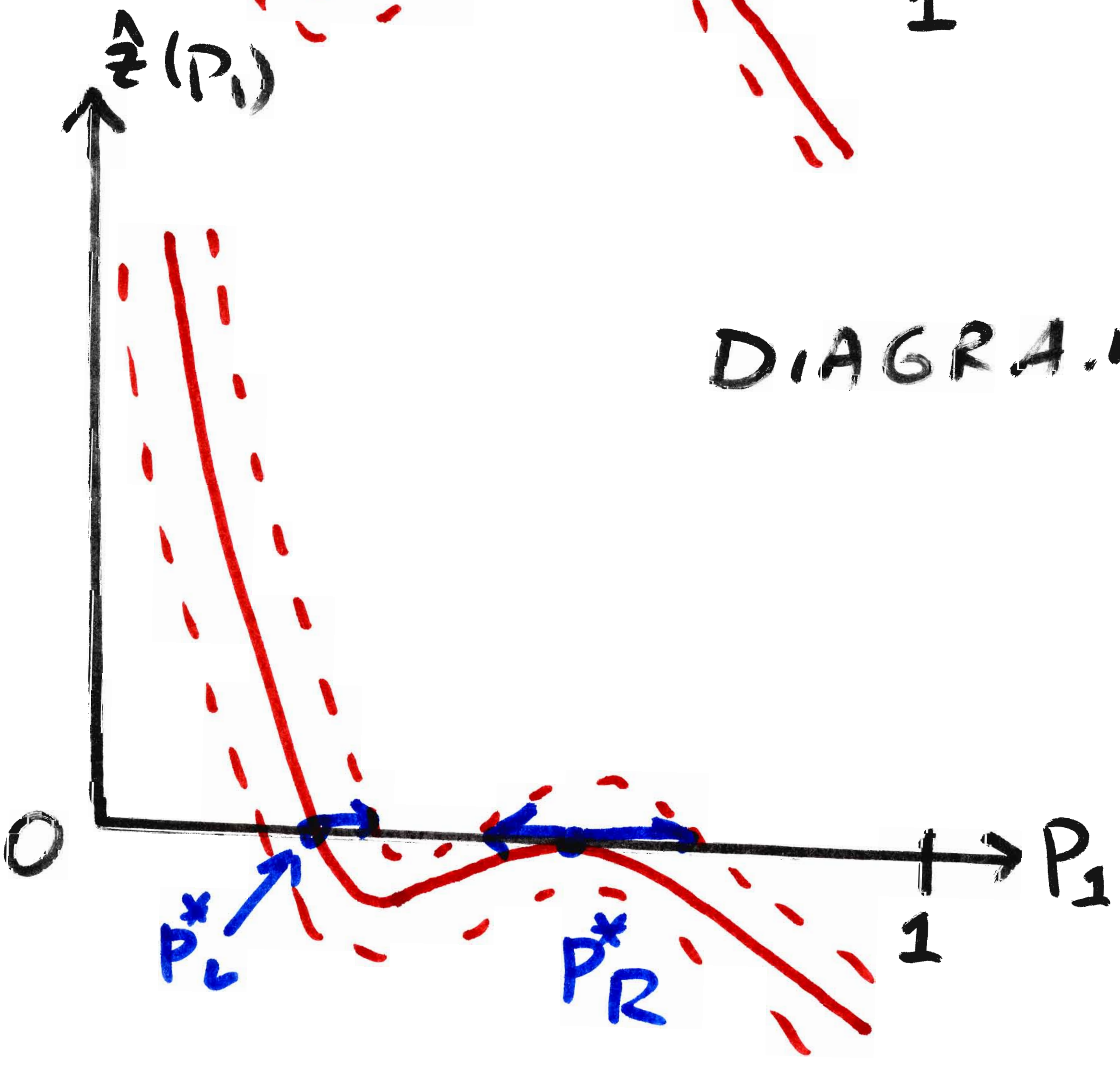
- Normalize $p_L = 1$ rather than $p \in \Delta$. Price is represented by

$$\hat{p} = (p_1, \ldots, p_{L-1}) \in \mathbf{R}^{L-1}_{++}$$

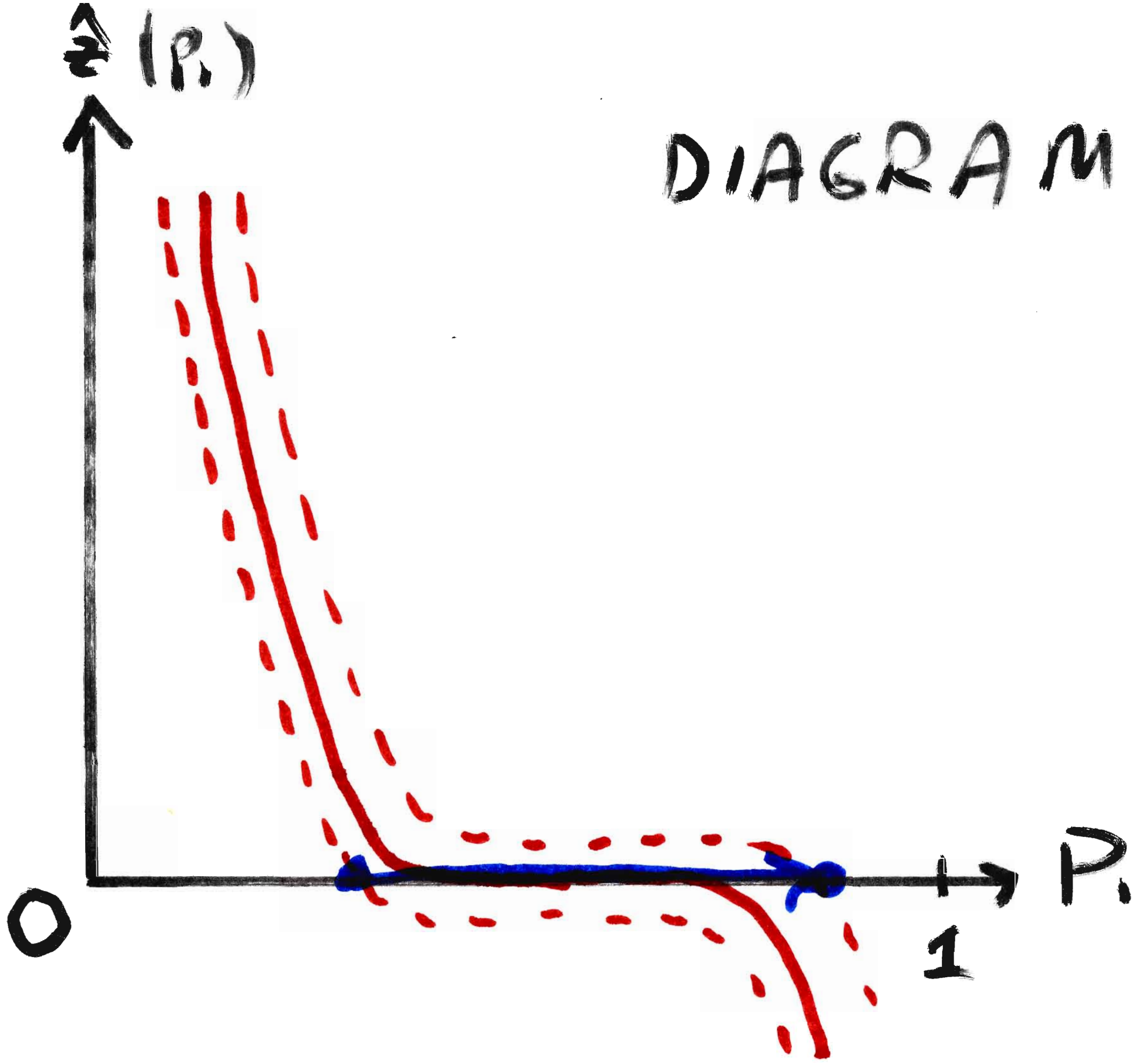
# DIAGRAM I




# DIAGRAM II



# DIAGRAM III



- Let

$$\hat{z}(\hat{p}) = (z_1(\hat{p}, 1), \ldots, z_{L-1}(\hat{p}, 1))$$

- Walras' Law with Equality and $p \gg 0$ implies that

$$\hat{z}(\hat{p}) = 0 \Leftrightarrow z(\hat{p}, 1) = 0$$

- Observe that

$$\hat{z} : \mathbf{R}^{L-1}_{++} \to \mathbf{R}^{L-1}$$

  so $D\hat{z}$, the Jacobian matrix of $\hat{z}$, is $(L-1) \times (L-1)$

- *Definition*: An equilibrium price $p^*$ is *regular* if

$$\det D\hat{z}|_{\hat{p}^*} \neq 0$$

  This is equivalent to

$$\text{rank } \; Dz|_{p^*} = L - 1$$

  A *regular economy* is an economy for which every equilibrium price is regular.

- *Maintained Hypotheses for Remainder of 17.D*:

  - $z$ satisfies the hypotheses of the Debreu-Gale-Kuhn-Nikaido Lemma.

  - $z$ is homogeneous of degree zero, i.e.

$$\forall_{p \in \mathbf{R}^L_{++}, \lambda > 0} z(\lambda p) = z(p)$$

    Caution: $\hat{z}$ is *not* homogeneous because it is a representation of a normalized price ($p_L = 1$).

  - $z$ is $C^1$. This appears technical, but it's strong and has economic consequences because it rules out boundary consumptions: demand necessarily has a kink at the price where demand first hits boundary. This can be weakened to allow boundary consumptions, and the theorems more or less hold.

**Proposition 2 (17.D.1)** *In a regular economy, (normalized) Walrasian Equilibrium prices are locally unique, and there are only finitely many equilibria.*

**Proof:** Suppose $\hat{z}(\hat{p}^*) = 0$. Since the economy is regular, $D\hat{z}|_{\hat{p}^*}$ is nonsingular. By the Inverse Function Theorem, there is a neighborhood $U$ of $\hat{p}^*$ and a neighborhood $V$ of 0 and a $C^1$ function $h : V \to U$, $h$ is one-to-one and onto such that

$$\forall_{v \in V}\ \hat{z}(h(v)) = v,\ \forall_{u \in U}\ h(\hat{z}(u)) = u$$

If $u \in U$ and $\hat{z}(u) = 0$,

$$\begin{aligned} u &= h(\hat{z}(u)) \\ &= h(0) \\ \hat{p}^* &= h(\hat{z}(\hat{p}^*)) \\ &= h(0) \end{aligned}$$

Since $h$ is a function, $u = \hat{p}^*$, so Equilibrium is locally unique.

Now, we show that there are a finite number of equilibria.

*Claim*: There is a compact set $\hat{K} \subset \mathbf{R}^{L-1}_{++}$ such that if $\hat{z}(\hat{p}^*) = 0$, then $\hat{p}^* \in \hat{K}$.

Suppose the claim is not true. Define $\phi : \mathbf{R}^{L-1}_{++} \to \Delta^0$ by

$$\phi(\hat{p}) = \frac{(\hat{p}, 1)}{|\hat{p}_1 + \ldots + \hat{p}_{L-1} + 1|}$$

Observe that $\phi$ is one-to-one and onto, is continuous, and has continuous inverse. Let

$$K_n = \left\{ p \in \Delta^0 : p_\ell \geq \frac{1}{n} \ (1 \leq \ell \leq L) \right\}$$

Then $\phi^{-1}(K_n)$ is the continuous image of a compact set, hence a compact subset of $\mathbf{R}^{L-1}_{++}$, so the set of equilibrium prices cannnot be contained in $\phi^{-1}(K_n)$. Thus, we may find a sequence $\hat{p}^*_n$ of equilibrium prices such that $\phi(\hat{p}^*_n) \notin K_n$, hence

$$\phi(\hat{p}^*_n) \to p \in \Delta \setminus \Delta^0$$

By the Boundary Condition,

$$|z(\phi(\hat{p}^*_n))| \to \infty$$

but

$$\begin{aligned} z(\phi(\hat{p}^*_n)) &= z(\hat{p}^*_n, 1) \text{ (homogeneity of degree zero)} \\ &= 0 \text{ (since } \hat{z}(\hat{p}^*_n) = 0) \end{aligned}$$

a contradiction which proves the claim.

Since $\hat{z}$ is continuous,

$$\begin{aligned} E &= \left\{ \hat{p} \in \mathbf{R}^{L-1}_{++} : \hat{z}(\hat{p}) = 0 \right\} \\ &= \hat{z}^{-1}(\{0\}) \\ &= \mathbf{R}^{L-1}_{++} \setminus \hat{z}^{-1}\left( \mathbf{R}^{L-1} \setminus \{0\} \right) \end{aligned}$$

is closed; since $E$ is a closed subset of the compact set $K$, $E$ is compact. For each $\hat{p} \in E$, we may find $\delta_{\hat{p}} > 0$ such that

$$E \cap B(\hat{p}, \delta_{\hat{p}}) = \{\hat{p}\}$$

The collection

$$\{B(\hat{p}, \delta_{\hat{p}}) : \hat{p} \in E\}$$

is an open cover of $E$, hence has a finite subcover. Since each element of this finite subcover contains exactly one element of $E$, $E$ is finite. ■

**The Index Theorem**

- Throughout, we will denote a price in $\Delta = \{p \in \mathbf{R}^L_+ : \sum_{\ell=1}^L p_\ell = 1\}$ by $p$, and the associated price in $\mathbf{R}^{L-1}_{++}$ (with the assumption that the price of good $L$ has been normalized to 1) by $\hat{p}$.

- *Definition*: If $p^*$ is a regular equilibrium price, define

$$\text{index } (p^*) = (-1)^{L-1} \text{sign det } D\hat{z}|_{\hat{p}^*}$$

- For $L = 2$,

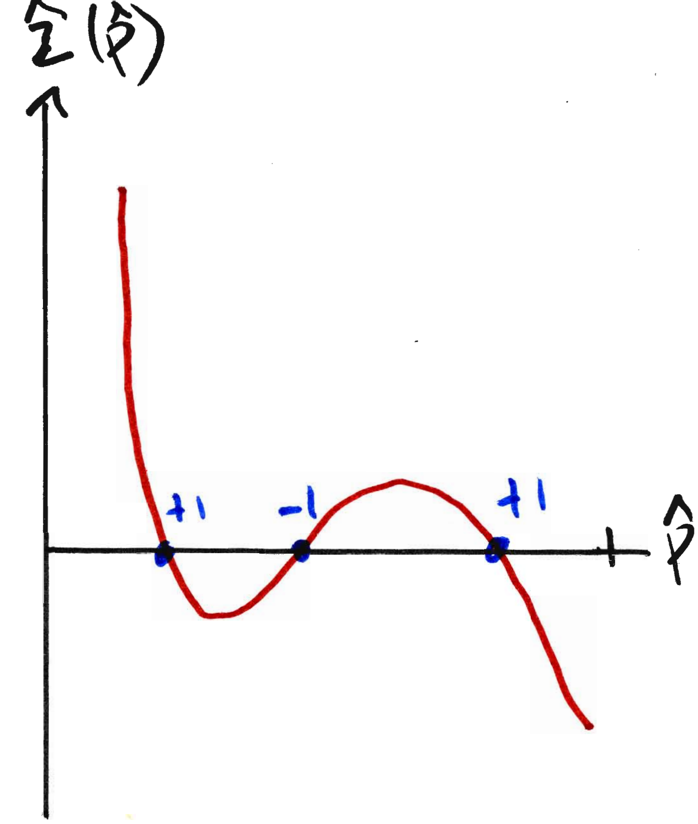
$$\begin{aligned} \text{index } (p^*) &= (-1)^1 \text{sign det } D\hat{z}|_{\hat{p}^*} \\ &= -\text{sign } \hat{z}'(\hat{p}^*) \end{aligned}$$

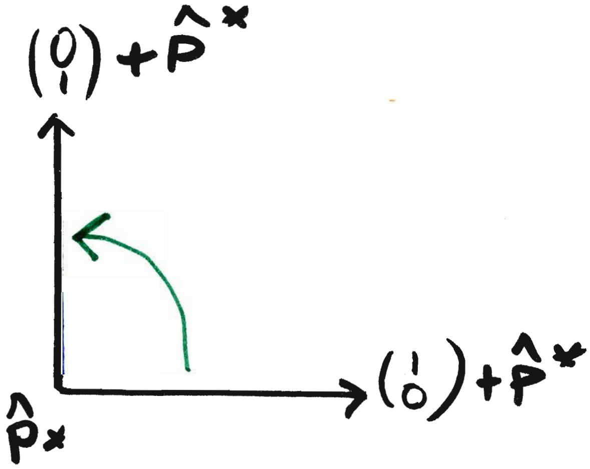
1. index $(p^*) = +1$ means that $\hat{z}'(\hat{p}^*) < 0$; that means demand is downward sloping, so we are in the "normal" case in which an increase in $\hat{z}$, the excess demand for good 1, results in an increase in the equilibrium price of good 1.
2. index $(p^*) = -1$ means that $\hat{z}'(\hat{p}^*) > 0$; that means demand is upward sloping, so we are in the "abnormal" case in which an increase in $\hat{z}$, the excess demand for good 1, results in an decrease in the equilibrium price of good 1.

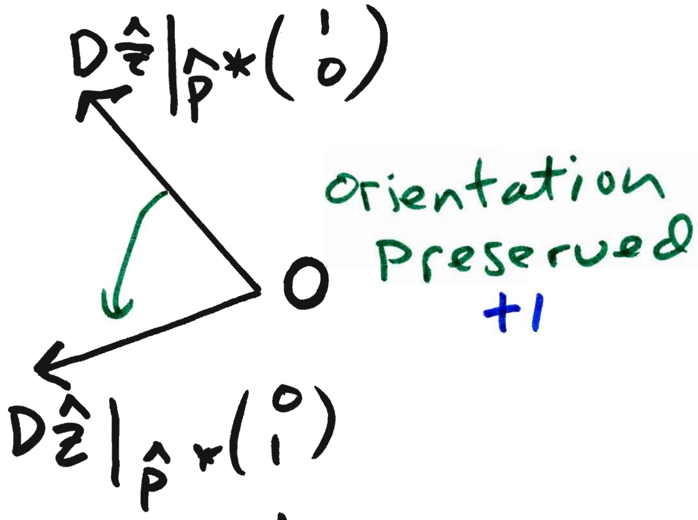
- For $L = 3$,

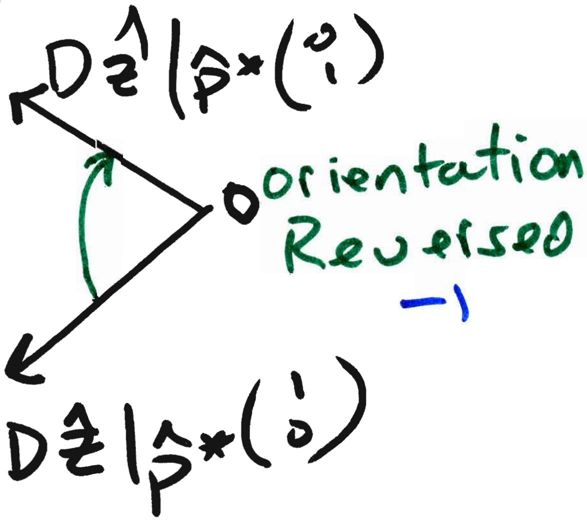
$$\begin{aligned} \text{index } (p^*) &= (-1)^2 \text{sign det } D\hat{z}|_{\hat{p}^*} \\ &= \text{sign det } D\hat{z}|_{\hat{p}^*} \end{aligned}$$

The sign of the determinant is +1 if orientation is preserved, -1 if orientation is reversed:


$\hat{z}(\hat{p})$
+1
-1
+1
$\hat{p}$


$\binom{0}{1} + \hat{p}^*$
$\binom{1}{0} + \hat{p}^*$
$\hat{p}^*$


$D\hat{z}|_{\hat{p}^*}\binom{1}{0}$
$D\hat{z}|_{\hat{p}^*}\binom{0}{1}$
0
orientation
Preserved
+1


$D\hat{z}|_{\hat{p}^*}\binom{0}{1}$
$D\hat{z}|_{\hat{p}^*}\binom{1}{0}$
0
orientation
Reversed
-1

–

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ is obtained from } \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

by a counterclockwise rotation.

$$D\hat{z}|_{\hat{p}^*} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ is obtained from } D\hat{z}|_{\hat{p}^*} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

by a rotation; counterclockwise (orientation preserved) if

$$\det D\hat{z}|_{\hat{p}^*} > 0, \ \text{index}\,(\hat{p}^*) = +1$$

and clockwise (orientation reversed) if

$$\det D\hat{z}|_{\hat{p}^*} < 0, \ \text{index}\,(\hat{p}^*) = -1$$

- For $L = 4$,

$$\begin{aligned} \text{index}\,(p^*) &= (-1)^3 \text{sign } \det D\hat{z}|_{\hat{p}^*} \\ &= -\text{sign } \det D\hat{z}|_{\hat{p}^*} \end{aligned}$$

The sign of the determinant is +1 if orientation is preserved, -1 if orientation is reversed:

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$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

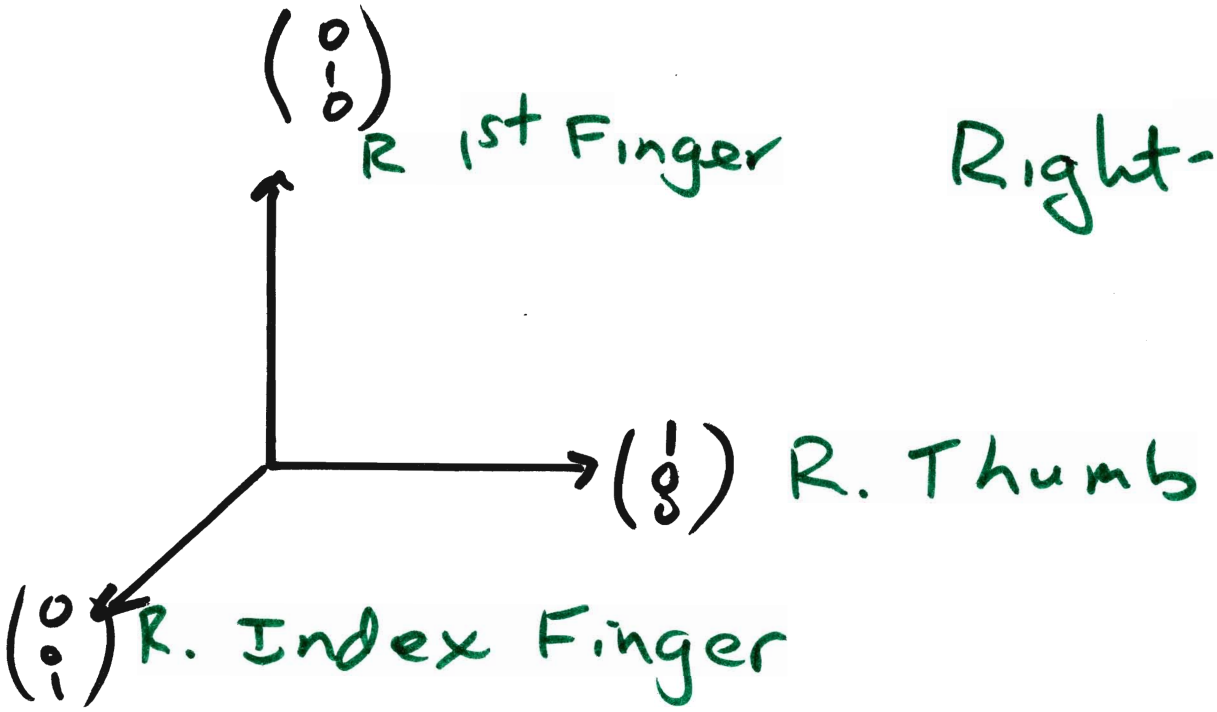
is a right-handed system.

$$\det D\hat{z}|_{\hat{p}^*} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \det D\hat{z}|_{\hat{p}^*} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \det D\hat{z}|_{\hat{p}^*} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

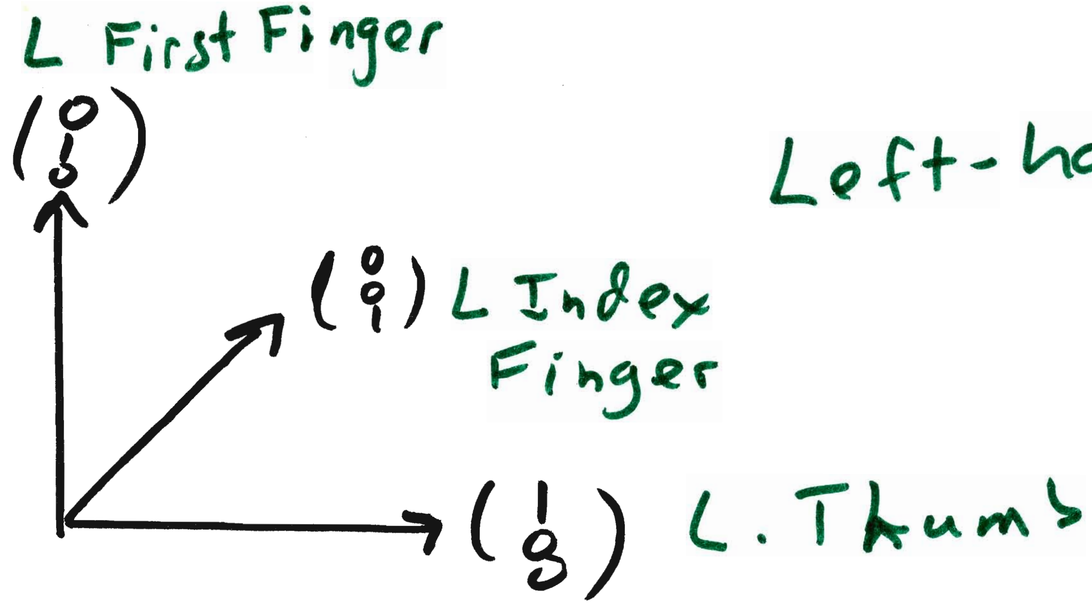
is right-handed (orientation preserved) if

$$\det D\hat{z}|_{\hat{p}^*} > 0, \ \text{index}\,(\hat{p}^*) = -1$$

Right-handed




Left-handed



and left-handed (orientation reversed) if

$$\det D\hat{z}|_{\hat{p}^*} < 0, \text{ index } (\hat{p}^*) = +1$$

- *Connection to Tatonnement Stability*: Consider the Tatonnement Price Dynamics

$$\frac{d\hat{p}}{dt} = \hat{z}(\hat{p}) \tag{1}$$

  - This is a nonlinear differential equation, but we can approximate its behavior near an equilibrium price $\hat{p}^*$ by considering the linear differential equation

$$\frac{d\hat{p}}{dt} = D\hat{z}|_{\hat{p}^*} (\hat{p} - \hat{p}^*) \tag{2}$$

  - Let $\lambda_1, \ldots, \lambda_{L-1}$ be the eigenvalues of $D\hat{z}|_{\hat{p}^*}$.
  - *Fact*:

$$\det D\hat{z}|_{\hat{p}^*} = \prod_{\ell=1}^{L-1} \lambda_\ell$$

    This is obvious if the matrix is diagonalizable, but is true in general.

    * Some of the eigenvalues are real; the others come in conjugate pairs.
    * If $a + bi$ and $a - bi$ are a conjugate pair of eigenvalues

$$(a + bi)(a - bi) = a^2 + b^2 > 0$$

    * Thus,

$$\text{sign } \det D\hat{z}|_{\hat{p}^*} = \prod_{\lambda_\ell \in \mathbf{R}} \text{sign } (\lambda_\ell)$$

      is the product of the signs of the real eigenvalues.
    * Each complex eigenvalue represents a rotation, which does not change orientation.

    * Each real, negative eigenvalue represents a change of orientation. Orientation is unchanged if there are an even number of real, negative eigenvalues.

- Equation (1) is locally asymptotically stable near $\hat{p}^*$ if all solutions to Equation (2) converge to $\hat{p}^*$, which is true if and only if

$$\Re(\lambda_1) < 0, \ldots, \Re(\lambda_{L-1}) < 0$$

  If $\Re(\lambda_\ell) > 0$ for any $\ell$, then Equation (1) is not locally asymptotically stable.

  Suppose $L-1$ is odd. Since there are an even number of complex eigenvalues, there are an odd number of real eigenvalues, so if all of them are negative, the determinant is negative and

$$\text{index }(\hat{p}^*) = (-1)^{L-1}\text{sign det } D\hat{z}|_{\hat{p}^*} = +1$$

  On the other hand, suppose $L-1$ is even. Since there are an even number of complex eigenvalues, there are an even number of real eigenvalues, so if all of them are negative, the determinant is positive and

$$\text{index }(\hat{p}^*) = (-1)^{L-1}\text{sign det } D\hat{z}|_{\hat{p}^*} = +1$$

  Thus, we have

$$\text{Tatonnement Stability near } \hat{p}^* \Rightarrow \text{index }(\hat{p}^*) = +1$$

  but the converse is false. Thus, the Index Theorem lets us quickly determine that some equilibria are unstable, and allow us to concentrate a search for stable equilibria on those with index +1, which *might* be stable.

**Theorem 3 (Index Theorem)** *For any regular economy,*

$$\sum_{\hat{p}^* \in \mathbf{R}^{L-1}_{++}, \hat{z}(\hat{p}^*)=0} \text{index}\,(\hat{p}^*) = +1$$

**Corollary 4** *For any regular economy, there are an odd number of equilibria. Since* 0 *is even, every regular economy has an equilibrium.*

*Intuition behind Index Theorem*: index $(\hat{p}^*)$ indicates the direction in which $\hat{z}$ passes through zero near $\hat{p}^*$. The Boundary Condition implies that $\hat{z}$ starts on one side of zero and ends up on the other side of zero, so every equilibrium price with index -1 must be paired with an equilibrium price with index +1, and exactly one equilibrium price with index +1 must be left unpaired.

**Genericity: Almost All Economies are Regular**

*Review notion of Lebesgue measure zero from 204*: This is a natural formulation of the notion that $A$ is a small set:

"If you choose $x \in \mathbf{R}^n$ at random,

the probability that $x \in A$ is zero."

*Regular and Critical Points and Values:*

Suppose $X \subseteq \mathbf{R}^n$ is open. Suppose $f : X \to \mathbf{R}^m$ is differentiable at $x \in X$. Then $df_x \in L(\mathbf{R}^n, \mathbf{R}^m)$, so

$$\text{rank}\,(df_x) \leq \min\{m, n\}$$

- $x$ is a *regular point* of $f$ if rank $(df_x) = \min\{m, n\}$.
- $x$ is a *critical point* of $f$ if rank $(df_x) < \min\{m, n\}$.
- $y$ is a *critical value* of $f$ if there exists $x \in X$, $f(x) = y$, $x$ is a critical point of $y$.

- $y$ is a *regular value* of $f$ if $y$ is not a critical value of $f$ (notice this has the counterintuitive implication that if $y \notin f(X)$, then $y$ is automatically a regular value of $f$).

A function may have many critical points; for example, if a function is constant on an interval, then every element of the interval is a critical point. But it can't have many critical *values*.

**Theorem 5 (2.4, Sard's Theorem)** *Let $X \subseteq \mathbf{R}^n$ be open, $f : X \to \mathbf{R}^m$, $f$ is $C^r$ with $r \geq 1+\max\{0, n-m\}$. Then the set of all critical values of $f$ has Lebesgue measure zero.*

Recall that our definition of critical point differed from de la Fuente's in the case $m > n$. If $m > n$, then every $x \in X$ is critical using de la Fuente's definition, because

$$\text{rank } Df(x) \leq n < m$$

Consequently, every $y \in f(X)$ is a critical value, using de la Fuente's definition. This does not contradict Sard's Theorem, since one can show that $f(X)$ is a set of Lebesgue measure zero when $m > n$ and $f \in C^1$. The Transversality Theorem is a particularly convenient formulation of Sard's Theorem for our purposes:

**Theorem 6 (2.5', Transversality Theorem)** *Let*

$$\begin{aligned} X \subseteq \mathbf{R}^n \textit{ and } \Omega &\subseteq \mathbf{R}^p \textit{ be open} \\ F : X \times \Omega &\to \mathbf{R}^m \in C^r \\ &\quad \textit{with } r \geq 1 + \max\{0, n-m\} \end{aligned}$$

*Suppose that*

$$F(x, \omega) = 0 \Rightarrow DF(x, \omega) \textit{ has rank } m.$$

*Then there is a set $\Omega_0 \subseteq \Omega$ such that $\Omega \setminus \Omega_0$ has Lebesgue measure zero such that*

$$\omega \in \Omega_0,\ F(x,\omega) = 0 \Rightarrow D_x F(x,\omega) \textit{has rank } m$$

*In particular, if $m = n$, and $\omega_0 \in \Omega_0$,*

- *there is a local implicit function*

  $$x^*(\omega)$$

  *characterized by*

  $$F(x^*(\omega),\omega) = 0$$

  *where $x^*$ is a $C^r$ function of $\omega$*

- *the equilibrium correspondence*

  $$\omega \to \{x : F(x,\omega) = 0\}$$

  *is lower hemicontinuous at $\omega_0$.*

*Remark:* If $n < m$, rank $D_x F(x,\omega) \leq \min\{m,n\} = n < m$. Therefore,

$$
\begin{aligned}
& (F(x,\omega) = 0 \Rightarrow DF(x,\omega) \text{ has rank m}) \\
\Rightarrow\ & \text{for all } \omega \text{ except for a set of Lebesgue measure zero} \\
& F(x,\omega) = 0 \text{ has no solution}
\end{aligned}
$$