University of California, Berkeley
Economics 204–First Midterm Test
Tuesday August 25, 2003; 6-9pm
Each question is worth 20% of the total
Please use separate bluebooks for Parts I and II

Part I

1. Prove that
$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

2. Consider the function
$$f(x,y) = e^{x^2-6xy+y^2}$$
Recall that $\frac{d}{dz}e^z = e^z$.
   (a) Compute the first order conditions for a local maximum or minimum of f. Find the unique (x^*, y^*) at which the first order conditions are satisfied.
   (b) Find the second order Taylor series expansion of f at the point (x^*, y^*) determined in part (a); your answer should involve a symmetric matrix A representing the quadratic terms in the expansion.
   (c) Diagonalize the matrix A you found in part (b). Find an orthonormal basis $\{v_1, v_2\}$ of $\mathbf{R}^2$ such that the quadratic terms of the Taylor expansion can be written as
$$g((x^*, y^*) + \gamma_1 v_1 + \gamma_2 v_2) = \lambda_1(\gamma_1)^2 + \lambda_2(\gamma_2)^2$$
   Use this information to determine whether f has a local maximum, a local minimum, or neither, at (x^*, y^*) and to describe the level sets of f near (x^*, y^*).

3. Give a proof that does *not* involve sequential compactness of the following

   Theorem: Let (X, d) and (Y, ρ) be metric spaces, and $f : X \to Y$ a continuous function. If C is a compact set in (X, d), then $f(C)$ is compact in (Y, ρ).

Part II

4. Let $C([0,1])$ denote the set of real-valued, continuous functions from $[0,1]$ to $\mathbf{R}$, and consider the metric

$$d(f,g) = \sup\{|f(t) - g(t)| : t \in [0,1]\}$$

Let

$$X = \{f \in C([0,1]) : \sup\{|f(t)| : t \in [0,1]\} \leq 1\}$$

Show that X is not compact.

5. Consider the parametrized utility function

$$u : \mathbf{R}^2_{++} \times \mathbf{R}^2_{++} \to \mathbf{R}, \quad u(x,\omega) = \omega_1 x_1 + \omega_2 x_2 + x_1 x_2$$

Here, $\mathbf{R}^2_{+} = \{x \in \mathbf{R}^2 : x_1 \geq 0, x_2 \geq 0\}$, $\mathbf{R}^2_{++} = \{x \in \mathbf{R}^2 : x_1 > 0, x_2 > 0\}$, $x = (x_1, x_2)$ denotes a consumption vector in $\mathbf{R}^2_{++}$, $\omega = (\omega_1, \omega_2)$ denotes a parameter vector in $\mathbf{R}^2_{++}$, and $I \in \mathbf{R}_{++}$ denotes income.

(a) Define demand $Z : \mathbf{R}^2_{++} \times \mathbf{R}_{++} \times \mathbf{R}^2_{++} \to \mathbf{R}^2_{+}$ by $Z(p, I, \omega)$ maximizes $u(x, \omega)$ subject to $x \in \mathbf{R}^2_{+}$, $p \cdot x = I$. You may assume without proof that $Z(p, I, \omega)$ is uniquely defined. Assuming that $Z(p, I, \omega) \in \mathbf{R}^2_{++}$, write down the first order conditions for the maximization problem that defines $Z(p, I, \omega)$.

(b) Find a function

$$F : \mathbf{R}^2_{++} \times \mathbf{R}^2_{++} \times \mathbf{R}_{++} \times \mathbf{R}^2_{++} \to \mathbf{R}^2$$

such that if $Z(p, I, \omega) \in \mathbf{R}^2_{++}$, $Z(p, I, \omega)$ satisfies $F(x, p, I, \omega) = 0$, i.e. $F(Z(p, I, \omega), p, I, \omega) = 0$. *Hint:* The first component of F should encode the Lagrange multiplier condition, and the second component should encode the budget constraint, that you found in part (a).

(c) Use the Implicit Function Theorem to show that if $Z(p^*, I^*, \omega^*) \in \mathbf{R}^2_{++}$, then there is an open set U containing (p^*, I^*, ω^*) such that Z is a C^1 function on U. Compute the Jacobian matrix DZ.