Econ 204 – Problem Set 1

Due 11:59pm Friday August 1 on Gradescope

1. Use induction to prove the following statements are true for all $n \in \mathbb{N}$:
   (a) $2^{2n} - 1$ is divisible by 3.
   (b) $1 + \frac{1}{\sqrt{2}} + \ldots + \frac{1}{\sqrt{n}} \leq 2\sqrt{n}$
2. Let $f : \mathbb{R} \to \mathbb{R}$ be defined as follows. For each function, determine whether it is (i) injective and (ii) surjective:
   (a) $f(x) = 2x + 1$
   (b) $f(x) = x^2$
   (c) $f(x) = e^x$
   (d) $f(x) = \dfrac{1}{1 + x^2}$
3. Do there exist sets X and Y such that the function $f : X \to Y$: $f(x) = x^2$ is a bijection? If yes, specify examples and if no, justify your answer.
4. In the following examples, show that the sets A and B are numerically equivalent by finding a specific bijection between the two.
   (a) $A = [0, 1],\ B = [10, 20]$
   (b) $A = [0, 1],\ B = [0, 1)$
   (c) $A = (-1, 1),\ B = \mathbb{R}$
5. Let R_1, R_2, and R_3 be binary relations on the set of real numbers $\mathbb{R}$, defined as follows:
   - R_1: $a \sim b$ if and only if $a = b$
   - R_2: $a \sim b$ if and only if $a \geq b$
   - R_3: $a \sim b$ if and only if $a > b$

   (a) For each relation, determine whether it is:
      i. Reflexive
      ii. Symmetric
      iii. Transitive
      iv. An equivalence relation

   (b) If R_i is an equivalence relation, describe the quotient set $\mathbb{R}/R_i$.
6. In this exercise we will practice working with sets whose elements are sets as well. For this, we will need the following definition:

   Sigma-Algebra: Let Ω be a set and $\mathcal{F} \subseteq 2^\Omega$ be a collection of subsets of Ω. We say that $\mathcal{F}$ is a sigma-algebra if the following properties hold:
   - $\Omega \in \mathcal{F}$
   - If $A \in \mathcal{F}$, then $A^C \in \mathcal{F}$.

- If $\{A_n\}_{n\in\mathbb{N}}$ is a countable collection of sets such that $\forall n \in \mathbb{N}\ A_n \in \mathcal{F}$, then $\cup_{n\in\mathbb{N}} A_n \in \mathcal{F}$.

(a) Prove that if $\mathcal{F}$ is a sigma-algebra and $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$.

(b) Prove that if $\mathcal{F}$ is a sigma-algebra, then $\varnothing \in \mathcal{F}$

(c) Prove that $\{\varnothing, \Omega\}$ is a sigma-algebra. Argue that this is the smallest sigma-algebra over the set Ω.

(d) Prove that 2^Ω is a sigma-algebra. Argue that this is the largest sigma-algebra over the set Ω.

(e) Prove that if $\mathcal{F}_1, \mathcal{F}_2$ are sigma-algebras, then $\mathcal{F}_1 \cap \mathcal{F}_2$ is a sigma-algebra.

(f) Prove that if $\{\mathcal{F}_a\}_{a\in\mathcal{A}}$ is a collection of sigma-algebras, then $\cap_{a\in\mathcal{A}}\mathcal{F}_a$ is a sigma-algebra. (Note that we have made no restriction on the set $\mathcal{A}$.)

(g) Prove or provide a counterexample to the following statement: If $\mathcal{F}_1, \mathcal{F}_2$ are sigma-algebras, then $\mathcal{F}_1 \cup \mathcal{F}_2$ is a sigma-algebra.

(h) Let $\Omega = \{1, 2, 3\}$. List all the possible sigma-algebras over Ω. (There are surprisingly few).

7. Let $X \subseteq \mathbb{R}$. We say that a function $f : X \to \mathbb{R}$ is bounded if its image $f(X) \subseteq \mathbb{R}$ is a bounded set. We then write $\sup_f = \sup f(X)$ and $\inf_f = \inf f(X)$.

   (a) Show that if $f, g : X \to \mathbb{R}$ are bounded, $f + g : X \to \mathbb{R}$ is bounded

   (b) Show that $(f + g)(X) \subseteq f(X) + g(X)$ and provide an example in which the inclusion is strict. [1]

   (c) Show that $\sup_{f+g} \leq \sup_f + \sup_g$ and $\inf_{f+g} \geq \inf_f + \inf_g$

   (d) Provide an example for which the inequalities in the previous item are strict.

   (e) Show that if $f, g : X \to \mathbb{R}$ are bounded, $f \cdot g : X \to \mathbb{R}$ is bounded

   (f) Show that $(f \cdot g)(X) \subseteq f(X) \cdot g(X)$ [2]

   (g) Show that, if f and g are both positive[3], then $\sup_{f\cdot g} \leq \sup_f \cdot \sup_g$ and $\inf_{f\cdot g} \geq \inf_f \cdot \inf_g$

   (h) Provide an example for which the inequalities in the previous item are strict.

   (i) Provide a counterexample for item g) if the functions are not positive.

   (j) Show that if f is positive, $\sup_{f^2} = (\sup_f)^2$

[1]Given $A, B \subseteq \mathbb{R}$ non-empty and bounded, we define $A + B = \{z \in \mathbb{R} | z = x + y, x \in A, y \in B\}$
[2]Given $A, B \subseteq \mathbb{R}$ non-empty and bounded, we define $A \cdot B = \{z \in \mathbb{R} | z = x \cdot y, x \in A, y \in B\}$.
[3]f is positive if $\forall x \in X\ f(x) \geq 0$