Econ 204 – Problem Set 2

Due 11:59pm Tuesday August 5 on Gradescope

1. For each of the following sequences, state the $\liminf_{n\to\infty} x_n$ and $\limsup_{n\to\infty} x_n$ (no proof or justification is required):

   (a) $x_n = \frac{1}{n}$

   (b) $x_n = (-1)^n$

   (c) $x_n = \begin{cases} n, & \text{if } n \text{ is odd} \\ -\sqrt{n}, & \text{if } n \text{ is even} \end{cases}$

   (d) $x_n = \sin(n)$

2. For some metric space (X, d), take any two sets $A, B \subset X$ such that $\text{int}(A) = \text{int}(B) = \emptyset$, and A is closed. Prove that $\text{int}(A \cup B) = \emptyset$.

3. Given $A, B \subseteq \mathbb{R}^n$, we define the sum of these two sets by:

$$A + B = \{a + b \mid a \in A, b \in B\}$$

   Prove or find a counterexample to the following statements (assume the standard metric):

   (a) If either A or B is open, then $A + B$ is open.

   (b) If both A and B are closed, then $A + B$ is closed.

4. For each of the following sequences, find all cluster points (in $\mathbb{R}$ with the standard metric, no justification required):

   (a) $x_n = (-1)^n + \frac{1}{n}$

   (b) A sequence enumerating all rationals in $(0, 1)$

   (c) $x_n = n$ for $n < 10$, and $x_n = 1$ for $n \geq 10$

5. Let (X, d) be a metric space. Assume $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ are uniformly continuous on (X, d) and $(\mathbb{R}, |\cdot|)$, with $|\cdot|$ the standard metric.

   (a) Show that $f+g : X \to \mathbb{R}$ is uniformly continuous, where $(f+g)(x) = f(x)+g(x)$.

   (b) Show that $\max\{f, g\} : X \to \mathbb{R}$ is uniformly continuous, where $\max\{f, g\}(x) = \max\{f(x), g(x)\}$.

   (c) Give a counterexample to the following statement: $f \cdot g : X \to \mathbb{R}$ *is uniformly continuous on* (X, d) *and* $(\mathbb{R}, |\cdot|)$, where $f \cdot g = f(x) \cdot g(x)$.

6. For each of the following sets, find the limit points and isolated points (in $\mathbb{R}$ with the standard metric, no justification required):

   (a) $A = \{1, 2, 3\}$

   (b) $A = \left\{1 - \frac{1}{n} : n \in \mathbb{N}\right\}$

   (c) $A = \mathbb{Q} \cap (0, 1)$

7. A function $f : X \to Y$ is open if $\forall A \subset X$ such that A is open, $f(A)$ is open. Show that any continuous open function from $\mathbb{R}$ into $\mathbb{R}$ is strictly monotonic (assume the standard metric).