Econ 204 – Problem Set 4

GSI - Anna Vakarova

August 7, 2025

Invertibility

Problem 1

Let $T : X \to Y$ be a linear transformation between two finite-dimensional vector spaces with respective bases V and W. Prove that

1. If T is invertible, then $Mtx_{W,V}(T)$ is invertible and $Mtx_{V,W}(T^{-1}) = (Mtx_{W,V}(T))^{-1}$.
2. If $Mtx_{W,V}(T)$ is invertible, then T is invertible and $Mtx_{V,W}(T^{-1}) = (Mtx_{W,V}(T))^{-1}$.

Problem 2

The norm on the space of square matrices $R^{n\times n}$ is defined as follows: for every $A \in R^{n\times n}$

$$||A|| = \sup\{\ ||Ax||_{R^n} : x \in R^n \text{ and } ||x||_{R^n} = 1\}$$

We can also define a metric d on the space of $n \times n$ matrices using this norm:

$$d(A, B) = ||A - B||$$

Take as given that $\det : \mathbb{R}^{n\times n} \to \mathbb{R}$ is continuous. Use the continuity of the determinant to prove that the set of all invertible matrices is an open, dense subset of all square matrices.
Note: if you are not familiar with the norm/metric on the space of square matrices, in this problem you only need to use the following properties

- for any constant $c \in R$, $||cA|| = |c|||A||$
- the norm of the identity matrix I is 1, $||I|| = 1$

Also, to show that the set $S \subset X$ is dense in X, for any $x \in X$, construct a sequence $\{s_n\}_{n\in N}$, $s_n \in S$ for all n, such that $s_n \to x$ in the respective metric.

Invariant Subspaces

Problem 3

Let $T : \mathbb{R}^m \to \mathbb{R}^n$ be a linear transformation between finite-dimensional vector spaces over a field $\mathbb{R}$. Then

1. There is no injective linear transformation $T : \mathbb{R}^m \to \mathbb{R}^n$ if $m > n$.
2. There is no surjective linear transformation $T : \mathbb{R}^m \to \mathbb{R}^n$ if $n > m$.
3. There is an isomorphism (i.e., a bijective linear transformation) $T : \mathbb{R}^m \to \mathbb{R}^n$ if and only if $m = n$.

Problem 4

Let A be an $n \times n$ matrix.

1. Show that if λ is an eigenvalue of A, then λ^k is an eigenvalue of A^k for $k \in \mathbb{N}$.
2. Show that if λ is an eigenvalue of the matrix A and A is invertible, then $1/\lambda$ is an eigenvalue of A^{-1}.
3. Find an expression for $\det(A)$ in terms of the eigenvalues of A.
4. The *eigenspace* of an eigenvalue λ_i of A is the kernel of $A - \lambda_i I$ (all $x \in R^n$ such that $(A - \lambda_i I)x = 0$). Show that the eigenspace of any eigenvalue λ_i of A is a vector subspace of R^n.

Quotient Space

Problem 5

Let X be a normed vector space and V a proper closed subspace. Denote elements of the quotient vector space X/V by $x + V$, with $x \in X$.

1. Show that the quantity $\|x + V\| = \inf_{v \in V} \|x - v\|_X$ is a norm on X/V.
2. Show that for any $\epsilon > 0$ there exists $x \in X$ satisfying $\|x\|_X = 1$ such that $\|x + V\| > 1 - \epsilon$.
3. Show that the natural projection map $\pi : X \to X/V$ has norm equal to 1.
4. (optional, will not be graded) A normed vector space X is complete iff for any sequence of elements x_n satisfying $\sum_{n=1}^{\infty} \|x_n\|_X < \infty$, the series $\sum_{n=1}^{\infty} (x_n)$ converges in X. Use this statement without proving it to show that if X is complete, then so is X/V.

Linear Maps between Normed Spaces

Problem 6

Let X be a normed vector space. Let $T : X \to R$ be a linear map. Prove that T is bounded if and only if $T^{-1}(\{0\})$ is closed.