Econ 204 – Problem Set 6

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1 Fixed points

Problem 1

For a metric space (X, m) where m is the metric function, the distance between a point $a \in X$ and a subset $B \subset X$ is defined by $d(b, A) = \inf_{b \in B} m(a, b)$. Given two nonempty compact subsets $A, B \subseteq \mathbb{R}^n$, the Hausdorff distance between them is

$$D_H(A, B) = \max \left\{ \sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A) \right\}$$

Let $S, T \subset \mathbb{R}^n$, S compact. Define $f : S \to 2^T$ to be a continuous Hausdorff limit if it has a compact graph and there is a sequence $\{f_n\}$ of continuous functions from S to T whose graphs

$$\{(x, y) \mid y = f_n(x)\} \subseteq S \times T$$

converge to the graph of f

$$\{(x, y) \mid f(x) \ni y\} \subseteq S \times T$$

in Hausdorff distance. Show that every continuous Hausdorff limit $f : T \to 2^T$ from a compact convex subset of $\mathbb{R}^n$ has a fixed point.

Problem 2

There are m items and n players. Each player has a utility function $u_i(x_i)$ defined on each subset $x_i \subseteq \{1, 2, \dots, m\}$. Assume $1 \leq u_i(x_i) \leq 2$ for all i and x_i.

There are k allocations, $A(1), A(2), \dots, A(k)$. Allocation $A(j)$ assigns $A_i^{(j)} \subseteq \{1, 2, \dots, m\}$ to player i, where $A_i^{(j)} \cap A_h^{(j)} = \emptyset$ for any $h \neq i$.

A *mixed allocation* is defined as a probability distribution over the possible allocations: $p = (p_1, p_2, \dots, p_k) \in \mathcal{P}$, such that $\sum_j p_j = 1$ and $p_j \geq 0$ for all j. Denote the set of mixed allocations by P.

Given p, player i's expected utility is given by

$$\sum_j p_j u_i(A_i^{(j)}).$$

Fix $\epsilon > 0$. Let W be the set of vector weights

$$W = \left\{ w = (w_1, w_2, \dots, w_n) \;\middle|\; \sum_i w_i = 1 \text{ and } w_i \geq \epsilon \text{ for all } i \right\}.$$

A mapping Φ takes $(p, w) \in P \times W$ and maps it into $\{P(w), \bar{w}(p, w)\} \subset P(w) \times W$, where $P(w) \subset P$ is defined as

$$P(w) = \{p \in P \;\Big|\; p \in \arg\max \sum_{i \in N} w_i \sum_{j \in K} p_j u_i(A_i^{(j)})\}$$

and $\bar{w} : P \times W \to W$ is a continuous function.

1. Let
$$A(w) = \{j \mid A^{(j)} \text{ maximizes } \sum_i w_i u_i(A_i^{(j)}) \text{ over all allocations}\}.$$
Prove that $p' \in P(w)$ iff $p'_j > 0$ implies that $j \in A(w)$.

2. Given a mixed allocation $p \in P$, prove that the set of W such that $p \in P(w)$ is a convex closed set.

3. Prove that for any sequence $(w^t, p^t$ with $\lim_{t \to \infty} w^t = w$ and $\lim_{t \to \infty} p^t = p$, if for every t, $p(t) \in P(w^t)$, then $p \in P(w)$. In other words, $P(w)$ has a closed graph.

4. Conclude that the mapping Φ has a fixed point.

2 Separating Hyperplane Theorem

Problem 3

1. Let A and B be disjoint nonempty convex subsets of $\mathbb{R}^n$ and suppose $p \in R^n$ is a non-zero vector that separates A and B with $p \cdot a \geq p \cdot b$ for all $a \in A, b \in B$. If A includes a set of the form $\{x\} + \mathbb{R}^n_{++}$, then $p > 0$.
Hint: proof by contradiction.

2. Let C be a nonempty convex subset of a vector space, and let $f_1, \dots, f_m : C \to \mathbb{R}$ be concave. Letting $f = (f_1, \dots, f_m) : C \to \mathbb{R}^m$, exactly one of the following is true:

   a
   $$\exists \bar{x} \in C \text{ such that } f(\bar{x}) \gg 0$$

   b
   $$\exists p > 0 \text{ such that } p \cdot f(x) \leq 0 \text{ for all } x \in C$$

3 Differential equations

Problem 4

Solve the following differential equation: $y'' - 5y' + 4y = e^{4x}$. Concretely, provide (i) the general solution of the homogeneous differential equation, and (ii) the particular and general solutions of the inhomogeneous differential equation. Solve explicitly for the constants using the following initial conditions: $y(0) = 3$, $y(0)' = \frac{19}{3}$.