Economics 204 Summer/Fall 2025
Lecture 15–Friday August 15, 2025
Revised to add initial condition on page 9

Second Order Linear Differential Equations

Consider the second order differential equation $y'' = cy + by'$ with $b, c \in \mathbf{R}$.

Rewrite this as a *first order* linear differential equation in two variables:

$$\begin{aligned} \bar{y}(t) &= \begin{pmatrix} y(t) \\ y'(t) \end{pmatrix} \\ \bar{y}'(t) &= \begin{pmatrix} y'(t) \\ y''(t) \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ c & b \end{pmatrix} \begin{pmatrix} y(t) \\ y'(t) \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ c & b \end{pmatrix} \bar{y} \end{aligned}$$

The eigenvalues are $\frac{b \pm \sqrt{b^2+4c}}{2}$, the roots of the equation $\lambda^2 - b\lambda - c = 0$. The qualitative behavior of the solutions can be explicitly described from the coefficients b and c, by determining whether the eigenvalues are real or complex, and whether the real parts are negative, zero, or positive. See Section 6 of the Differential Equations Handout.

Example Consider the second order linear differential equation

$$y'' = 2y + y'$$

As above, let

$$\bar{y} = \begin{pmatrix} y \\ y' \end{pmatrix}$$

so the equation becomes

$$\bar{y}' = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix} \bar{y}$$

The eigenvalues are the roots of the characteristic polynomial

$$\lambda^2 - \lambda - 2 = 0$$

Eigenvalues and corresponding eigenvectors are given by

$$\lambda_1 = 2 \;\; v_1 = (1, 2)$$
$$\lambda_2 = -1 \;\; v_2 = (1, -1)$$

From this information alone, we know the qualitative properties of the solutions are as given in the phase plane diagram (see Figure 1):

- Solutions are roughly hyperbolic in shape with asymptotes along the eigenvectors. Along the eigenvector v_1, the solutions flow off to infinity; along the eigenvector v_2, the solutions converge to zero.
- Solutions flow in directions consistent with flows along asymptotes

- On the y-axis, we have $y' = 0$, which means that everywhere on the y-axis (except at the stationary point 0), the solution must have a vertical tangent.
- On the y'-axis, we have $y = 0$, so we have

$$y'' = 2y + y' = y'$$

  Thus, above the y-axis, $y'' = y' > 0$, so y' is increasing along the direction of the solution; below the y-axis, $y'' = y' < 0$, so y' is decreasing along the direction of the solution.
- Along the line $y' = -2y$, $y'' = 2y - 2y = 0$, so y' achieves a minimum or maximum where it crosses that line.

The general solution is given by

$$\begin{aligned}
\begin{pmatrix} y(t) \\ y'(t) \end{pmatrix} &= Mtx_{U,V}(id) \begin{pmatrix} e^{2(t-t_0)} & 0 \\ 0 & e^{-(t-t_0)} \end{pmatrix} Mtx_{V,U}(id) \begin{pmatrix} y(t_0) \\ y'(t_0) \end{pmatrix} \\
&= \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} e^{2(t-t_0)} & 0 \\ 0 & e^{-(t-t_0)} \end{pmatrix} \begin{pmatrix} 1/3 & 1/3 \\ 2/3 & -1/3 \end{pmatrix} \begin{pmatrix} y(t_0) \\ y'(t_0) \end{pmatrix} \\
&= \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} \frac{e^{2(t-t_0)}}{3} & \frac{e^{2(t-t_0)}}{3} \\ \frac{2e^{-(t-t_0)}}{3} & -\frac{e^{-(t-t_0)}}{3} \end{pmatrix} \begin{pmatrix} y(t_0) \\ y'(t_0) \end{pmatrix} \\
&= \begin{pmatrix} \frac{e^{2(t-t_0)}+2e^{-(t-t_0)}}{3} & \frac{e^{2(t-t_0)}-e^{-(t-t_0)}}{3} \\ \frac{2e^{2(t-t_0)}-2e^{-(t-t_0)}}{3} & \frac{2e^{2(t-t_0)}+e^{-(t-t_0)}}{3} \end{pmatrix} \begin{pmatrix} y(t_0) \\ y'(t_0) \end{pmatrix}
\end{aligned}$$

$$= \begin{pmatrix} \frac{y(t_0)+y'(t_0)}{3} e^{2(t-t_0)} + \frac{2y(t_0)-y'(t_0)}{3} e^{-(t-t_0)} \\ \\ \frac{2y(t_0)+2y'(t_0)}{3} e^{2(t-t_0)} + \frac{-2y(t_0)+y'(t_0)}{3} e^{-(t-t_0)} \end{pmatrix}$$

The general solution has two real degrees of freedom; a specific solution is determined by specifying initial conditions $y(t_0)$ and $y'(t_0)$.

Because

$$\bar{y} = \begin{pmatrix} y \\ y' \end{pmatrix}$$

it is easier to find the general solution by setting

$$y(t) = C_1 e^{2(t-t_0)} + C_2 e^{-(t-t_0)}$$

Then

$$\begin{aligned} y(t_0) &= C_1 + C_2 \\ y'(t) &= 2C_1 e^{2(t-t_0)} - C_2 e^{-(t-t_0)} \\ y'(t_0) &= 2C_1 - C_2 \\ C_1 &= \frac{y(t_0) + y'(t_0)}{3} \\ C_2 &= \frac{2y(t_0) - y'(t_0)}{3} \\ y(t) &= \frac{y(t_0) + y'(t_0)}{3} e^{2(t-t_0)} + \frac{2y(t_0) - y'(t_0)}{3} e^{-(t-t_0)} \end{aligned}$$

Inhomogeneous Linear Differential Equations with Nonconstant Coefficients

Consider the inhomogeneous linear differential equation

$$y' = M(t)y + H(t) \tag{1}$$

where M is continuous function from t to the set of $n \times n$ matrices; and H is continuous function from t to $\mathbf{R}^n$.

There is a close relationship between solutions of the *inhomogeneous* linear differential equation (1) and the associated *homogeneous* linear differential equation

$$y' = M(t)y \tag{2}$$

Theorem 1 *The general solution of the inhomogeneous linear differential equation (1) is*

$$y_h + y_p$$

where y_h is the general solution of the homogeneous linear differential equation (2) and y_p is any particular solution of the inhomogeneous linear differential equation (1).

Proof: Fix any particular solution y_p of inhomogeneous equation (1). Suppose y_h is any solution of the corresponding homogeneous equation (2). Let $y_i(t) = y_h(t) + y_p(t)$.

$$\begin{aligned} y_i'(t) &= y_h'(t) + y_p'(t) \\ &= M(t)y_h(t) + M(t)y_p(t) + H(t) \\ &= M(t)(y_h(t) + y_p(t)) + H(t) \\ &= M(t)y_i(t) + H(t) \end{aligned}$$

so y_i is solution of inhomogeneous equation (1).

Conversely, suppose y_i is any solution of inhomogenous equation (1). Let $y_h(t) = y_i(t) - y_p(t)$.

$$\begin{aligned} y_h'(t) &= y_i'(t) - y_p'(t) \\ &= M(t)y_i(t) + H(t) - M(t)y_p(t) - H(t) \\ &= M(t)(y_i(t) - y_p(t)) \\ &= M(t)y_h(t) \end{aligned}$$

so y_h is solution of homogeneous equation (2) and $y_i = y_h + y_p$. ∎

Remark: To find general solution of inhomogeneous equation:

1. Find general solution of homogeneous equation;
2. Find a particular solution of inhomogeneous equation;
3. Add these to get general solution of inhomogeneous equation

In analogy with how we define e^x for $x \in \mathbf{R}$, for an $n \times n$ matrix M we define

$$e^M = \sum_{k=0}^{\infty} \frac{M^k}{k!} = I + M + \frac{M^2}{2} + \cdots$$

and

$$e^{tM} = \sum_{k=0}^{\infty} \frac{t^k M^k}{k!}$$

The matrix exponential satisfies many properties analogous to the exponential function in **R** and **C**. Here are a few of the most important properties, which can be established fairly directly from the definitions above.

- if D is a diagonal matrix with diagonal $d_1, \ldots, d_n$,

$$e^D = \begin{pmatrix} e^{d_1} & 0 & \cdots & 0 \\ 0 & e^{d_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{d_n} \end{pmatrix}$$

- $e^{A+B} = e^A e^B$ if $AB = BA$ (this is not necessarily valid for matrices A and B that do not commute)
- $e^{P^{-1}AP} = P^{-1}e^A P$
- $g(t) = e^{tM}$ is differentiable (in fact C^∞) and $g'(t) = Me^{tM}$

In particular, notice that if M is diagonalizable, so $M = P^{-1}DP$ for a diagonal matrix D, then

$$e^M = P^{-1}e^D P = P^{-1} \begin{pmatrix} e^{d_1} & 0 & \cdots & 0 \\ 0 & e^{d_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{d_n} \end{pmatrix} P$$

This observation will help tie the form of the general solution we establish below for linear differential equations with constant coefficients to the results established in lecture 14 for the diagonalizable case.

Theorem 2 *Consider the inhomogeneous linear differential equation (1), and suppose that $M(t)$ is a constant matrix M, independent of t. A particular solution of the inhomogeneous linear differential equation (1), satisfying the initial condition $y_p(t_0) = y_0$, is given by*

$$y_p(t) = e^{(t-t_0)M} y_0 + \int_{t_0}^{t} e^{(t-s)M} H(s)\, ds \qquad (3)$$

Proof: We verify that y_p solves (3):

$$\begin{aligned}
y_p(t) &= e^{(t-t_0)M} y_0 + \int_{t_0}^{t} e^{(t-s)M} H(s)\, ds \\
&= e^{(t-t_0)M} y_0 + \int_{t_0}^{t} e^{(t-t_0)M} e^{-(s-t_0)M} H(s)\, ds \\
&= e^{(t-t_0)M} \left(y_0 + \int_{t_0}^{t} e^{-(s-t_0)M} H(s)\, ds \right) \\
y_p'(t) &= M e^{(t-t_0)M} \left(y_0 + \int_{t_0}^{t} e^{-(s-t_0)M} H(s)\, ds \right) \\
&\quad + e^{(t-t_0)M} \left(e^{-(t-t_0)M} H(t) \right) \\
&= M y_p(t) + H(t) \\
y_p(t_0) &= e^{(t_0-t_0)M} y_0 + \int_{t_0}^{t_0} e^{(s-t_0)M} H(s)\, ds \\
&= y_0
\end{aligned}$$

∎

Example Consider the inhomogeneous linear differential equation

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}, \mathbf{y}(\mathbf{0}) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

By Theorem 2, a particular solution is given by

$$\begin{aligned}
&y_p(t) \\
&= e^{(t-t_0)M} y_0 + \int_{t_0}^t e^{(t-s)M} H(s)\, ds \\
&= \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \int_0^t \begin{pmatrix} e^{(t-s)} & 0 \\ 0 & e^{-(t-s)} \end{pmatrix} \begin{pmatrix} \sin s \\ \cos s \end{pmatrix} ds \\
&= \begin{pmatrix} e^t \\ e^{-t} \end{pmatrix} + \int_0^t \begin{pmatrix} e^{t-s} \sin s \\ e^{s-t} \cos s \end{pmatrix} ds \\
&= \begin{pmatrix} e^t \left(1 + \int_0^t e^{-s} \sin s\, ds\right) \\ e^{-t} \left(1 + \int_0^t e^{s} \cos s\, ds\right) \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
&\int_0^t e^{-s} \sin s\, ds \\
&= -e^{-s} \sin s\Big|_0^t - \int_0^t -e^{-s} \cos s\, ds \\
&= -e^{-t} \sin t + e^0 \sin 0 + \int_0^t e^{-s} \cos s\, ds \\
&= -e^{-t} \sin t + -e^{-s} \cos s\Big|_0^t - \int_0^t -e^{-s}(-\sin s)\, ds \\
&= -e^{-t} \sin t + -e^{-t} \cos t + e^0 \cos 0 - \int_0^t e^{-s} \sin s\, ds \\
&= -e^{-t}(\sin t + \cos t) + 1 - \int_0^t e^{-s} \sin s\, ds
\end{aligned}$$

$$\begin{aligned}
&2\int_0^t e^{-s} \sin s\, ds \\
&= -e^{-t}(\sin t + \cos t) + 1
\end{aligned}$$

$$\begin{aligned}
&\int_0^t e^{-s} \sin s\, ds \\
&= \frac{-e^{-t}(\sin t + \cos t) + 1}{2}
\end{aligned}$$

$$
\begin{aligned}
&\int_0^t e^s \cos s\, ds \\
&= e^s \cos s\big|_0^t - \int_0^t e^s(-\sin s)\, ds \\
&= e^t \cos t - e^0 \cos 0 + \int_0^t e^s \sin s\, ds \\
&= e^t \cos t - 1 + e^s \sin s\big|_0^t - \int_0^t e^s \cos s\, ds \\
&= e^t \cos t - 1 + e^t \sin t + e^0 \sin 0 - \int_0^t e^s \cos s\, ds \\
&= e^t(\sin t + \cos t) - 1 - \int_0^t e^s \cos s\, ds
\end{aligned}
$$

$$
\begin{aligned}
2\int_0^t e^s \cos s\, ds &= e^t(\sin t + \cos t) - 1 \\
\int_0^t e^s \cos s\, ds &= \frac{e^t(\sin t + \cos t) - 1}{2}
\end{aligned}
$$

$$
\begin{aligned}
y_p(t) &= \begin{pmatrix} e^t\,(1 + \int_0^t e^{-s} \sin s\, ds) \\ e^{-t}\,(1 + \int_0^t e^s \cos s\, ds) \end{pmatrix} \\
&= \begin{pmatrix} e^t \left(1 + \frac{-e^{-t}(\sin t + \cos t) + 1}{2}\right) \\ e^{-t} \left(1 + \frac{e^t(\sin t + \cos t) - 1}{2}\right) \end{pmatrix} \\
&= \begin{pmatrix} e^t \left(\frac{3 - e^{-t}(\sin t + \cos t)}{2}\right) \\ e^{-t} \left(\frac{1 + e^t(\sin t + \cos t)}{2}\right) \end{pmatrix} \\
&= \begin{pmatrix} \frac{3e^t - \sin t - \cos t)}{2} \\ \frac{e^{-t} + \sin t + \cos t}{2} \end{pmatrix}
\end{aligned}
$$

Thus, the general solution of the original inhomogeneous equation is given by

$$
\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} C_1 e^t \\ C_2 e^{-t} \end{pmatrix} + \begin{pmatrix} \frac{3e^t - \sin t - \cos t}{2} \\ \frac{e^{-t} + \sin t + \cos t}{2} \end{pmatrix}
$$

$$= \begin{pmatrix} D_1 e^t - \frac{\sin t + \cos t}{2} \\ D_2 e^{-t} + \frac{\sin t + \cos t}{2} \end{pmatrix}$$

where D_1 and D_2 are arbitrary real constants.

Nonlinear Differential Equations–Linearization

Nonlinear differential equations are very difficult to solve in closed form. Specific techniques solve special classes of equations. Numerical methods compute numerical solutions of any ordinary differential equation. Linearization provides qualitative information about the solutions of nonlinear autonomous equations. The idea is to find stationary points of the equation, then study solutions of linearized equation near the stationary points. This gives a reasonably reliable guide to behavior of solutions of original nonlinear equation.

Example: Pendulum The equation of motion of a frictionless pendulum is a nonlinear autonomous differential equation

$$y'' = -\alpha^2 \sin y, \ \alpha > 0$$

Here, y is the angle between the pendulum and a vertical line. The fact that the motion follows this differential equation is obtained by resolving the downward force of gravity into two components, one tangent to the curve the pendulum follows and one which is parallel to the pendulum; the latter component is canceled by the pendulum rod.

This has much in common with all cyclical processes, including processes such as business cycles. This equation very difficult to solve exactly because of nonlinearity.

Define

$$\bar{y}(t) = \begin{pmatrix} y(t) \\ y'(t) \end{pmatrix}$$

so differential equation becomes

$$\bar{y}'(t) = \begin{pmatrix} y_2(t) \\ -\alpha^2 \sin y_1(t) \end{pmatrix}$$

Let

$$F(\bar{y}) = \begin{pmatrix} y_2 \\ -\alpha^2 \sin y_1 \end{pmatrix}$$

Solve for stationary points: points $\bar{y}$ such that $F(\bar{y}) = 0$:

$$\begin{aligned} F(\bar{y}) = 0 \ &\Rightarrow \ \begin{pmatrix} y_2 \\ -\alpha^2 \sin y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ &\Rightarrow \ \sin y_1 = 0 \text{ and } y_2 = 0 \\ &\Rightarrow \ y_1 = n\pi \text{ and } y_2 = 0 \end{aligned}$$

so set of stationary points is

$$\{(n\pi, 0) : n \in \mathbf{Z}\}$$

Linearize the equation around each of the stationary points: take the first order Taylor polynomial for F:

$$
\begin{aligned}
F(n\pi + h, 0 + k) + o(|h| + |k|) & \\
= \; F(n\pi, 0) + \begin{pmatrix} \frac{\partial F_1}{\partial y_1} & \frac{\partial F_1}{\partial y_2} \\ & \\ \frac{\partial F_2}{\partial y_1} & \frac{\partial F_2}{\partial y_2} \end{pmatrix} \begin{pmatrix} h \\ \\ k \end{pmatrix} & \\
= \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -\alpha^2 \cos n\pi & 0 \end{pmatrix} \begin{pmatrix} h \\ k \end{pmatrix} & \\
= \begin{pmatrix} 0 & 1 \\ (-1)^{n+1}\alpha^2 & 0 \end{pmatrix} \begin{pmatrix} h \\ k \end{pmatrix} &
\end{aligned}
$$

- For n even, the eigenvalues are solutions to

$$
\begin{aligned}
\lambda^2 + \alpha^2 &= 0 \\
\text{so } \lambda_1 &= i\alpha, \; \lambda_2 = -i\alpha
\end{aligned}
$$

  Close to $(n\pi, 0)$ for n even, the solutions spiral around the stationary point. For $y_2 = y_1' > 0$, y_1 is increasing, so the solutions move in a clockwise direction.

- For n odd, the eigenvalues solve $\lambda^2 - \alpha^2 = 0$, so the eigenvalues and eigenvectors are

$$
\begin{aligned}
\lambda_1 &= \alpha, \; \lambda_2 = -\alpha \\
v_1 &= (1, \alpha), \; v_2 = (1, -\alpha)
\end{aligned}
$$

  Close to $(n\pi, 0)$ for n odd, the solutions are roughly hyperbolic in shape; along v_2, they converge to the stationary point, while along v_1, they diverge from the stationary point. The solutions of the linearized equation tend to infinity along v_1. The stationary point $(n\pi, 0)$ with n odd corresponds to the pendulum pointing vertically upwards.

- From this information alone, we know the qualitative properties of the solutions of the linearized equation are as given in the phase plane diagram in Figure 2; the solutions of the original equation will closely follow these near the stable points:
  - On the y-axis, we have $y' = 0$, which means that everywhere on the y-axis (except at the stationary points), the solution must have a vertical tangent.
  - Solve $y'' = -\alpha^2 \sin y = 0$, so $y = n\pi$; thus, at $y = n\pi$, the derivative of y' is zero, so the tangent to the curve is horizontal.
- If the initial value of $|y_2|$ is sufficiently large, the solutions of the original equation no follow longer closed curves; this corresponds to the pendulum going "over the top" rather than oscillating back and forth.

Nonlinear Differential Equations–Stability

Linearization provides information about qualitative properties of solutions of nonlinear differential equations near the stationary points. Suppose y_s is a stationary point:

- If eigenvalues of linearized equation at y_s all have strictly negative real parts, there exists $\varepsilon > 0$ such that, if $|y(0) - y_s| < \varepsilon$, then $\lim_{t\to\infty} y(t) = y_s$; all solutions of the original nonlinear equation which start sufficiently close to the stationary point y_s converge to y_s.
- If eigenvalues of the linearized equation at y_s all have strictly positive real parts, no solution of the original nonlinear equation converges to y_s.
- If eigenvalues of the linearized equation at y_s all have real part zero, then the solutions of linearized equation are closed curves around y_s. This tells us little about the solutions of nonlinear equation. They may
  - follow closed curves around y_s
  - converge to y_s
  - converge to a limit closed curve around y_s
  - diverge from y_s
  - converge to y_s along certain directions and diverge from y_s along other directions.

Determining Behavior of Solutions when Eigenvalues have Real Part Zero

Example Consider the initial value problem

$$\begin{pmatrix} y_1'(t) \\ y_2'(t) \end{pmatrix} = \begin{pmatrix} -9y_2(t) + 4y_1^3(t) + 4y_1(t)y_2^2(t) \\ 4y_1(t) + 9y_1^2(t)y_2(t) + 9y_2^3(t) \end{pmatrix},$$

$$y_1(0) = 3,\ y_2(0) = 0 \tag{4}$$

$y_s = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is a stationary point. Linearization around y_s is

$$y'(t) = \begin{pmatrix} 0 & -9 \\ 4 & 0 \end{pmatrix} y$$

The characteristic equation is $\lambda^2 + 36 = 0$, so the matrix has distinct eigenvalues $\lambda_1 = 6i$ and $\lambda_2 = -6i$; since both have real part zero, we know the solutions of the linearized differential equation follows closed curves around zero. Eigenvectors are

$$v_1 = \begin{pmatrix} 3i/2 \\ 1 \end{pmatrix} \text{ and } v_2 = \begin{pmatrix} -3i/2 \\ 1 \end{pmatrix}$$

so change of basis matrices are

$$Mtx_{U,V}(id) = \begin{pmatrix} 3i/2 & -3i/2 \\ 1 & 1 \end{pmatrix} \text{ and } Mtx_{V,U}(id) = \begin{pmatrix} -i/3 & 1/2 \\ i/3 & 1/2 \end{pmatrix}$$

Then the solution of the linearized initial value problem is

$$y = \begin{pmatrix} 3i/2 & -3i/2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e^{6ti} & 0 \\ 0 & e^{-6ti} \end{pmatrix} \begin{pmatrix} -i/3 & 1/2 \\ i/3 & 1/2 \end{pmatrix} \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$

$$
\begin{aligned}
&= \begin{pmatrix} 3i/2 & -3i/2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -ie^{6ti}/3 & e^{6ti}/2 \\ ie^{-6ti}/3 & e^{-6ti}/2 \end{pmatrix} \begin{pmatrix} 3 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} (e^{6ti}+e^{-6ti})/2 & (e^{6ti}-e^{-6ti})3i/4 \\ (e^{-6ti}-e^{6ti})i/3 & (e^{6ti}+e^{-6ti})/2 \end{pmatrix} \begin{pmatrix} 3 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} \cos 6t & -3(\sin 6t)/2 \\ 2(\sin 6t)/3 & \cos 6t \end{pmatrix} \begin{pmatrix} 3 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 3\cos 6t \\ 2\sin 6t \end{pmatrix}
\end{aligned}
$$

since

$$
\begin{aligned}
e^{6ti} + e^{-6ti} &= \cos 6t + i\sin 6t + \cos(-6t) + i\sin(-6t) \\
&= \cos 6t + i\sin 6t + \cos 6t - i\sin 6t \\
&= 2\cos 6t \\
e^{6ti} - e^{-6ti} &= \cos 6t + i\sin 6t - \cos(-6t) - i\sin(-6t) \\
&= \cos 6t + i\sin 6t - \cos 6t + i\sin 6t \\
&= 2i\sin 6t
\end{aligned}
$$

Notice that

$$
\begin{aligned}
\frac{y_1^2(t)}{9} + \frac{y_2^2(t)}{4} &= \frac{9\cos^2 6t}{9} + \frac{4\sin^2 6t}{4} \\
&= \cos^2 6t + \sin^2 6t \\
&= 1
\end{aligned}
$$

so the solution of the linearized initial value problem is a closed curve running counterclockwise around the ellipse with principal axes along the y_1 and y_2 axes, of length 3 and 2 respectively.

Let

$$G(y) = \frac{y_1^2}{9} + \frac{y_2^2}{4}$$

and compute $\frac{dG(y(t))}{dt}$:

$$\begin{aligned}
\frac{dG(y(t))}{dt} &= \begin{pmatrix} \frac{\partial G}{\partial y_1} & \frac{\partial G}{\partial y_2} \end{pmatrix} \begin{pmatrix} y_1'(t) \\ y_2'(t) \end{pmatrix} \\
&= \begin{pmatrix} \frac{2y_1(t)}{9} & \frac{y_2(t)}{2} \end{pmatrix} \begin{pmatrix} -9y_2(t) + 4y_1^3(t) + 4y_1(t)y_2^2(t) \\ 4y_1(t) + 9y_1^2(t)y_2(t) + 9y_2^3(t) \end{pmatrix} \\
&= -2y_1(t)y_2(t) + 8y_1^4(t)/9 + 8y_1^2(t)y_2^2(t)/9 \\
&\quad +2y_1(t)y_2(t) + 9y_1^2(t)y_2^2(t)/2 + 9y_2^4(t)/2 \\
&= 8y_1^4(t)/9 + 97y_1^2(t)y_2^2(t)/18 + 9y_2^4(t)/2 \\
&> 0
\end{aligned}$$

- $y'(t)$ is tangent to the solution at every t, and $y'(t)$ always points outside the level curve of G through $y(t)$, as in Figure 4.
- Solution of initial value problem (4) spirals outward, always moving to higher level curves of G.
- For $G(y) \geq 1$ (i.e., outside the ellipse which the solution of the linearized initial value problem follows), easy to see that

$$8y_1^4(t)/9+97y_1^2(t)y_2^2(t)/18+9y_2^4(t)/2 > \frac{8}{9}\left(y_1^2(t) + y_2^2(t)\right)^2$$

so $\frac{dG(y(t))}{dt}$ is uniformly bounded away from zero, so $G(y(t)) = G(y(0)) + \int_0^t \frac{dG(y(s))}{ds} ds \to \infty$ as $t \to \infty$.

- Linear terms become dwarfed by the higher order terms, which will determine whether the solution continues to spiral as it heads off into the distance.

Consider instead the initial value problem

$$\begin{pmatrix} y_1'(t) \\ y_2'(t) \end{pmatrix} = \begin{pmatrix} -9y_2(t) - 4y_1^3(t) - 4y_1(t)y_2^2(t) \\ 4y_1(t) - 9y_1^2(t)y_2(t) - 9y_2^3(t) \end{pmatrix},$$
$$y_1(0) = 3, \; y_2(0) = 0 \tag{5}$$

The linearized initial value problem has not changed. As before, compute

$$\begin{aligned}
\frac{dG(y(t))}{dt} &= \begin{pmatrix} \frac{\partial G}{\partial y_1} & \frac{\partial G}{\partial y_2} \end{pmatrix} \begin{pmatrix} y_1'(t) \\ y_2'(t) \end{pmatrix} \\
&= \begin{pmatrix} \frac{2y_1(t)}{9} & \frac{y_2(t)}{2} \end{pmatrix} \begin{pmatrix} -9y_2(t) - 4y_1^3(t) - 4y_1(t)y_2^2(t) \\ 4y_1(t) - 9y_1^2(t)y_2(t) - 9y_2^2(t) \end{pmatrix} \\
&= -2y_1(t)y_2(t) - 8y_1^4(t)/9 - 8y_1^2(t)y_2^2(t)/9 \\
&\quad +2y_1(t)y_2(t) - 9y_1^2(t)y_2^2(t)/2 - 9y_2^4(t)/2 \\
&= -8y_1^4(t)/9 - 97y_1^2(t)y_2^2(t)/18 - 9y_2^4(t)/2 \\
&< 0
\end{aligned}$$

- $y'(t)$ is tangent to the solution at every t, and $y'(t)$ always points inside the level curve of G through $y(t)$, as in Figure 4.

- the solution of initial value problem (5) spirals inward, always moving to lower level curves of G.
- **Claim**: $y(t) \to \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ as $t \to \infty$.
  - Note $\frac{dG(y(t))}{dt} < 0$ except at origin, so for all $C > 0$,

    $$\alpha = \sup\left\{\frac{dG(y(t))}{dt} : C \leq G(y(t)) \leq G(y(0))\right\} < 0$$

    since $\{y : C \leq G(y) \leq G(y(0))\}$ is compact.
  - If $G(y(t)) \geq C$ for all t,

    $$\begin{aligned} G(y(t)) &= G(y(0)) + \int_0^t \frac{dG(y(s))}{ds}\, ds \\ &\leq G(y(0)) + \alpha t \\ &\to -\infty \text{ as } t \to \infty \end{aligned}$$

    contradiction.
  - Thus, $G(y(t)) \to 0$ and the solution of initial value problem (5) converges to stationary point $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ as $t \to \infty$.

In initial value problems (4) and (5), we were lucky to some extent.

- We took G to be function whose level sets are the solutions of the linearized differential equation, and found tangent to the solution always pointed outside the level curve in (4) and always pointed inside the level curve in (5).

- It is not hard to construct examples in which tangent points outward at some points and inward at others, so the value $G(y(t))$ is not monotonic.
  - May be able to show by calculation that $G(y(t)) \to \infty$, so the solution disappears off into the distance
  - May be able to show by calculation that $G(y(t)) \to 0$, so the solution converges to the stationary point.
  - Alternative method is to choose a *different* function G, whose level sets are not solutions of linearized equation, but for which one can prove that $\frac{dG(y(t))}{dt}$ is always positive or always negative; this is called Liapunov's Second Method.

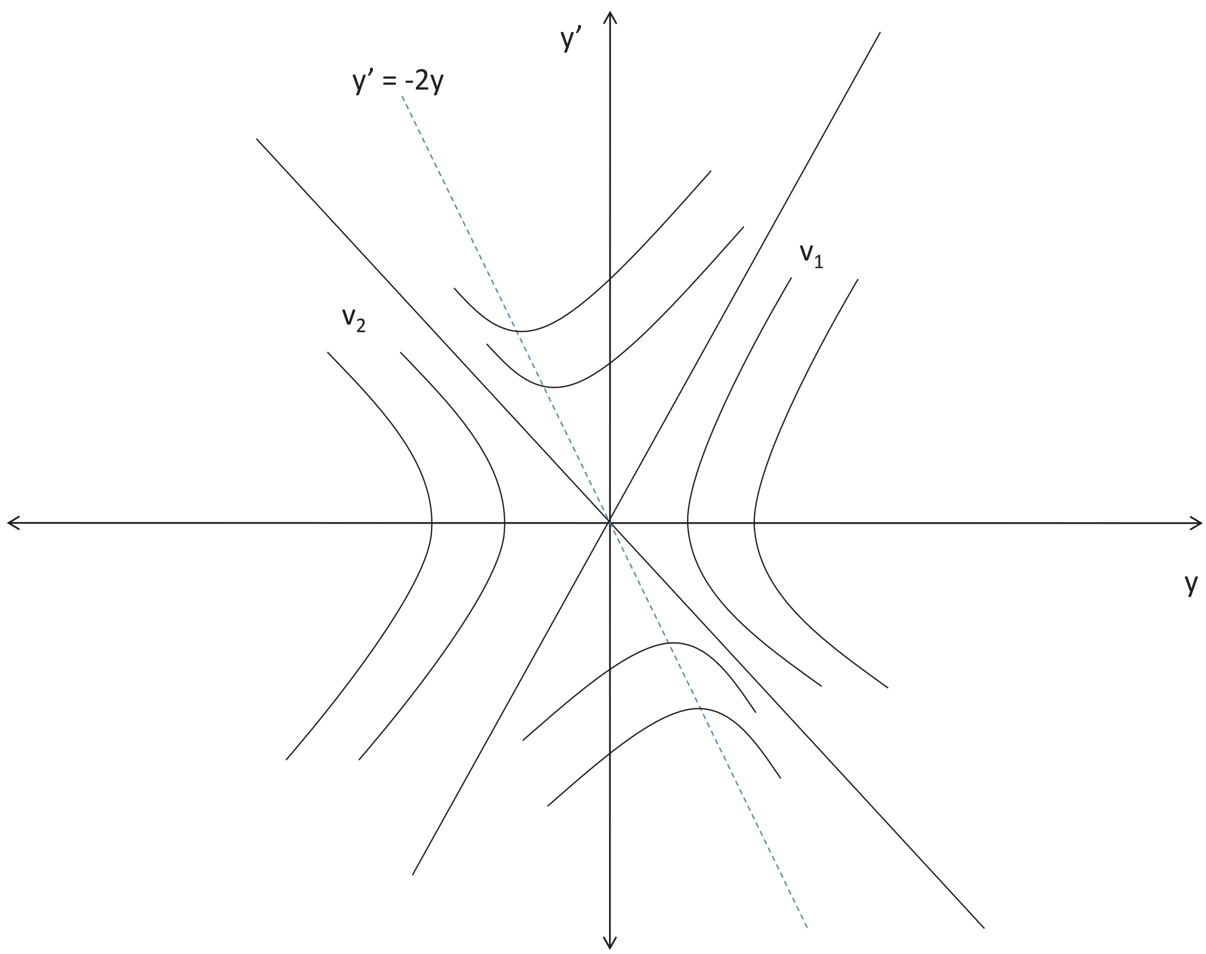

Figure 1: Phase plane diagram for $y'' = 2y + y'$.

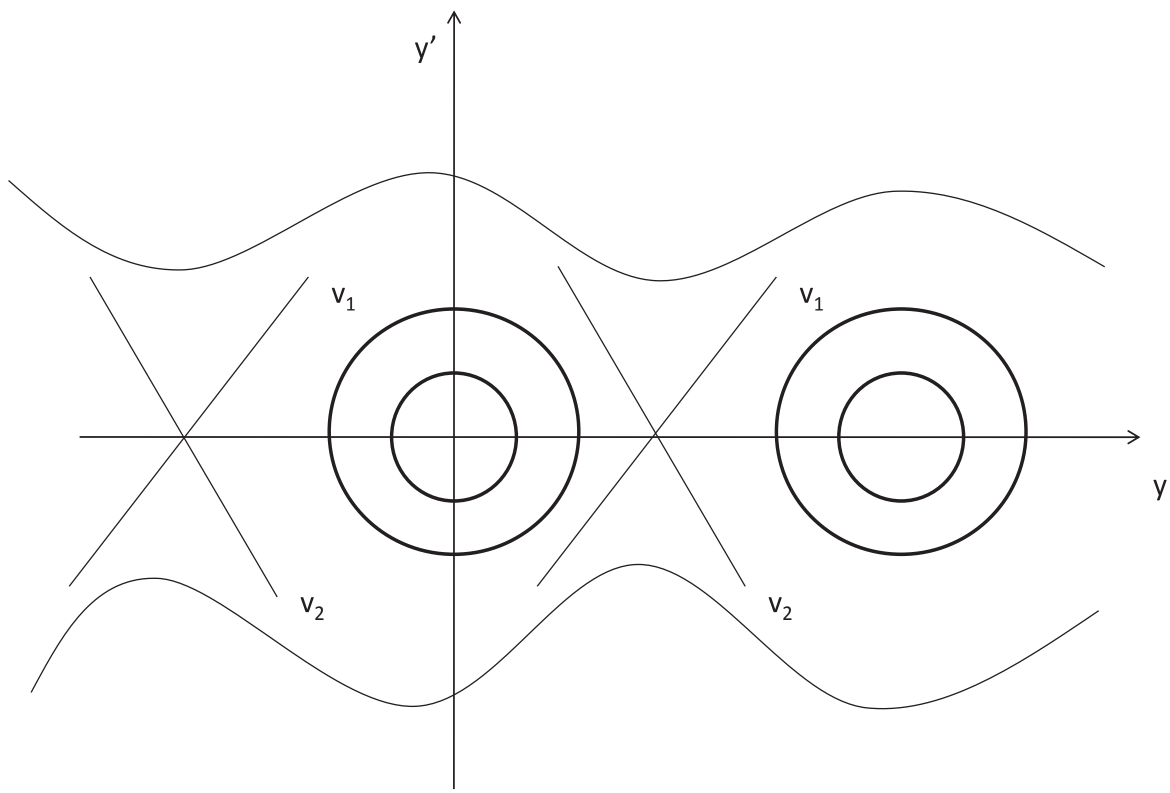

Figure 2: Phase plane diagram for $y'' = \alpha^2 \sin y$.

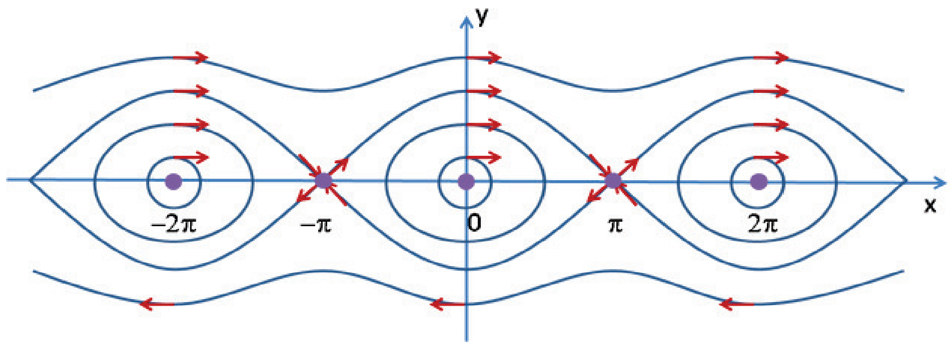

Figure 3: Phase plane diagram for $y'' = \alpha^2 \sin y$.

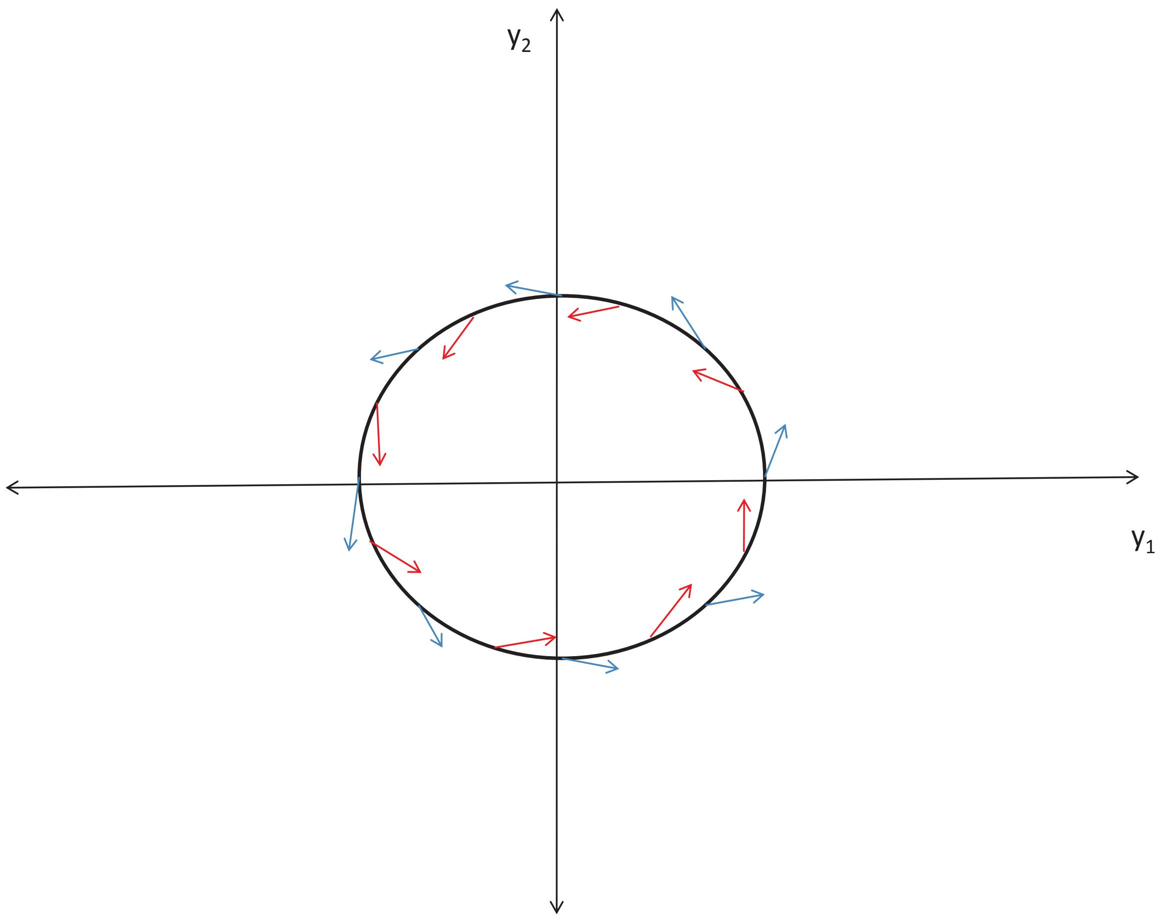

Figure 4: Behavior of solutions when eigenvalues have real part zero.