Economics 204 Summer/Fall 2025
Lecture 2–Tuesday July 29, 2025
Revised: See footnote 4.

Section 1.4. Cardinality (cont.)

Theorem 1 (Cantor) $2^{\mathbf{N}}$, *the set of all subsets of* $\mathbf{N}$, *is* not *countable.*

Proof: Suppose $2^{\mathbf{N}}$ is countable. Then there is a bijection $f : \mathbf{N} \to 2^{\mathbf{N}}$. Let $A_m = f(m)$. We create an infinite matrix, whose $(m,n)^{th}$ entry is 1 if $n \in A_m$, 0 otherwise:

			$\mathbf{N}$					
			1	2	3	4	5	$\cdots$
	$A_1 =$	$\emptyset$	**0**	0	0	0	0	$\cdots$
	$A_2 =$	$\{1\}$	1	**0**	0	0	0	$\cdots$
$2^{\mathbf{N}}$	$A_3 =$	$\{1,2,3\}$	1	1	**1**	0	0	$\cdots$
	$A_4 =$	$\mathbf{N}$	1	1	1	**1**	1	$\cdots$
	$A_5 =$	$2\mathbf{N}$	0	1	0	1	**0**	$\cdots$
	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Now, on the main diagonal, change all the 0s to 1s and vice versa:

			$\mathbf{N}$				
		1	2	3	4	5	$\cdots$
$A_1 =$	$\emptyset$	**1**	0	0	0	0	$\cdots$
$A_2 =$	$\{1\}$	1	**1**	0	0	0	$\cdots$
$2^{\mathbf{N}}$ $A_3 =$	$\{1,2,3\}$	1	1	**0**	0	0	$\cdots$
$A_4 =$	$\mathbf{N}$	1	1	1	**0**	1	$\cdots$
$A_5 =$	$2\mathbf{N}$	0	1	0	1	**1**	$\cdots$
$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

The coding on the diagonal represents a subset of $\mathbf{N}$ which differs from each of the A_m, contradiction. It is important that we go along the diagonal. We need to define a set $A \subseteq \mathbf{N}$ which is different from $f(1), f(2), \ldots$. To define a set, we need to specify exactly what its elements are, and we do this by taking one entry from each column and one entry from each row. The entry from column n tells us whether or not n is in the set, and the entry in row m is used to ensure that $A \neq A_m$.

More formally, let

$$t_{mn} = \begin{cases} 1 & \text{if } n \in A_m \\ 0 & \text{if } n \notin A_m \end{cases}$$

Let $A = \{m \in \mathbf{N} : t_{mm} = 0\}$. (This is the set described by changing all the codings on the diagonal.)

$$\begin{aligned} m \in A &\Leftrightarrow t_{mm} = 0 \\ &\Leftrightarrow m \notin A_m \\ 1 \in A &\Leftrightarrow 1 \notin A_1 \text{ so } A \neq A_1 \\ 2 \in A &\Leftrightarrow 2 \notin A_2 \text{ so } A \neq A_2 \\ &\vdots \\ m \in A &\Leftrightarrow m \notin A_m \text{ so } A \neq A_m \end{aligned}$$

Therefore, $A \neq f(m)$ for any m, so f is not onto, contradiction. ∎

Remark: Notice that $2^{\mathbf{N}}$ is not finite (why not?), so this result shows that $2^{\mathbf{N}}$ is uncountable. Thus there are fundamentally more subsets of $\mathbf{N}$ than elements of $\mathbf{N}$. One can show that $2^{\mathbf{N}}$ is numerically equivalent to $\mathbf{R}$, so there are fundamentally more real numbers than rational numbers.

Remark: See the Appendix for some additional facts about cardinality.

Section 1.5: Algebraic Structures

Here we define abstract objects that have much of the algebraic structure of $\mathbf{R}$, with notions of addition, subtraction, multiplication and division.

Field Axioms

Definition 2 A *field* $\mathcal{F} = (F, +, \cdot)$ is a 3-tuple consisting of a set F and two binary operations $+, \cdot : F \times F \to F$ such that

1. Associativity of $+$:

$$\forall \alpha, \beta, \gamma \in F,\ (\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$$

2. Commutativity of $+$:

$$\forall \alpha, \beta \in F,\ \alpha + \beta = \beta + \alpha$$

3. Existence of additive identity:[1]

$$\exists! 0 \in F \text{ s.t. } \forall \alpha \in F,\ \alpha + 0 = 0 + \alpha = \alpha$$

4. Existence of additive inverse:[2]

$$\forall \alpha \in F\ \exists! (-\alpha) \in F \text{ s.t. } \alpha + (-\alpha) = (-\alpha) + \alpha = 0$$

We define $\alpha - \beta = \alpha + (-\beta)$.

[1] This says that the element 0 behaves like the real number zero; it need not be the real number zero. Indeed, F need not be a subset of $\mathbf{R}$.

[2] We write $\alpha + (-\alpha)$ rather than $\alpha - \alpha$ because subtraction has not yet been defined. In fact, we define $\alpha - \beta$ to be $\alpha + (-\beta)$.

5. Associativity of $\cdot$:

$$\forall \alpha, \beta, \gamma \in F,\ (\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma)$$

6. Commutativity of $\cdot$:

$$\forall \alpha, \beta \in F,\ \alpha \cdot \beta = \beta \cdot \alpha$$

7. Existence of multiplicative identity:[3]

$$\exists! 1 \in F \text{ s.t. } 1 \neq 0 \text{ and } \forall \alpha \in F,\ \alpha \cdot 1 = 1 \cdot \alpha = \alpha$$

8. Existence of multiplicative inverse:

$$\forall \alpha \in F \text{ s.t. } \alpha \neq 0\ \exists! \alpha^{-1} \in F \text{ s.t. } \alpha \cdot \alpha^{-1} = \alpha^{-1} \cdot \alpha = 1$$

We define $\frac{\alpha}{\beta} = \alpha\beta^{-1}$.

9. Distributivity of multiplication over addition:

$$\forall \alpha, \beta, \gamma \in F,\ \alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma$$

Examples:

- $\mathbf{R}$
- $\mathbf{C} = \{x + iy : x, y \in \mathbf{R}\}$. $i^2 = -1$, so

$(x+iy)(w+iz) = xw+ixz+iwy+i^2yz = (xw-yz)+i(xz+wy$

[3]This says that the element 1 behaves like the real number one; it need not be the real number one. Again, F need not even be a subset of $\mathbf{R}$.

- $\mathbf{Q}$: $\mathbf{Q} \subset \mathbf{R}$, $\mathbf{Q} \neq \mathbf{R}$. $\mathbf{Q}$ is closed under $+, \cdot$, taking additive and multiplicative inverses; the field axioms are inherited from the field axioms on $\mathbf{R}$, so $\mathbf{Q}$ is a field.
- $\mathbf{N}$ is not a field: no additive identity.
- $\mathbf{Z}$ is not a field; no multiplicative inverse for 2.
- $\mathbf{Q}(\sqrt{2})$, the smallest field containing $\mathbf{Q} \cup \{\sqrt{2}\}$. Take $\mathbf{Q}$, add $\sqrt{2}$, and close up under $+$, $\cdot$, taking additive and multiplicative inverses. One can show

$$\mathbf{Q}(\sqrt{2}) = \{q + r\sqrt{2} : q, r \in \mathbf{Q}\}$$

  For example,[4]

$$\left(q + r\sqrt{2}\right)^{-1} = \frac{q}{q^2 - 2r^2} - \frac{r}{q^2 - 2r^2}\sqrt{2}$$

- A *finite field*: $F_2 = (\{0, 1\}, +, \cdot)$ where

$$\begin{array}{rcl} & 0+0 = 0 & \\ 0+1 = 1+0 = 1 & & 0\cdot 1 = 1\cdot 0 = 0 \\ & 1+1 = 0 & \end{array} \qquad \begin{array}{rcl} & 0\cdot 0 = 0 & \\ & & \\ & 1\cdot 1 = 1 & \end{array}$$

  (*"Arithmetic mod 2"*)

[4]Note that if $q^2 - 2r^2 = 0$, then $(q - r\sqrt{2})(q + r\sqrt{2}) = 0$ so either $(q + r\sqrt{2}) = 0$ and there is no multiplicative inverse; or $(q - r\sqrt{2}) = 0$, in which case the left side is $(2r\sqrt{2})^{-1} = \frac{\sqrt{2}}{4r}$.

Vector Space Axioms

Here we define abstract objects that "behave like $\mathbf{R}^n$".

Definition 3 A *vector space* is a 4-tuple $(V, F, +, \cdot)$ where V is a set of elements, called *vectors*, F is a field, $+$ is a binary operation on V called vector addition, and $\cdot : F \times V \to V$ is called scalar multiplication, satisfying

1. Associativity of $+$:

$$\forall x, y, z \in V,\ (x + y) + z = x + (y + z)$$

2. Commutativity of $+$:

$$\forall x, y \in V,\ x + y = y + x$$

3. Existence of vector additive identity:[5]

$$\exists! 0 \in V \text{ s.t. } \forall x \in V,\ x + 0 = 0 + x = x$$

4. Existence of vector additive inverse:

$$\forall x \in V\ \exists!(-x) \in V \text{ s.t. } x + (-x) = (-x) + x = 0$$

We define $x - y$ to be $x + (-y)$.

5. Distributivity of scalar multiplication over vector addition:

$$\forall \alpha \in F, x, y \in V,\ \alpha \cdot (x + y) = \alpha \cdot x + \alpha \cdot y$$

[5]Note that $0 \in V$ and $0 \in F$ are different.

6. Distributivity of scalar multiplication over scalar addition:

$$\forall \alpha, \beta \in F, x \in V \quad (\alpha + \beta) \cdot x = \alpha \cdot x + \beta \cdot x$$

7. Associativity of $\cdot$:

$$\forall \alpha, \beta \in F, x \in V \quad (\alpha \cdot \beta) \cdot x = \alpha \cdot (\beta \cdot x)$$

8. Multiplicative identity:

$$\forall x \in V \quad 1 \cdot x = x$$

(Note that 1 is the multiplicative identity in F; $1 \notin V$)

We often say "V is a vector space over F".

Examples:

1. $\mathbf{R}^n$ over $\mathbf{R}$.
2. $\mathbf{R}$ is a vector space over $\mathbf{Q}$:

   (scalar multiplication) $q \cdot r = qr$ (product in $\mathbf{R}$)

   $\mathbf{R}$ is not finite-dimensional over $\mathbf{Q}$, i.e. $\mathbf{R}$ is not $\mathbf{Q}^n$ for any $n \in \mathbf{N}$.
3. $\mathbf{R}$ is a vector space over $\mathbf{R}$.
4. $\mathbf{Q}(\sqrt{2})$ is a vector space over $\mathbf{Q}$. As a vector space, it is $\mathbf{Q}^2$; as a field, you need to take the funny field multiplication.

5. $\mathbf{Q}(\sqrt[3]{2})$, as a vector space over $\mathbf{Q}$, is $\mathbf{Q}^3$.
6. $(F_2)^n$ is a *finite* vector space over F_2.
7. $C([0,1])$, the space of all continuous real-valued functions on $[0,1]$, is a vector space over $\mathbf{R}$.
   - vector addition:
     $$(f+g)(t) = f(t) + g(t)$$
     Note we define the function $f+g$ by specifying what value it takes for each $t \in [0,1]$.
   - scalar multiplication:
     $$(\alpha f)(t) = \alpha(f(t))$$
   - vector additive identity: 0 is the function which is identically zero: $0(t) = 0$ for all $t \in [0,1]$.
   - vector additive inverse:
     $$(-f)(t) = -(f(t))$$

Section 1.6 Axioms for R

One can show that $\mathbf{R}$ is characterized by the following axioms, that is, that there exists a field that satisfies these axioms, and this is what we define as the set of real numbers.

1. $\mathbf{R}$ is a field with the usual operations $+$, $\cdot$, additive identity 0, and multiplicative identity 1.

2. **Order Axiom:** There is a complete ordering $\leq$, i.e. $\leq$ is reflexive, transitive, antisymmetric ($\alpha \leq \beta, \beta \leq \alpha \Rightarrow \alpha = \beta$) with the property that

$$\forall \alpha, \beta \in \mathbf{R} \text{ either } \alpha \leq \beta \text{ or } \beta \leq \alpha$$

The order is compatible with $+$ and $\cdot$, i.e.

$$\forall \alpha, \beta, \gamma \in \mathbf{R} \begin{cases} \alpha \leq \beta \Rightarrow \alpha + \gamma \leq \beta + \gamma \\ \alpha \leq \beta, 0 \leq \gamma \Rightarrow \alpha\gamma \leq \beta\gamma \end{cases}$$

$\alpha \geq \beta$ means $\beta \leq \alpha$.
$\alpha < \beta$ means $\alpha \leq \beta$ and $\alpha \neq \beta$.

3. **Completeness Axiom:** Suppose $L, H \subseteq \mathbf{R}$, $L \neq \emptyset \neq H$ satisfy

$$\forall \ell \in L, h \in H \ \ell \leq h$$

Then

$$\exists \alpha \in \mathbf{R} \text{ s.t. } \forall \ell \in L, h \in H \ \ell \leq \alpha \leq h$$

$$\begin{array}{ccccc} & & \alpha & & \\ & L & \downarrow & H & \\ - - - -) & & \cdot & & (- - - - \end{array}$$

The Completeness Axiom differentiates $\mathbf{R}$ from $\mathbf{Q}$: $\mathbf{Q}$ satisfies all the axioms for $\mathbf{R}$ except the Completeness Axiom.

The most useful consequence of the Completeness Axiom (and often used as an alternative axiom) is the **Supremum Property**.

Definition 4 Suppose $X \subseteq \mathbf{R}$. We say u is an *upper bound* for X if

$$x \leq u \ \ \forall x \in X$$

and ℓ is a *lower bound* for X if

$$\ell \leq x \ \ \forall x \in X$$

X is *bounded above* if there is an upper bound for X, and *bounded below* if there is a lower bound for X.

Definition 5 Suppose X is bounded above. The *supremum* of X, written $\sup X$, is the least upper bound for X, i.e. $\sup X$ satisfies

$$\sup X \geq x \ \ \forall x \in X \text{ ($\sup X$ is an upper bound)}$$

$$\forall y < \sup X \ \exists x \in X \text{ s.t. } x > y \text{ (there is no smaller upper bound)}$$

Analogously, suppose X is bounded below. The *infimum* of X, written $\inf X$, is the greatest lower bound for X, i.e. $\inf X$ satisfies

$$\inf X \leq x \ \ \forall x \in X \text{ ($\inf X$ is a lower bound)}$$

$$\forall y > \inf X \ \exists x \in X \text{ s.t. } x < y \text{ (there is no greater lower bound)}$$

If X is not bounded above, write $\sup X = \infty$. If X is not bounded below, write $\inf X = -\infty$. By convention, $\sup \emptyset = -\infty$, $\inf \emptyset = +\infty$.

The Supremum Property: Every nonempty set of real numbers that is bounded above has a supremum, which is a real number. Every nonempty set of real numbers that is bounded below has an infimum, which is a real number.

Note: $\sup X$ need not be an element of X. For example, $\sup(0,1) = 1 \notin (0,1)$.

Theorem 6 (Theorem 6.8, plus . . .) *The Supremum Property and the Completeness Axiom are equivalent.*

Proof: Assume the Completeness Axiom. Let $X \subseteq \mathbf{R}$ be a nonempty set which is bounded above. Let U be the set of all upper bounds for X. Since X is bounded above, $U \neq \emptyset$. If $x \in X$ and $u \in U$, $x \leq u$ since u is an upper bound for X. So

$$x \leq u \ \ \forall x \in X, u \in U$$

By the Completeness Axiom,

$$\exists \alpha \in \mathbf{R} \text{ s.t. } x \leq \alpha \leq u \ \ \forall x \in X, u \in U$$

α is an upper bound for X, and it is less than or equal to every other upper bound for X, so it is the least upper bound for X, so $\sup X = \alpha \in \mathbf{R}$. The case in which X is bounded below is similar. Thus, the Supremum Property holds.

Conversely, assume the Supremum Property. Suppose $L, H \subseteq \mathbf{R}$, $L \neq \emptyset \neq H$, and

$$\ell \leq h \ \ \forall \ell \in L, h \in H$$

Since $L \neq \emptyset$ and L is bounded above (by any element of H), $\alpha = \sup L$ exists and is real. By the definition of supremum, α is an upper bound for L, so

$$\ell \leq \alpha \ \ \forall \ell \in L$$

Suppose $h \in H$. Then h is an upper bound for L, so by the definition of supremum, $\alpha \leq h$. Therefore, we have shown that

$$\ell \leq \alpha \leq h \ \ \forall \ell \in L, h \in H$$

so the Completeness Axiom holds. ∎

Theorem 7 (Archimedean Property, Theorem 6.10 +

$$\forall x, y \in \mathbf{R}, y > 0 \ \exists n \in \mathbf{N} \ s.t. \ ny = \underbrace{(y + \cdots + y)}_{n \ times} > x$$

Proof: Exercise. This is a nice exercise in proof by contradiction, using the Supremum Property. ∎

Theorem 8 (Intermediate Value Theorem) *Suppose $f : [a, b] \to \mathbf{R}$ is continuous, and $f(a) < d < f(b)$. Then there exists $c \in (a, b)$ such that $f(c) = d$.*

Proof: Later, we will give a slick proof. Here, we give a bare-hands proof using the Supremum Property. Let

$$B = \{x \in [a, b] : f(x) < d\}$$

$a \in B$, so $B \neq \emptyset$; $B \subseteq [a, b]$, so B is bounded above. By the Supremum Property, $\sup B$ exists and is real so let $c = \sup B$. Since $a \in B$, $c \geq a$. $B \subseteq [a, b]$, so $c \leq b$. Therefore, $c \in [a, b]$. (See Figure 1.)

We claim that $f(c) = d$. If not, suppose $f(c) < d$. Then since $f(b) > d$, $c \neq b$, so $c < b$. Let $\varepsilon = \frac{d - f(c)}{2} > 0$. Since f is continuous at c, there exists $\delta > 0$ such that

$$\begin{array}{rcl} |x - c| < \delta \Rightarrow |f(x) - f(c)| & < & \varepsilon \\ \Rightarrow \quad f(x) & < & f(c) + \varepsilon \\ & = & f(c) + \frac{d - f(c)}{2} \\ & = & \frac{f(c) + d}{2} \\ & < & \frac{d + d}{2} \\ & = & d \end{array}$$

so $(c, c + \delta) \subseteq B$, so $c \neq \sup B$, contradiction. (See Figure 2.)

Suppose $f(c) > d$. Then since $f(a) < d$, $a \neq c$, so $c > a$. Let $\varepsilon = \frac{f(c)-d}{2} > 0$. Since f is continuous at c, there exists $\delta > 0$ such that

$$\begin{array}{rcl} |x-c| < \delta \Rightarrow |f(x)-f(c)| & < & \varepsilon \\ \Rightarrow \quad f(x) & > & f(c) - \varepsilon \\ & = & f(c) - \frac{f(c)-d}{2} \\ & = & \frac{f(c)+d}{2} \\ & > & \frac{d+d}{2} \\ & = & d \end{array}$$

so $(c-\delta, c+\delta) \cap B = \emptyset$. So either there exists $x \in B$ with $x \geq c + \delta$ (in which case c is not an upper bound for B) or $c - \delta$ is an upper bound for B (in which case c is not the least upper bound for B); in either case, $c \neq \sup B$, contradiction. (See Figure 3.)

Since $f(c) \not< d$, $f(c) \not> d$, and the order is complete, $f(c) = d$. Since $f(a) < d$ and $f(b) > d$, $a \neq c \neq b$, so $c \in (a, b)$. ∎

Corollary 9 *There exists $x \in \mathbf{R}$ such that $x^2 = 2$.*

Proof: Let $f(x) = x^2$, for $x \in [0, 2]$. f is continuous. $f(0) = 0 < 2$ and $f(2) = 4 > 2$, so by the Intermediate Value Theorem, there exists $c \in (0, 2)$ such that $f(c) = 2$, i.e. $c^2 = 2$. ∎

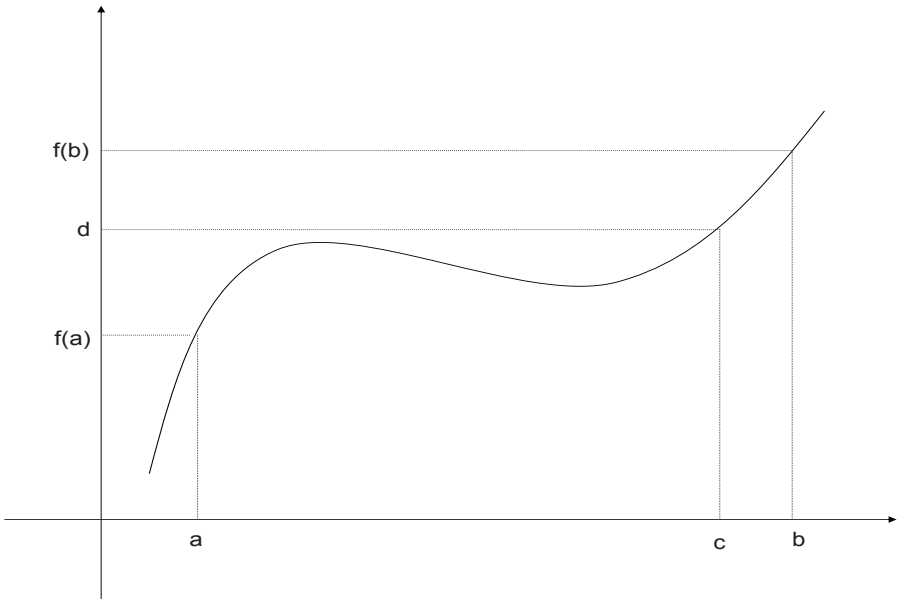

Figure 1: Intermediate Value Theorem. Claim: $d = f(c)$, where $c = \sup\{x \in [a, b] : f(x) < d\}$.

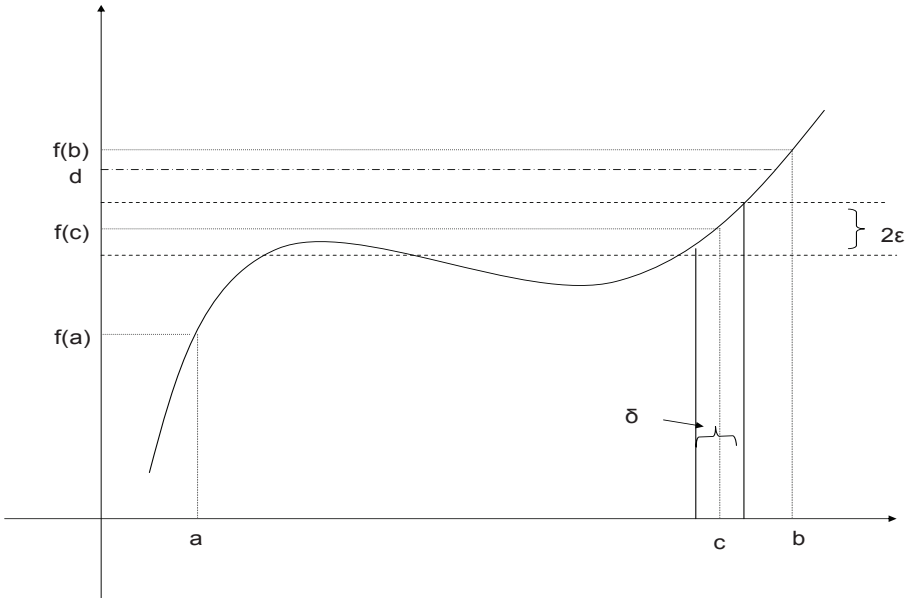

Figure 2: Intermediate Value Theorem. If $d > f(c)$, find $x > c$ with $f(c) < f(x) < f(d)$ using continuity.

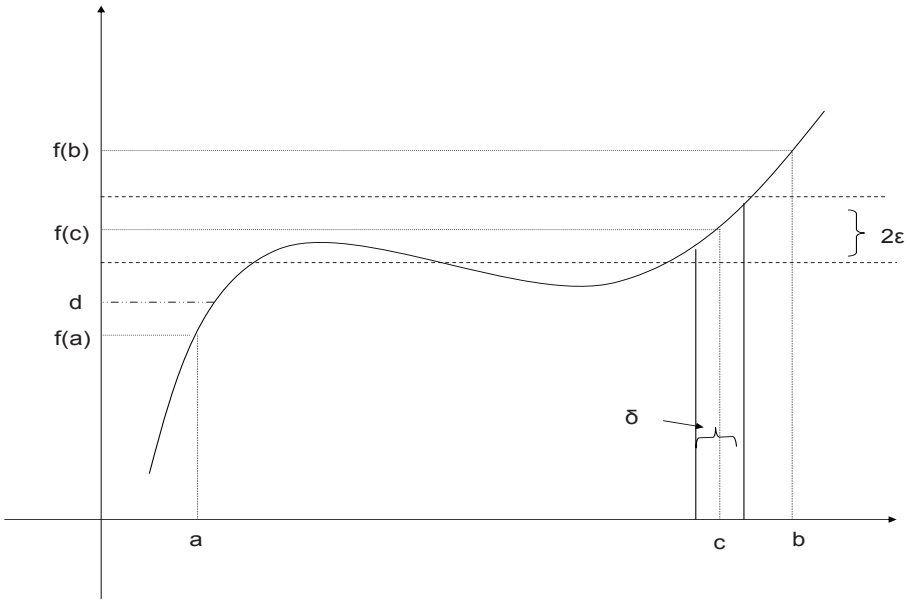

Figure 3: Intermediate Value Theorem. If $d < f(c)$, using continuity find $\delta > 0$ with $d < f(x)$ for any $x \in (c - \delta, c + \delta)$.

Read sections 1.6(c) (absolute values) and 1.7 (complex numbers) on your own.

Appendix: Some Facts About Cardinality

Here I record some additional facts about the notion of cardinality. This material is optional, as formalizing and proving some of these points goes well beyond what we can do in this class. In particular, to formalize some of these statements requires precise notions of "orders of infinity". Many of these properties can be shown as exercises given what we have done, however.

Recall we let $|A|$ denote the cardinality of a set A.

- if A is numerically equivalent to $\{1, \ldots, n\}$ for some $n \in \mathbf{N}$, then $|A| = n$.
- A and B are numerically equivalent if and only if $|A| = |B|$
- if $|A| = n$ and A is a proper subset of B (that is, $A \subseteq B$ and $A \neq B$) then $|A| < |B|$
- if A is countable and B is uncountable, then

$$n < |A| < |B| \quad \forall n \in \mathbf{N}$$

- if $A \subseteq B$ then $|A| \leq |B|$
- if $r : A \to B$ is 1-1, then $|A| \leq |B|$
- if B is countable and $A \subseteq B$, then A is at most countable, that is, A is either empty, finite, or countable
- if $r : A \to B$ is 1-1 and B is countable, then A is at most countable

We will make use of some of these facts in Lecture 8 when we discuss the notion of dimension of a linear space.

If you are interested in studying cardinality more formally, you might look at e.g. Willard *General Topology*.