

## Section 2.4. Open and Closed Sets

**Definition 1** Let  $(X, d)$  be a metric space. A set  $A \subseteq X$  is *open* if

$$\forall x \in A \exists \varepsilon > 0 \text{ s.t. } B_\varepsilon(x) \subseteq A$$

A set  $C \subseteq X$  is *closed* if  $X \setminus C$  is open.

See Figure 1.

**Example:**  $(a, b)$  is open in the metric space  $\mathbf{E}^1$  ( $\mathbf{R}$  with the usual Euclidean metric). Given

$x \in (a, b)$ ,  $a < x < b$ . Let

$$\varepsilon = \min\{x - a, b - x\} > 0$$

Then

$$\begin{aligned} y \in B_\varepsilon(x) &\Rightarrow y \in (x - \varepsilon, x + \varepsilon) \\ &\subseteq (x - (x - a), x + (b - x)) \\ &= (a, b) \end{aligned}$$

so  $B_\varepsilon(x) \subseteq (a, b)$ , so  $(a, b)$  is open.

Notice that  $\varepsilon$  depends on  $x$ ; in particular,  $\varepsilon$  gets smaller as  $x$  nears the boundary of the set.

**Example:** In  $\mathbf{E}^1$ ,  $[a, b]$  is closed.  $\mathbf{R} \setminus [a, b] = (-\infty, a) \cup (b, \infty)$  is a union of two open sets, which must be open.

**Example:** In the metric space  $[0, 1]$ ,  $[0, 1]$  is open. With  $[0, 1]$  as the underlying metric space,  $B_\varepsilon(0) = \{x \in [0, 1] : |x - 0| < \varepsilon\} = [0, \varepsilon)$ .

Thus, openness and closedness depend on the underlying metric space as well as on the set.

**Example:** Most sets are neither open nor closed. For example, in  $\mathbf{E}^1$ ,  $[0, 1] \cup (2, 3)$  is neither open nor closed.

**Example:** An open set may consist of a single point. For example, if  $X = \mathbf{N}$  and  $d(m, n) = |m - n|$ , then  $B_{1/2}(1) = \{m \in \mathbf{N} : |m - 1| < 1/2\} = \{1\}$ . Since 1 is the only element of the set  $\{1\}$  and  $B_{1/2}(1) = \{1\} \subseteq \{1\}$ , the set  $\{1\}$  is open.

**Example:** In any metric space  $(X, d)$  both  $\emptyset$  and  $X$  are open, and both  $\emptyset$  and  $X$  are closed. To see that  $\emptyset$  is open, note that the statement

$$\forall x \in \emptyset \exists \varepsilon > 0 B_\varepsilon(x) \subseteq \emptyset$$

is vacuously true since there aren't any  $x \in \emptyset$ . To see that  $X$  is open, note that since  $B_\varepsilon(x)$  is by definition  $\{z \in X : d(z, x) < \varepsilon\}$ , it is trivially contained in  $X$ . Since  $\emptyset$  is open,  $X$  is closed; since  $X$  is open,  $\emptyset$  is closed.

**Example:** Open balls are open sets. Suppose  $y \in B_\varepsilon(x)$ . Then  $d(x, y) < \varepsilon$ . Let  $\delta = \varepsilon - d(x, y) > 0$ . If  $d(z, y) < \delta$ , then

$$d(z, x) \leq d(z, y) + d(y, x)$$

$$\begin{aligned}
&< \delta + d(x, y) \\
&= \varepsilon - d(x, y) + d(x, y) \\
&= \varepsilon
\end{aligned}$$

so  $B_\delta(y) \subseteq B_\varepsilon(x)$ , so  $B_\varepsilon(x)$  is open.

**Theorem 2 (Thm. 4.2)** *Let  $(X, d)$  be a metric space. Then*

1.  $\emptyset$  and  $X$  are both open, and both closed.
2. The union of an arbitrary (finite, countable, or uncountable) collection of open sets is open.
3. The intersection of a finite collection of open sets is open.

**Proof:**

1. We have already shown this.
2. Suppose  $\{A_\lambda\}_{\lambda \in \Lambda}$  is a collection of open sets.

$$\begin{aligned}
x \in \bigcup_{\lambda \in \Lambda} A_\lambda &\Rightarrow \exists \lambda_0 \in \Lambda \text{ s.t. } x \in A_{\lambda_0} \\
&\Rightarrow \exists \varepsilon > 0 \text{ s.t. } B_\varepsilon(x) \subseteq A_{\lambda_0} \subseteq \bigcup_{\lambda \in \Lambda} A_\lambda
\end{aligned}$$

so  $\bigcup_{\lambda \in \Lambda} A_\lambda$  is open.

3. Suppose  $A_1, \dots, A_n \subseteq X$  are open sets. If  $x \in \bigcap_{i=1}^n A_i$ , then

$$x \in A_1, x \in A_2, \dots, x \in A_n$$

so

$$\exists \varepsilon_1 > 0, \dots, \varepsilon_n > 0 \text{ s.t. } B_{\varepsilon_1}(x) \subseteq A_1, \dots, B_{\varepsilon_n}(x) \subseteq A_n$$

Let

$$\varepsilon = \min\{\varepsilon_1, \dots, \varepsilon_n\} > 0$$

(Note this is where we need the fact that we are taking a finite intersection. The infimum of an infinite set of positive numbers could be zero. And the intersection of an infinite collection of open sets need not be open.)

Then

$$B_\varepsilon(x) \subseteq B_{\varepsilon_1}(x) \subseteq A_1, \dots, B_\varepsilon(x) \subseteq B_{\varepsilon_n}(x) \subseteq A_n$$

so

$$B_\varepsilon(x) \subseteq \bigcap_{i=1}^n A_i$$

which proves that  $\bigcap_{i=1}^n A_i$  is open.

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**Definition 3**     • The *interior* of  $A$ , denoted  $\text{int } A$ , is the largest open set contained in  $A$  (the union of all open sets contained in  $A$ ).

- The *closure* of  $A$ , denoted  $\bar{A}$ , is the smallest closed set containing  $A$  (the intersection of all closed sets containing  $A$ )
- The *exterior* of  $A$ , denoted  $\text{ext } A$ , is the largest open set contained in  $X \setminus A$ .
- The *boundary* of  $A$ , denoted  $\partial A = \overline{(X \setminus A)} \cap \bar{A}$

**Example:** Let  $A = [0, 1] \cup (2, 3)$ . Then

$$\text{int } A = (0, 1) \cup (2, 3)$$

$$\bar{A} = [0, 1] \cup [2, 3]$$

$$\text{ext } A = \text{int } (X \setminus A)$$

$$= (-\infty, 0) \cup (1, 2) \cup (3, +\infty)$$

$$\partial A = \overline{(X \setminus A)} \cap \bar{A}$$

$$= ((-\infty, 0] \cup [1, 2] \cup [3, +\infty)) \cap ([0, 1] \cup [2, 3])$$

$$= \{0, 1, 2, 3\}$$

**Theorem 4 (Thm. 4.13)** *A set  $A$  in a metric space  $(X, d)$  is closed if and only if*

$$\{x_n\} \subset A, x_n \rightarrow x \in X \Rightarrow x \in A$$

**Proof:**<sup>1</sup> Suppose  $A$  is closed. Then  $X \setminus A$  is open. Consider a convergent sequence  $x_n \rightarrow x \in X$ , with  $x_n \in A$  for all  $n$ . If  $x \notin A$ ,  $x \in X \setminus A$ , so there is some  $\varepsilon > 0$  such that  $B_\varepsilon(x) \subseteq X \setminus A$ . (See Figure 2.) Since  $x_n \rightarrow x$ , there exists  $N(\varepsilon)$  such that

$$n > N(\varepsilon) \Rightarrow x_n \in B_\varepsilon(x)$$

$$\Rightarrow x_n \in X \setminus A$$

$$\Rightarrow x_n \notin A$$

contradiction. Therefore,

$$x_n \in A, x_n \rightarrow x \in X \Rightarrow x \in A$$

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<sup>1</sup>This is different from the proof in de la Fuente: he puts the meat of the proof into Theorem 4.12

Conversely, suppose

$$\{x_n\} \subset A, x_n \rightarrow x \in X \Rightarrow x \in A$$

We need to show that  $A$  is closed, i.e.  $X \setminus A$  is open. Suppose not, so  $X \setminus A$  is not open.

Then there exists  $x \in X \setminus A$  such that for every  $\varepsilon > 0$ ,

$$B_\varepsilon(x) \not\subseteq X \setminus A$$

so there exists  $y \in B_\varepsilon(x)$  such that  $y \notin X \setminus A$ . Then  $y \in A$ , hence

$$B_\varepsilon(x) \cap A \neq \emptyset$$

See Figure 3. Construct a sequence  $\{x_n\}$  as follows: for each  $n$ , choose  $x_n \in B_{\frac{1}{n}}(x) \cap A$ .

Given  $\varepsilon > 0$ , we can find  $N(\varepsilon)$  such that  $N(\varepsilon) > \frac{1}{\varepsilon}$  by the Archimedean Property, so  $n > N(\varepsilon) \Rightarrow \frac{1}{n} < \frac{1}{N(\varepsilon)} < \varepsilon$ , so  $x_n \rightarrow x$ . Then  $\{x_n\} \subseteq A$ ,  $x_n \rightarrow x$ , so  $x \in A$ , contradiction.

Therefore,  $X \setminus A$  is open, so  $A$  is closed. ■

## Section 2.5. Limits of Functions

**Note:** Read this section of de la Fuente on your own.

Note that we may have  $\lim_{x \rightarrow a} f(x) = y$  even though

- $f$  is not defined at  $a$ ; or
- $f$  is defined at  $a$  but  $f(a) \neq y$ .

The existence and value of the limit depends on values of  $f$  *near*  $a$  but not *at*  $a$ .

## Section 2.6. Continuity in Metric Spaces

**Definition 5** Let  $(X, d)$  and  $(Y, \rho)$  be metric spaces. A function  $f : X \rightarrow Y$  is *continuous* at a point  $x_0 \in X$  if  $\forall \varepsilon > 0 \exists \delta(x_0, \varepsilon) > 0$  s.t.  $d(x, x_0) < \delta(x_0, \varepsilon) \Rightarrow \rho(f(x), f(x_0)) < \varepsilon$ .

$f$  is *continuous* if it is continuous at every element of its domain.

Note that  $\delta$  depends on  $x_0$  and  $\varepsilon$ .

This is a straightforward generalization of the definition of continuity in  $\mathbf{R}$ . Continuity at  $x_0$  requires:

- $f(x_0)$  is defined; and
- either
  - $x_0$  is an isolated point of  $X$ , i.e.  $\exists \varepsilon > 0$  s.t.  $B_\varepsilon(x) = \{x\}$ ; or
  - $\lim_{x \rightarrow x_0} f(x)$  exists and equals  $f(x_0)$

Suppose  $f : X \rightarrow Y$  and  $A \subseteq Y$ . Define  $f^{-1}(A) = \{x \in X : f(x) \in A\}$ .

**Theorem 6 (Thm. 6.14)** *Let  $(X, d)$  and  $(Y, \rho)$  be metric spaces, and  $f : X \rightarrow Y$ . Then  $f$  is continuous if and only if*

$$f^{-1}(A) \text{ is open in } X \forall A \subseteq Y \text{ s.t. } A \text{ is open in } Y$$

**Proof:**<sup>2</sup> Suppose  $f$  is continuous. Given  $A \subseteq Y$ ,  $A$  open, we must show that  $f^{-1}(A)$  is open in  $X$ . Suppose  $x_0 \in f^{-1}(A)$ . Let  $y_0 = f(x_0) \in A$ . Since  $A$  is open, we can find  $\varepsilon > 0$  such

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<sup>2</sup>We give a direct proof; de la Fuente works via closed sets.

that  $B_\varepsilon(y_0) \subseteq A$ . Since  $f$  is continuous, there exists  $\delta > 0$  such that

$$\begin{aligned} d(x, x_0) < \delta &\Rightarrow \rho(f(x), f(x_0)) < \varepsilon \\ &\Rightarrow f(x) \in B_\varepsilon(y_0) \\ &\Rightarrow f(x) \in A \\ &\Rightarrow x \in f^{-1}(A) \end{aligned}$$

so  $B_\delta(x_0) \subseteq f^{-1}(A)$ , so  $f^{-1}(A)$  is open. (See Figure 4.)

Conversely, suppose

$$f^{-1}(A) \text{ is open in } X \ \forall A \subseteq Y \text{ s.t. } A \text{ is open in } Y$$

We need to show that  $f$  is continuous. Let  $x_0 \in X$ ,  $\varepsilon > 0$ . Let  $A = B_\varepsilon(f(x_0))$ .  $A$  is an open ball, hence an open set, so  $f^{-1}(A)$  is open in  $X$ .  $x_0 \in f^{-1}(A)$ , so there exists  $\delta > 0$  such that  $B_\delta(x_0) \subseteq f^{-1}(A)$ . (See Figure 5.)

$$\begin{aligned} d(x, x_0) < \delta &\Rightarrow x \in B_\delta(x_0) \\ &\Rightarrow x \in f^{-1}(A) \\ &\Rightarrow f(x) \in A \\ &\Rightarrow \rho(f(x), f(x_0)) < \varepsilon \end{aligned}$$

Thus, we have shown that  $f$  is continuous at  $x_0$ ; since  $x_0$  is an arbitrary point in  $X$ ,  $f$  is continuous. ■

**Theorem 7 (Slightly weaker version of Thm. 6.10)** *Let  $(X, d_X)$ ,  $(Y, d_Y)$  and  $(Z, d_Z)$  be metric spaces. If  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$  are continuous, then  $g \circ f : X \rightarrow Z$  is continuous.*

**Proof:** Suppose  $A \subseteq Z$  is open. Since  $g$  is continuous,  $g^{-1}(A)$  is open in  $Y$ ; since  $f$  is continuous,  $f^{-1}(g^{-1}(A))$  is open in  $X$ .

We claim that

$$f^{-1}(g^{-1}(A)) = (g \circ f)^{-1}(A)$$

Observe

$$\begin{aligned} x \in f^{-1}(g^{-1}(A)) &\Leftrightarrow f(x) \in g^{-1}(A) \\ &\Leftrightarrow g(f(x)) \in A \\ &\Leftrightarrow (g \circ f)(x) \in A \\ &\Leftrightarrow x \in (g \circ f)^{-1}(A) \end{aligned}$$

which establishes the claim. This shows that  $(g \circ f)^{-1}(A)$  is open in  $X$ , so  $g \circ f$  is continuous.

■

**Definition 8** [Uniform Continuity] Suppose  $f : (X, d) \rightarrow (Y, \rho)$ .  $f$  is *uniformly continuous* if

$$\forall \varepsilon > 0 \exists \delta(\varepsilon) > 0 \text{ s.t. } \forall x_0 \in X, d(x, x_0) < \delta(\varepsilon) \Rightarrow \rho(f(x), f(x_0)) < \varepsilon$$

Notice the important contrast with continuity:  $f$  is continuous means

$$\forall x_0 \in X, \varepsilon > 0 \exists \delta(x_0, \varepsilon) > 0 \text{ s.t. } d(x, x_0) < \delta(x_0, \varepsilon) \Rightarrow \rho(f(x), f(x_0)) < \varepsilon$$

**Example:** Consider

$$f(x) = \frac{1}{x}, \quad x \in (0, 1]$$

$f$  is continuous (why?). We will show that  $f$  is **not** uniformly continuous. Fix  $\varepsilon > 0$  and  $x_0 \in (0, 1]$ . If  $x = \frac{x_0}{1+\varepsilon x_0}$ , then

$$\begin{aligned}
 1 + \varepsilon x_0 &> 1 \\
 x = \frac{x_0}{1 + \varepsilon x_0} &< x_0 \\
 \frac{1}{x} - \frac{1}{x_0} &> 0 \\
 |f(x) - f(x_0)| &= \left| \frac{1}{x} - \frac{1}{x_0} \right| \\
 &= \frac{1}{x} - \frac{1}{x_0} \\
 &= \frac{1 + \varepsilon x_0}{x_0} - \frac{1}{x_0} \\
 &= \frac{\varepsilon x_0}{x_0} \\
 &= \varepsilon
 \end{aligned}$$

Thus,  $\delta(x_0, \varepsilon)$  must be chosen small enough so that

$$\left| \frac{x_0}{1 + \varepsilon x_0} - x_0 \right| \geq \delta(x_0, \varepsilon)$$

$$\begin{aligned}
 \delta(x_0, \varepsilon) &\leq x_0 - \frac{x_0}{1 + \varepsilon x_0} \\
 &= \frac{\varepsilon(x_0)^2}{1 + \varepsilon x_0} \\
 &< \varepsilon(x_0)^2
 \end{aligned}$$

which converges to zero as  $x_0 \rightarrow 0$ . (See Figure 6.) So there is no  $\delta(\varepsilon)$  that will work for all  $x_0 \in (0, 1]$ .

**Example:** If  $f : \mathbf{R} \rightarrow \mathbf{R}$  and  $f'(x)$  is defined and uniformly bounded on an interval  $[a, b]$ , then  $f(x)$  is uniformly continuous on  $[a, b]$ . However, even a function with an unbounded

derivative may be uniformly continuous. Consider

$$f(x) = \sqrt{x}, \quad x \in [0, 1]$$

$f$  is continuous (why?). We will show that  $f$  is uniformly continuous. Given  $\varepsilon > 0$ , let  $\delta = \varepsilon^2$ . Then given any  $x_0 \in [0, 1]$ ,  $|x - x_0| < \delta$  implies by the Fundamental Theorem of Calculus

$$\begin{aligned} |f(x) - f(x_0)| &= \left| \int_{x_0}^x \frac{1}{2\sqrt{t}} dt \right| \\ &\leq \int_0^{|x-x_0|} \frac{1}{2\sqrt{t}} dt \\ &= \sqrt{|x - x_0|} \\ &< \sqrt{\delta} \\ &= \sqrt{\varepsilon^2} \\ &= \varepsilon \end{aligned}$$

Thus,  $f$  is uniformly continuous on  $[0, 1]$ , even though  $f'(x) \rightarrow \infty$  as  $x \rightarrow 0$ .

**Definition 9** Let  $X, Y$  be normed vector spaces,  $E \subseteq X$ .  $f : X \rightarrow Y$  is *Lipschitz on  $E$*  if

$$\exists K > 0 \text{ s.t. } \|f(x) - f(z)\|_Y \leq K\|x - z\|_X \quad \forall x, z \in E$$

$f$  is *locally Lipschitz* on  $E$  if

$$\forall x_0 \in E \exists \varepsilon > 0 \text{ s.t. } f \text{ is Lipschitz on } B_\varepsilon(x_0) \cap E$$

**Remark:** de la Fuente only defines Lipschitz and locally Lipschitz in the context of normed vector spaces. The notions can also be defined analogously in metric spaces as follows: Let

$(X, d)$  and  $(Y, \rho)$  be metric spaces,  $E \subseteq X$ .  $f : X \rightarrow Y$  is *Lipschitz on  $E$*  if

$$\exists K > 0 \text{ s.t. } \rho(f(x), f(z)) \leq Kd(x, z) \quad \forall x, z \in E$$

Similarly,  $f$  is *locally Lipschitz* on  $E$  if

$$\forall x_0 \in E \exists \varepsilon > 0 \text{ s.t. } f \text{ is Lipschitz on } B_\varepsilon(x_0) \cap E$$

Lipschitz continuity is stronger than either continuity or uniform continuity:

$$\text{locally Lipschitz} \Rightarrow \text{continuous}$$

$$\text{Lipschitz} \Rightarrow \text{uniformly continuous}$$

Every  $C^1$  function is locally Lipschitz. (Recall that a function  $f : \mathbf{R}^m \rightarrow \mathbf{R}^n$  is said to be  $C^1$  if all its first partial derivatives exist and are continuous.)

**Definition 10**<sup>3</sup> Let  $(X, d)$  and  $(Y, \rho)$  be metric spaces. A function  $f : X \rightarrow Y$  is called a *homeomorphism* if it is one-to-one, onto, continuous, and its inverse function is continuous.

Now suppose that  $f$  is a homeomorphism and  $U \subset X$ . Let  $g : Y \rightarrow X$  be the inverse of  $f$ , so  $g \circ f : X \rightarrow X$  is the identity on  $X$ , and  $f \circ g : Y \rightarrow Y$  is the identity on  $Y$ .

$$y \in g^{-1}(U) \Leftrightarrow g(y) = f^{-1}(y) \in U$$

$$\Leftrightarrow y \in f(U)$$

$$U \text{ open in } X \Rightarrow g^{-1}(U) \text{ is open in } (f(X), \rho)$$

$$\Rightarrow f(U) \text{ is open in } (f(X), \rho)$$

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<sup>3</sup>This is the standard definition; de la Fuente instead omits the requirement that  $f$  be onto, and requires that  $f^{-1}$  be continuous on  $f(X)$ . See the Corrections handout for a correction to Theorem 6.21

This says that  $(X, d)$  and  $(f(X), \rho|_{f(X)})$  are identical in terms of properties that can be characterized solely in terms of open sets; such properties are called “topological properties.”

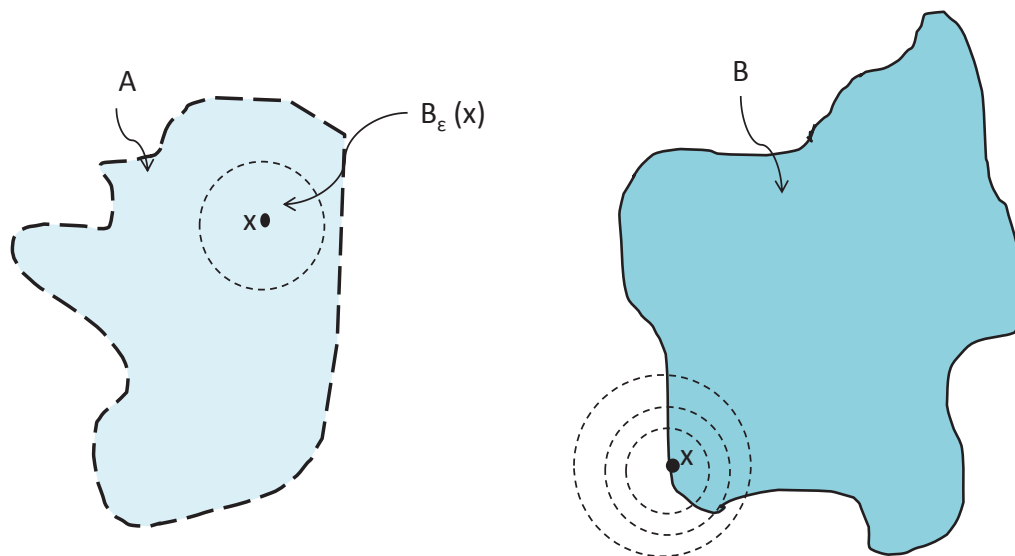


Figure 1:  $A$  is open: for every  $x \in A$  there is some  $\epsilon > 0$  such that  $B_\epsilon(x) \subseteq A$ .  $B$  is not open: for  $x$  depicted in the picture  $\nexists \epsilon > 0$  such that  $B_\epsilon(x) \subseteq B$ .

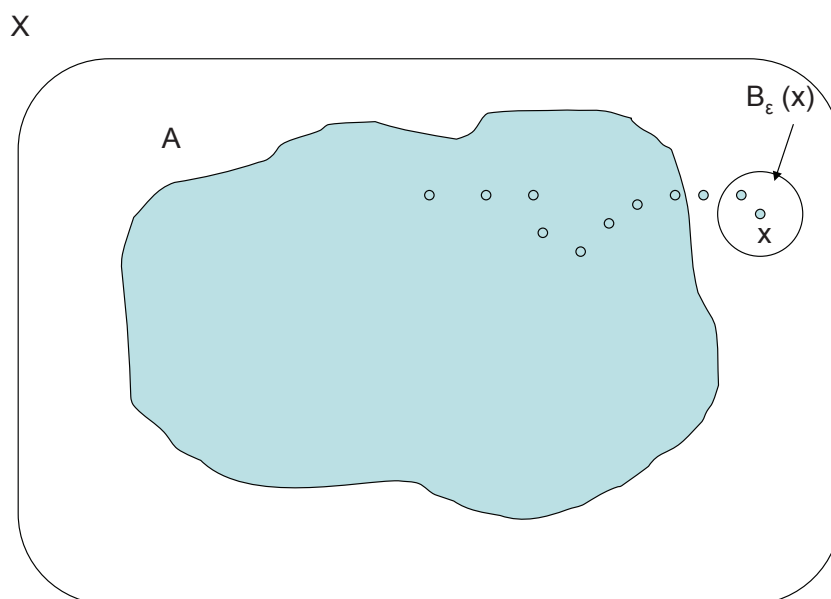


Figure 2: Sequences and closed sets

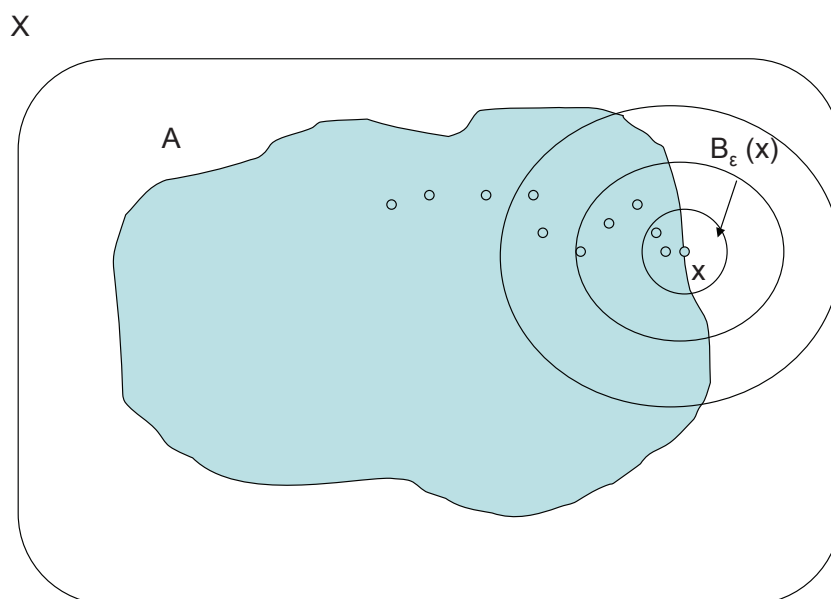


Figure 3: Sequences and closed sets

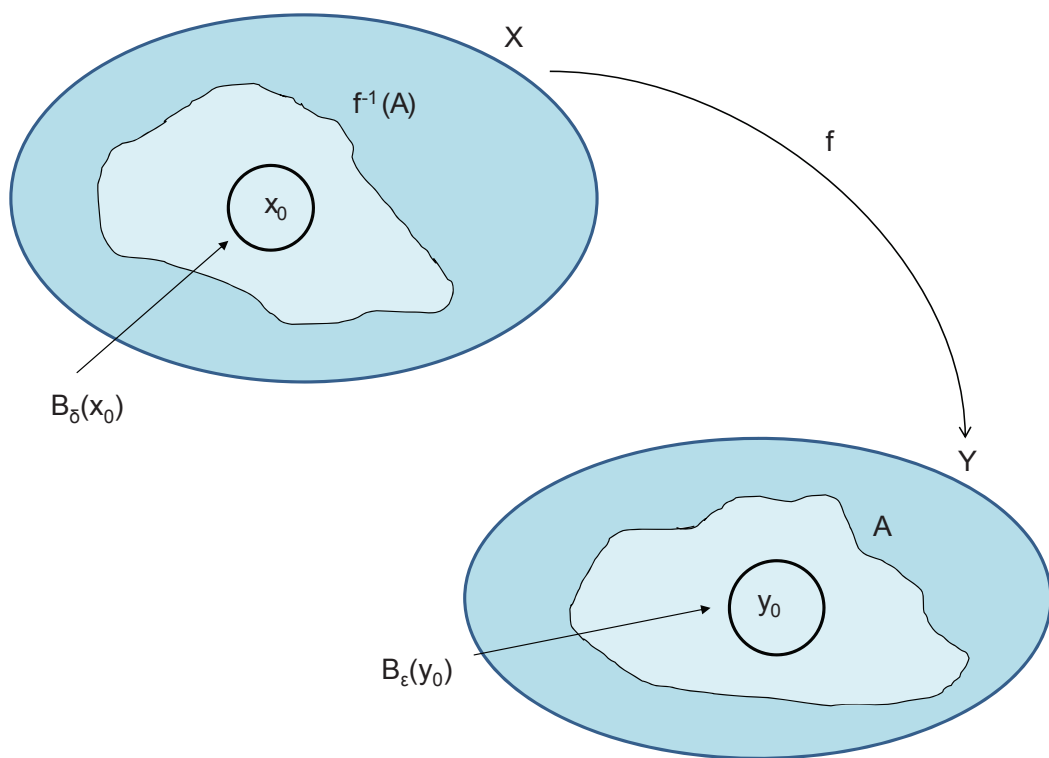


Figure 4: Proof of Theorem 6.

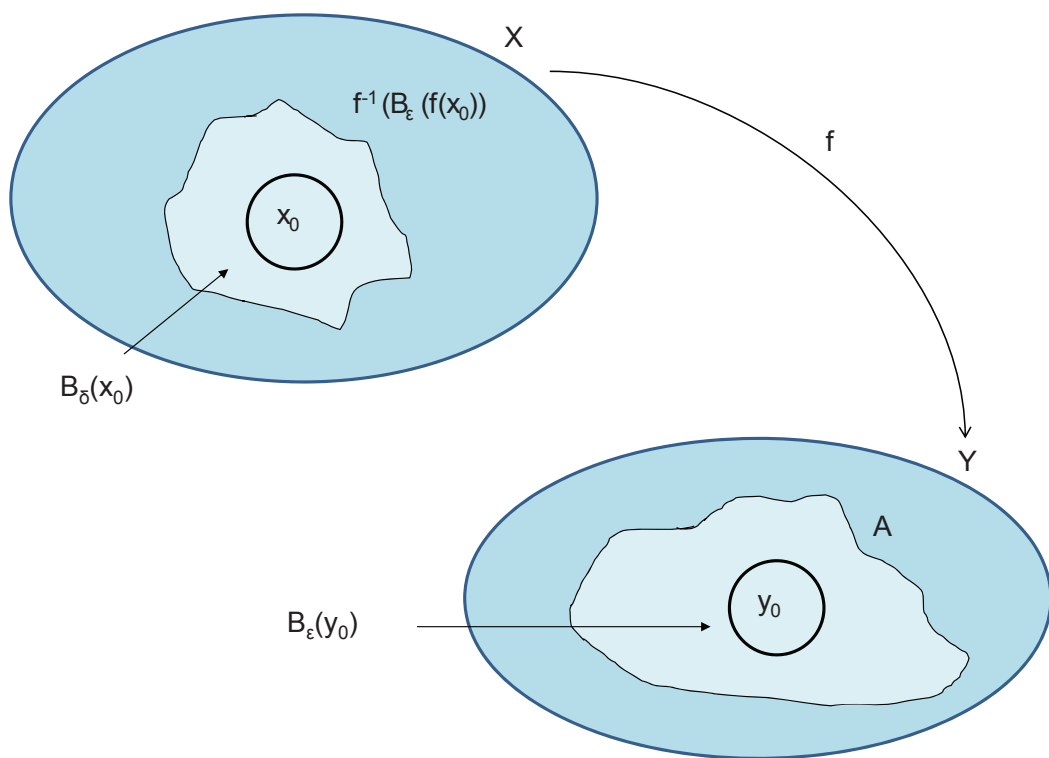


Figure 5: Proof of Theorem 6.

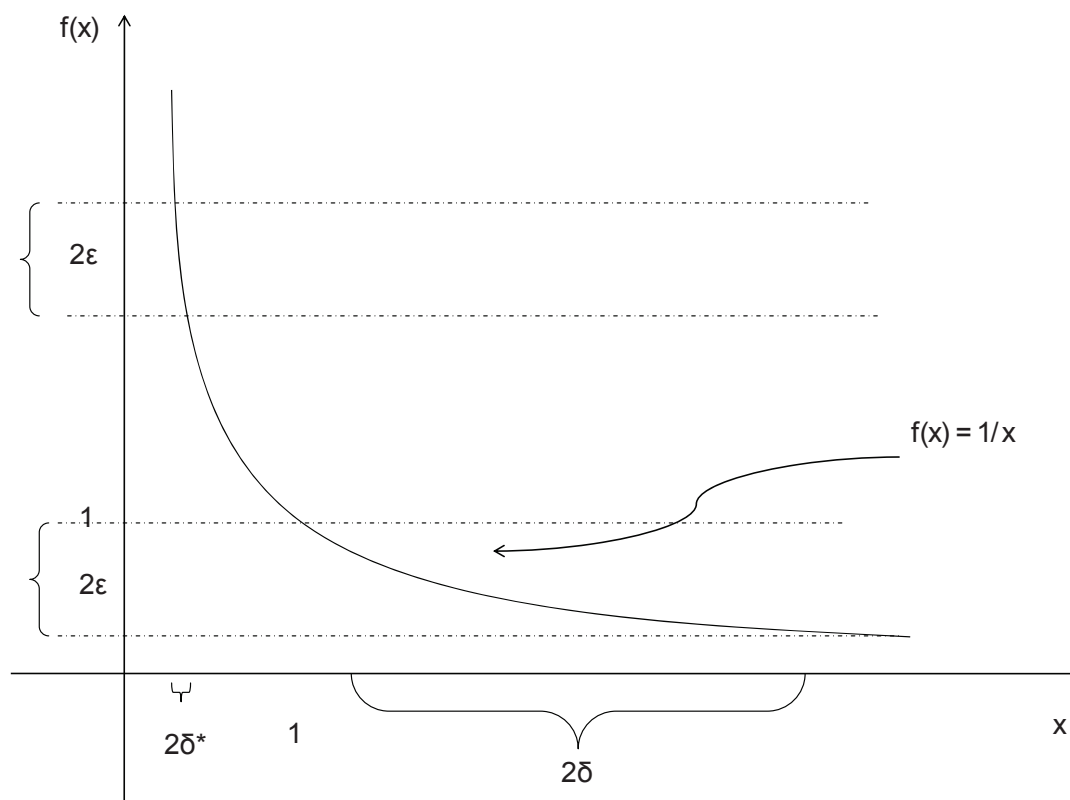


Figure 6:  $f(x) = \frac{1}{x}$  is not uniformly continuous.