Economics 204 Summer/Fall 2025

Lecture 6–Monday August 4, 2025

Section 2.8. Compactness

Definition 1 A collection of sets

$$\mathcal{U} = \{U_\lambda : \lambda \in \Lambda\}$$

in a metric space (X, d) is an *open cover* of A if U_λ is open for all $\lambda \in \Lambda$ and

$$\cup_{\lambda \in \Lambda} U_\lambda \supseteq A$$

Notice that Λ may be finite, countably infinite, or uncountable.

Definition 2 A set A in a metric space is *compact* if every open cover of A contains a finite subcover of A. In other words, if $\{U_\lambda : \lambda \in \Lambda\}$ is an open cover of A, there exist $n \in \mathbf{N}$ and $\lambda_1, \cdots, \lambda_n \in \Lambda$ such that

$$A \subseteq U_{\lambda_1} \cup \cdots \cup U_{\lambda_n}$$

It is important to understand what this definition does *not* say. In particular, it does not say "A has a finite open cover;" note that every set is contained in X, and X is open, so every set has a cover consisting of exactly one open set. Like the ε-δ definition of continuity, in which you are given an arbitrary $\varepsilon > 0$ and are challenged to specify an appropriate δ, here you are given an arbitrary open cover and challenged to specify a finite subcover of the given open cover.

Example: $(0,1]$ is not compact in $\mathbf{E}^1$. To see this, let

$$\mathcal{U} = \left\{U_m = \left(\frac{1}{m}, 2\right) : m \in \mathbf{N}\right\}$$

Then

$$\cup_{m\in\mathbf{N}} U_m = (0,2) \supset (0,1]$$

Given any finite subset $\{U_{m_1}, \ldots, U_{m_n}\}$ of $\mathcal{U}$, let

$$m = \max\{m_1, \ldots, m_n\}$$

Then

$$\cup_{i=1}^n U_{m_i} = U_m = \left(\frac{1}{m}, 2\right) \not\supseteq (0,1]$$

so $(0,1]$ is not compact. See Figure 1.

Note that this argument does not work for $[0,1]$. Given an open cover $\{U_\lambda : \lambda \in \Lambda\}$, there must be some $\lambda \in \Lambda$ such that $0 \in U_\lambda$, and therefore $U_\lambda \supseteq [0,\varepsilon)$ for some $\varepsilon > 0$, and a finite number of the U_m's we used to cover $(0,1]$ would cover the interval $(\varepsilon, 1]$. This is not a proof that $[0,1]$ is compact, since we need to show that *every* open cover has a finite subcover, but it is suggestive, and we will soon see that $[0,1]$ is indeed compact.

Example: $[0,\infty)$ is closed but not compact. To see that $[0,\infty)$ is not compact, let

$$\mathcal{U} = \{U_m = (-1, m) : m \in \mathbf{N}\}$$

Given any finite subset

$$\{U_{m_1}, \ldots, U_{m_n}\}$$

of $\mathcal{U}$, let

$$m = \max\{m_1, \ldots, m_n\}$$

Then

$$U_{m_1} \cup \cdots \cup U_{m_n} = (-1, m) \not\supseteq [0, \infty)$$

See Figure 2.

Theorem 3 (Thm. 8.14) *Every closed subset A of a compact metric space (X, d) is compact.*

Proof: Let $\{U_\lambda : \lambda \in \Lambda\}$ be an open cover of A. In order to use the compactness of X, we need to produce an open cover of X. There are two ways to do this:

$$\begin{aligned} U'_\lambda &= U_\lambda \cup (X \setminus A) \\ \Lambda' &= \Lambda \cup \{\lambda_0\},\ U_{\lambda_0} = X \setminus A \end{aligned}$$

We choose the first path, and let

$$U'_\lambda = U_\lambda \cup (X \setminus A)$$

See Figures 3 and 4.

Since A is closed, $X \setminus A$ is open; since U_λ is open, so is U'_λ. Then $x \in X \Rightarrow x \in A$ or $x \in X \setminus A$. If $x \in A$, $\exists \lambda \in \Lambda$ s.t. $x \in U_\lambda \subseteq U'_\lambda$. If instead $x \in X \setminus A$, then $\forall \lambda \in \Lambda$, $x \in U'_\lambda$. Therefore, $X \subseteq \cup_{\lambda \in \Lambda} U'_\lambda$, so $\{U'_\lambda : \lambda \in \Lambda\}$ is an open cover of X.

Since X is compact,

$$\exists \lambda_1, \ldots, \lambda_n \in \Lambda \text{ s.t. } X \subseteq U'_{\lambda_1} \cup \cdots \cup U'_{\lambda_n}$$

Then

$$\begin{aligned} a \in A &\Rightarrow a \in X \\ &\Rightarrow a \in U'_{\lambda_i} \text{ for some } i \\ &\Rightarrow a \in U_{\lambda_i} \cup (X \setminus A) \\ &\Rightarrow a \in U_{\lambda_i} \end{aligned}$$

so

$$A \subseteq U_{\lambda_1} \cup \cdots \cup U_{\lambda_n}$$

Thus A is compact. ■

As the second example above illustrates, a closed subset of a metric space need not be compact. The converse is always true, however.

Theorem 4 (Thm. 8.15) *If A is a compact subset of the metric space (X, d), then A is closed.*

Proof: Suppose by way of contradiction that A is not closed. Then $X \setminus A$ is not open, so we can find a point $x \in X \setminus A$ such that, for every $\varepsilon > 0$, $A \cap B_\varepsilon(x) \neq \emptyset$, and hence $A \cap B_\varepsilon[x] \neq \emptyset$. For $n \in \mathbf{N}$, let

$$U_n = X \setminus B_{1/n}[x]$$

See Figure 5. Each U_n is open, and

$$\cup_{n \in \mathbf{N}} U_n = X \setminus \{x\} \supseteq A$$

since $x \notin A$. Therefore, $\{U_n : n \in \mathbf{N}\}$ is an open cover for A. Since A is compact, there is a finite subcover $\{U_{n_1}, \ldots, U_{n_k}\}$. Let $n = \max\{n_1, \ldots, n_k\}$. Then

$$\begin{aligned} U_n &= X \setminus B_{1/n}[x] \\ &\supseteq X \setminus B_{1/n_j}[x] \ (j = 1, \ldots, k) \\ U_n &\supseteq \cup_{j=1}^k U_{n_j} \\ &\supseteq A \end{aligned}$$

But $A \cap B_{1/n}[x] \neq \emptyset$, so $A \not\subseteq X \setminus B_{1/n}[x] = U_n$. This is a contradiction, which proves that A is closed. ■

Next we look at a sequential notion of compactness.

Definition 5 A set A in a metric space (X, d) is *sequentially compact* if every sequence of elements of A contains a convergent subsequence whose limit lies in A.

This gives rise to a sequential characterization of compactness for metric spaces.

Theorem 6 (Thms. 8.5, 8.11) *A set A in a metric space (X, d) is compact if and only if it is sequentially compact.*

Proof: Suppose A is compact. We will show that A is sequentially compact. If not, we can find a sequence $\{x_n\}$ of elements of A such that no subsequence converges to *any* element of A. Recall that a is a cluster point of the sequence $\{x_n\}$ means that

$$\forall \varepsilon > 0 \ \ \{n : x_n \in B_\varepsilon(a)\} \text{ is infinite}$$

and this is equivalent to the statement that there is a subsequence $\{x_{n_k}\}$ converging to a. Thus, no element $a \in A$ can be a cluster point for $\{x_n\}$, and hence

$$\forall a \in A \;\; \exists \varepsilon_a > 0 \text{ s.t. } \{n : x_n \in B_{\varepsilon_a}(a)\} \text{ is finite} \tag{1}$$

Then

$$\{B_{\varepsilon_a}(a) : a \in A\}$$

is an open cover of A (if A is uncountable, it will be an uncountable open cover). Since A is compact, there is a finite subcover

$$\left\{B_{\varepsilon_{a_1}}(a_1), \ldots, B_{\varepsilon_{a_m}}(a_m)\right\}$$

Then

$$\begin{aligned}
\mathbf{N} &= \{n : x_n \in A\} \\
&\subseteq \left\{n : x_n \in \left(B_{\varepsilon_{a_1}}(a_1) \cup \cdots \cup B_{\varepsilon_{a_m}}(a_m)\right)\right\} \\
&= \{n : x_n \in B_{\varepsilon_{a_1}}(a_1)\} \cup \cdots \cup \{n : x_n \in B_{\varepsilon_{a_m}}(a_m)\}
\end{aligned}$$

so $\mathbf{N}$ is contained in a finite union of sets, each of which is finite by Equation (1). Thus, $\mathbf{N}$ must be finite, a contradiction which proves that A is sequentially compact.

For the converse, see de la Fuente. ■

Next we explore connections between compactness and notions of boundedness.

Definition 7 A set A in a metric space (X, d) is *totally bounded* if, for every $\varepsilon > 0$,

$$\exists x_1, \ldots, x_n \in A \text{ s.t. } A \subseteq \cup_{i=1}^{n} B_\varepsilon(x_i)$$

This is the standard definition; de la Fuente's definition is equivalent to this. See the comments in the *Corrections* handout for further discussions.

Example: Take $A = [0, 1]$ with the Euclidean metric. Given $\varepsilon > 0$, let $n > \frac{1}{\varepsilon}$. Then we may take

$$x_1 = \frac{1}{n}, x_2 = \frac{2}{n}, \ldots, x_{n-1} = \frac{n-1}{n}$$

Then $[0, 1] \subset \cup_{k=1}^{n-1} B_\varepsilon(\frac{k}{n})$.

Example: Consider $X = [0, 1]$ with the discrete metric

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

X is not totally bounded. To see this, take $\varepsilon = \frac{1}{2}$. Then for any x, $B_\varepsilon(x) = \{x\}$, so given any finite set $x_1, \ldots, x_n$,

$$\cup_{i=1}^{n} B_\varepsilon(x_i) = \{x_1, \ldots, x_n\} \not\supseteq [0, 1]$$

However, X is bounded because $X = B_2(0)$.

Note that any totally bounded set in a metric space (X, d) is also bounded. To see this, let $A \subset X$ be totally bounded. Then $\exists x_1, \ldots, x_n \in A$ such that $A \subset B_1(x_1) \cup \cdots \cup B_1(x_n)$. Let

$$M = 1 + d(x_1, x_2) + \cdots + d(x_{n-1}, x_n)$$

Then $M < \infty$. Now fix $a \in A$. We claim $d(a, x_1) < M$. To see this, notice that there is some $n_a \in \{1, \ldots, n\}$ for which $a \in B_1(x_{n_a})$. Then

$$d(a, x_1) \leq d(a, x_{n_a}) + \sum_{k=1}^{n} d(x_k, x_{k+1})$$

$$
\begin{aligned}
&< 1 + \sum_{k=1}^{n} d(x_k, x_{k+1}) \\
&= M
\end{aligned}
$$

See also Figure 6.

Remark 8 Fix ε and consider the open cover

$$\mathcal{U}_\varepsilon = \{B_\varepsilon(a) : a \in A\}$$

If A is compact, then every open cover of A has a finite subcover; in particular, $\mathcal{U}_\varepsilon$ must have a finite subcover, but this just says that A is totally bounded.

Theorem 9 (Thm. 8.16) *Let A be a subset of a metric space (X, d). Then A is compact if and only if A is complete and totally bounded.*

Proof: Here is a sketch of the proof; see de la Fuente for details. Compact implies totally bounded (Remark 8). Suppose $\{x_n\}$ is a Cauchy sequence in A. Since A is compact, A is sequentially compact, hence $\{x_n\}$ has a convergent subsequence $x_{n_k} \to a \in A$. Since $\{x_n\}$ is Cauchy, $x_n \to a$ (why?), so A is complete.

Conversely, suppose A is complete and totally bounded. Let $\{x_n\}$ be a sequence in A. Because A is totally bounded, we can extract a Cauchy subsequence $\{x_{n_k}\}$ (why?). Because A is complete, $x_{n_k} \to a$ for some $a \in A$, which shows that A is sequentially compact and hence compact. ■

From lecture 5, we know that a subset of a complete metric space is complete if and only if it is closed. So for a complete metric space, we have the following alternative characterization of compactness.

Corollary 10 *Let A be a subset of a complete metric space (X, d). Then A is compact if and only if it is closed and totally bounded.*

Notice that by putting these results together we conclude that a compact subset of a metric space must be closed and bounded.

Example: $[0, 1]$ is compact in $\mathbf{E}^1$. To see this, note that $\mathbf{E}^1$ is complete, and $[0, 1] \subset \mathbf{E}^1$ is closed and totally bounded.

In $\mathbf{R}^n$ we can simplify this characterization even further by the following extremely important results.

Theorem 11 (Thm. 8.19, Heine-Borel) *If $A \subseteq \mathbf{E}^1$, then A is compact if and only if A is closed and bounded.*

Proof: Let A be a closed, bounded subset of $\mathbf{R}$. Then $A \subseteq [a, b]$ for some interval $[a, b]$. Let $\{x_n\}$ be a sequence of elements of $[a, b]$. By the Bolzano-Weierstrass Theorem, $\{x_n\}$ contains a convergent subsequence with limit $x \in \mathbf{R}$. Since $[a, b]$ is closed, $x \in [a, b]$. Thus, we have shown that $[a, b]$ is sequentially compact, hence compact. A is a closed subset of $[a, b]$, hence A is compact.

Conversely, if A is compact, then A is closed and totally bounded, hence closed and bounded. ■

Theorem 12 (8.20, Heine-Borel) *If $A \subseteq \mathbf{E}^n$, then A is compact if and only if A is closed and bounded.*

Proof: See de la Fuente.■

Example: The closed interval

$$[a,b] = \{x \in \mathbf{R}^n : a_i \le x_i \le b_i \text{ for each } i = 1, \dots, n\}$$

is compact in $\mathbf{E}^n$ for any $a, b \in \mathbf{R}^n$.

Next we study the implications of compactness for continuous functions, and derive a general version of the Extreme Value Theorem.

Theorem 13 (Thm. 8.21) *Let (X,d) and (Y,ρ) be metric spaces. If $f : X \to Y$ is continuous and C is a compact subset of (X,d), then $f(C)$ is compact in (Y,ρ).*

Proof: There is a proof in de la Fuente using sequential compactness. Here we give an alternative proof using directly the open cover definition of compactness:

Let $\{U_\lambda : \lambda \in \Lambda\}$ be an open cover of $f(C)$. For each $c \in C$, $f(c) \in f(C)$ so $f(c) \in U_{\lambda_c}$ for some $\lambda_c \in \Lambda$, that is, $c \in f^{-1}(U_{\lambda_c})$. Thus the collection $\{f^{-1}(U_\lambda) : \lambda \in \Lambda\}$ is a cover of C; in addition, since f is continuous, each set $f^{-1}(U_\lambda)$ is open in C, so $\{f^{-1}(U_\lambda) : \lambda \in \Lambda\}$ is an open cover of C. Since C is compact, there is a finite subcover

$$\left\{f^{-1}(U_{\lambda_1}), \dots, f^{-1}(U_{\lambda_n})\right\}$$

of C. Given $x \in f(C)$, there exists $c \in C$ such that $f(c) = x$, and $c \in f^{-1}(U_{\lambda_i})$ for some i, so $x \in U_{\lambda_i}$. Thus, $\{U_{\lambda_1}, \dots, U_{\lambda_n}\}$ is a finite subcover of $f(C)$, so $f(C)$ is compact. ■

Corollary 14 (Thm. 8.22, Extreme Value Theorem) *Let C be a compact set in a metric space (X, d), and suppose $f : C \to \mathbf{R}$ is continuous. Then f is bounded on C and attains its minimum and maximum on C.*

Proof: Since C is compact and f is continuous, $f(C) \subset \mathbf{R}$ is compact, hence closed and bounded. Let $M = \sup f(C)$; $M < \infty$. Then $\forall m > 0$ there exists $y_m \in f(C)$ such that

$$M - \frac{1}{m} \leq y_m \leq M$$

So $y_m \to M$ and $\{y_m\} \subseteq f(C)$. Since $f(C)$ is closed, $M \in f(C)$, i.e. there exists $c \in C$ such that $f(c) = M = \sup f(C)$, so f attains its maximum at c. The proof for the minimum is similar.■

Theorem 15 (Thm. 8.24) *Let (X, d) and (Y, ρ) be metric spaces, C a compact subset of X, and $f : C \to Y$ a continuous function. Then f is uniformly continuous on C.*

Proof: Fix $\varepsilon > 0$. We ignore X and consider f as defined on the metric space (C, d). Given $c \in C$, find $\delta(c) > 0$ such that

$$x \in C,\ d(x, c) < 2\delta(c) \Rightarrow \rho(f(x), f(c)) < \frac{\varepsilon}{2}$$

Let

$$U_c = B_{\delta(c)}(c)$$

Then

$$\{U_c : c \in C\}$$

is an open cover of C. Since C is compact, there is a finite subcover

$$\{U_{c_1}, \ldots, U_{c_n}\}$$

Let

$$\delta = \min\{\delta(c_1), \ldots, \delta(c_n)\}$$

Given $x, y \in C$ with $d(x, y) < \delta$, note that $x \in U_{c_i}$ for some $i \in \{1, \ldots, n\}$, so $d(x, c_i) < \delta(c_i)$.

$$\begin{aligned} d(y, c_i) &\leq d(y, x) + d(x, c_i) \\ &< \delta + \delta(c_i) \\ &\leq \delta(c_i) + \delta(c_i) \\ &= 2\delta(c_i) \end{aligned}$$

so

$$\begin{aligned} \rho(f(x), f(y)) &\leq \rho(f(x), f(c_i)) + \rho(f(c_i), f(y)) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

which proves that f is uniformly continuous.■

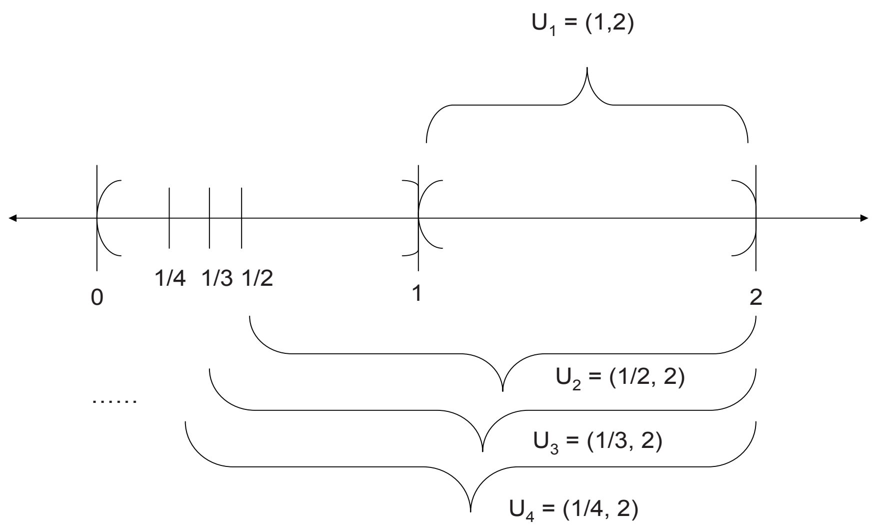

Figure 1: $(0, 1]$ is not compact: $\{U_n : n \in \mathbf{N}\}$ covers $(0, 1]$ but has no finite subcover.

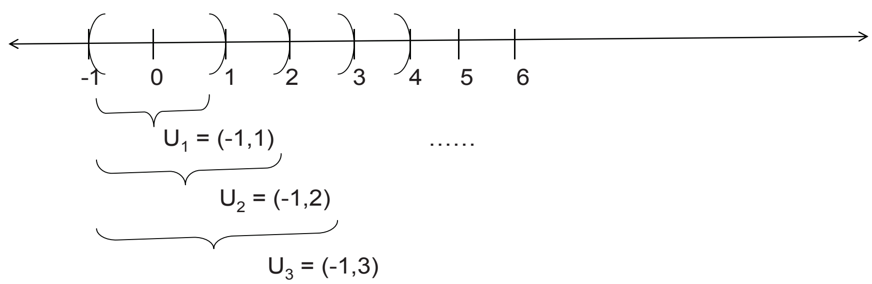

Figure 2: $[0, \infty)$ is closed but not compact: $\{U_n : n \in \mathbf{N}\}$ covers $[0, \infty)$ but has no finite subcover.

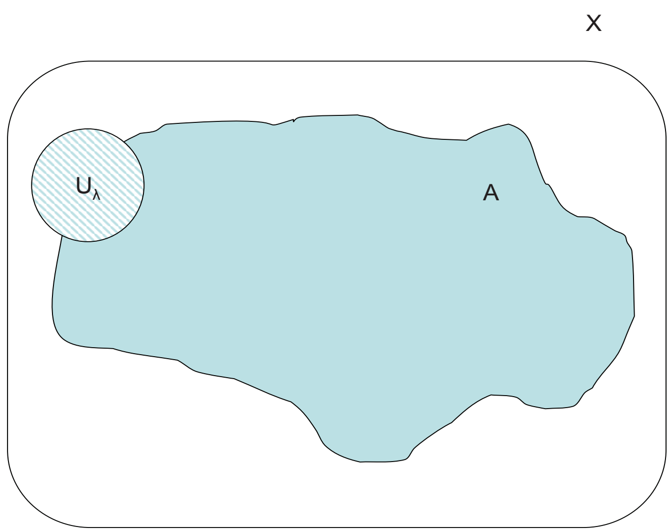

Figure 3: $\{U_\lambda : \lambda \in \Lambda\}$ is an open cover of A.

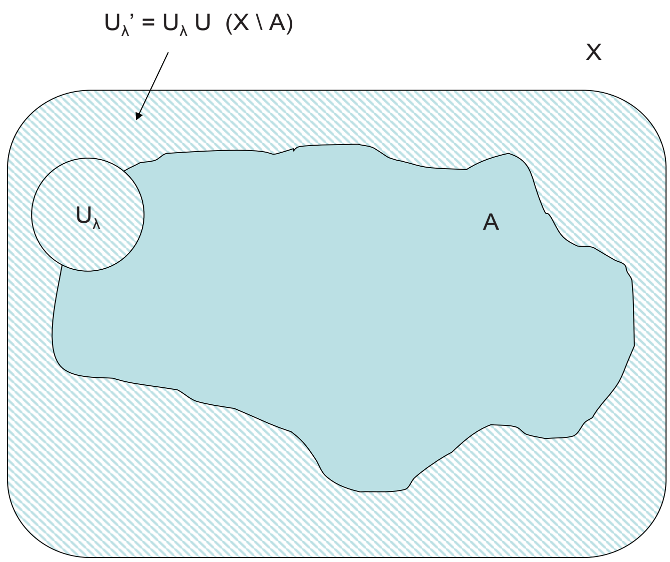

Figure 4: $\{U'_\lambda : \lambda \in \Lambda\}$ is an open cover of X with $U'_\lambda = U_\lambda \cup (X \setminus A)$.

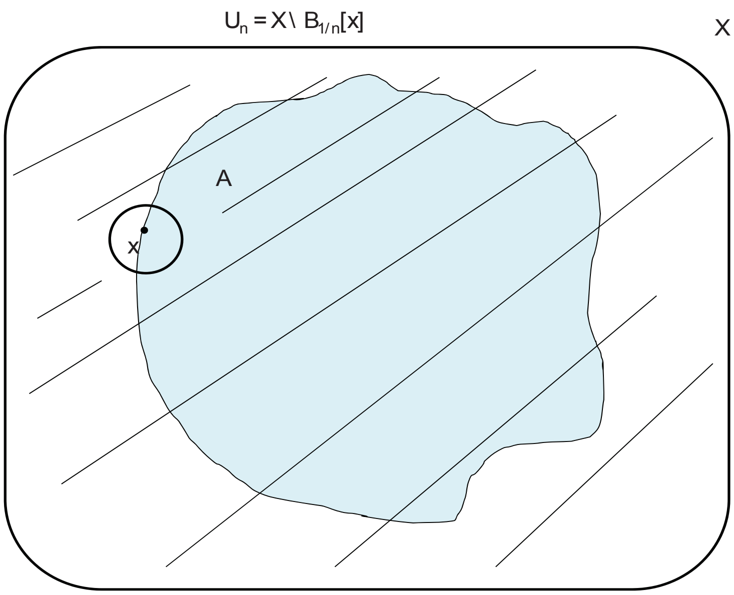

Figure 5: $\{U_n : n \in \mathbf{N}\}$ with $U_n = X \setminus B_{\frac{1}{n}}[x]$ is an open cover of A.

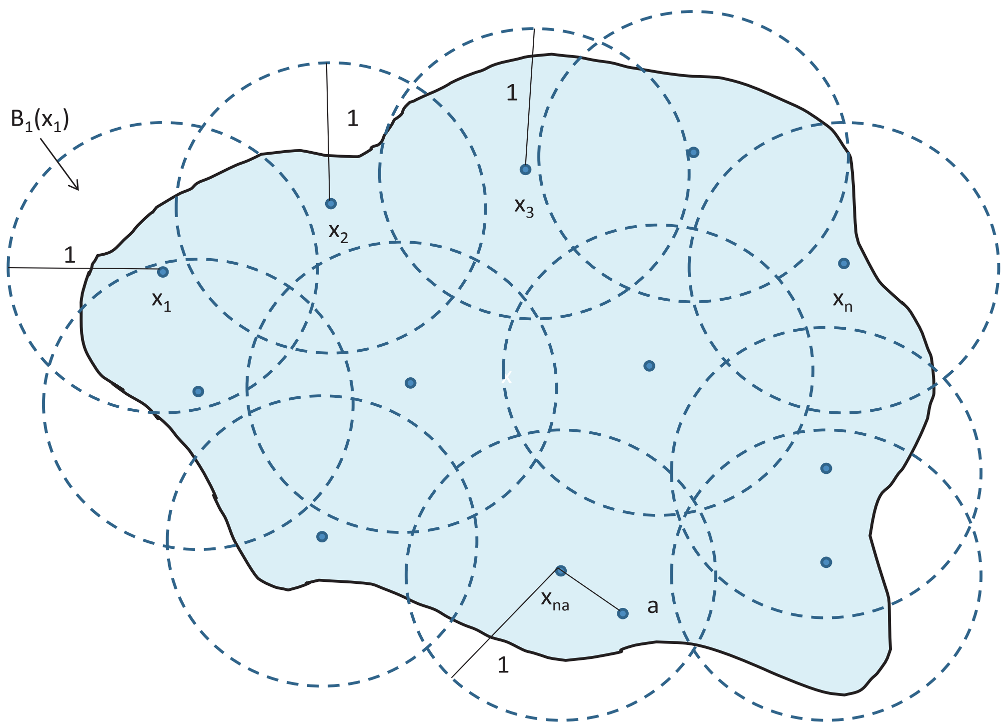

Figure 6: Every totally bounded subset of a metric space is bounded.