Economics 204–Final Exam–August 18, 2009, 9am-12pm
Each of the four questions is worth 25% of the total
Please use three *separate* blue/greenbooks, one for each of the three Parts

Part I

1. Prove that if X is a compact metric space and C is a closed subset of X, then C is compact.

2. Consider the function

$$f(x,y) = 4x^2 + 3y^2 + 2xy + (x+y)^4$$

   (a) Show that $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is a critical point of f.

   (b) Determine whether f has a local max, a local min, or neither at $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

   (c) Does f have a global max, a global min, or neither at $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$?

Part II

3. Consider the function $z : (0,\infty) \times \mathbf{R} \to \mathbf{R}$ defined by

$$z(p,\alpha) = \log p + \frac{2}{9p} - p + \alpha$$

and the equilibrium correspondence

$$E(\alpha) = \{p \in (0,\infty) : z(p,\alpha) = 0\}$$

Here, $\log t$ denotes the natural logarithm of t, so that $\frac{d}{dt}\log t = \frac{1}{t}$. In answering the following questions, it may be useful to you to know that $\lim_{t\to 0, t>0} t\log t = 0$ and that $\lim_{t\to\infty} \frac{\log t}{t} = 0$.

   (a) Let

$$A = \left\{ \alpha \in \mathbf{R} : \exists p^* \in (0,\infty) \ \left((z(p^*,\alpha) = 0) \wedge \left(\left.\frac{\partial z}{\partial p}\right|_{(p^*,\alpha)} = 0 \right) \right) \right\}$$

   In other words, A is the set of α such that 0 is a critical value of $z(\cdot,\alpha)$. Using the Transversality Theorem, show that A is a set of Lebesgue measure zero.

   (b) Using the Implicit Function Theorem, prove directly from the definition that if $\alpha \notin A$, then the equilibrium correspondence E is lower hemicontinuous at α.

   (c) Prove that the equilibrium correspondence E is upper hemicontinuous at all α.

   (d) Find the local minima and local maxima of $z(\cdot,\alpha)$.

(e) Sketch the equilibrium correspondence E.

Part III

4. Suppose U is a basis (not necessarily the standard basis) of $\mathbf{R}^n$ and $T \in L(\mathbf{R}^n, \mathbf{R}^n)$. Suppose $Mtx_U(T)$ is symmetric. Consider the quadratic form $f(x) = T(x) \cdot x$, where $\cdot$ denotes the usual dot product in $\mathbf{R}^n$.

   (a) Suppose first that U is the standard basis of $\mathbf{R}^n$.

      i. What can you say about the eigenvalues and eigenvectors of the linear transformation T?

      ii. Is there a basis W of $\mathbf{R}^n$ in which $Mtx_W(T)$ is diagonal?

      iii. Give conditions on the eigenvalues of T that ensure that $f(x)$ has a global maximum, a global minimum, or a saddle at $x = 0$.

   (b) Suppose now that U is an orthonormal basis, but not necessarily the standard basis, of $\mathbf{R}^n$. Which of your answers in part (a) still hold? Explain why.

   (c) Suppose now that U is any basis, but not necessarily an orthonormal basis, of $\mathbf{R}^n$. Which of your answers in part (a) still hold? Explain why.