Economics 204–Final Exam–August 29, 2005, 6-9pm
Each question is worth 20% of the total
Please use separate bluebooks for each of the three Parts

Part I

1. (a) Define the term "contraction."

   (b) State the Contraction Mapping Theorem.

   (c) Give the following portion of the proof of the Contraction Mapping Theorem: Given a contraction, start with an arbitrary point, and show how to construct a sequence of points and prove that the sequence converges to a limit which is a candidate fixed point. You need not prove that the limit is in fact a fixed point.

2. Suppose $T \in L(\mathbf{R}^n, \mathbf{R}^n)$ is a linear transformation, and let V be any basis of $\mathbf{R}^n$. Show that λ is an eigenvalue of T if and only if λ is an eigenvalue of $Mtx_V(T)$.

Part II

3. Consider the function

$$f(x,y) = x^3 + y^3 + 6x^2 + 8y^2 - 2\sqrt{3}xy + (2\sqrt{3} - 15)x + (2\sqrt{3} - 19)y$$

   (a) Compute the first order conditions for a local maximum or minimum of f. Show that the first order conditions are satisfied at the point $(x_0, y_0) = (1, 1)$.

   (b) Compute $D^2 f(x_0, y_0)$ and give the quadratic Taylor polynomial for f at the point (x_0, y_0).

   (c) Find the eigenvalues of $D^2 f(x_0, y_0)$ and determine whether f has a local max, a local min, or a saddle at (x_0, y_0).

   (d) Does f have a global max, a global min, or neither, at (x_0, y_0)?

   (e) Find an orthonormal basis for $\mathbf{R}^2$ consisting of eigenvectors $D^2 f(x_0, y_0)$. Rewrite the quadratic Taylor polynomial for f at the point (x_0, y_0) in terms of this basis.

   (f) Use the quadratic Taylor polynomial found in part (d) to describe the approximate shape of the level sets of f near the point (x_0, y_0).

Part III

4. Show that a compact subset S of a metric space (X, d) is closed. To get full credit, you must directly use the open cover definition of compactness.

Please turn over

5. Consider the sequence of functions $f_n : [0,1] \to \mathbf{R}$ defined by

$$f_n(t) = \begin{cases} nt & \text{if } t \in \left[0, \frac{1}{n}\right] \\ 1 & \text{if } t \in \left(\frac{1}{n}, 1\right] \end{cases}$$

(a) Show that f_n is continuous for each n.

(b) Show that for each $t \in [0,1]$, $\lim_{n\to\infty} f_n(t)$ exists.

(c) Define the function $f : [0,1] \to \mathbf{R}$ by

$$f(t) = \lim_{n\to\infty} f_n(t)$$

Show that f is not continuous.

(d) We say a sequence of functions $f_n : [0,1] \to \mathbf{R}$ *converges uniformly* to a function $f : [0,1] \to \mathbf{R}$ if

$$\forall_{\varepsilon>0} \exists_{N\in\mathbf{N}} \forall_{t\in[0,1]}\ n > N \Rightarrow |f_n(t) - f(t)| < \varepsilon$$

Does f_n converge uniformly to f? Justify your answer with a proof.