

## Section 2.8, Compactness

**Definition 1** A collection of sets

$$\mathcal{U} = \{U_\lambda : \lambda \in \Lambda\}$$

in a metric space  $(X, d)$  is an *open cover* of  $A$  if  $U_\lambda$  is open for all  $\lambda \in \Lambda$  and

$$\cup_{\lambda \in \Lambda} U_\lambda \supseteq A$$

( $\Lambda$  may be finite, countably infinite, or uncountable.)

A set  $A$  in a metric space is *compact* if every open cover of  $A$  contains a finite subcover of  $A$ . In other words, if  $\{U_\lambda : \lambda \in \Lambda\}$  is an open cover of  $A$ , there exist  $n \in \mathbf{N}$  and  $\lambda_1, \dots, \lambda_n \in \Lambda$  such that

$$A \subseteq U_{\lambda_1} \cup \dots \cup U_{\lambda_n}$$

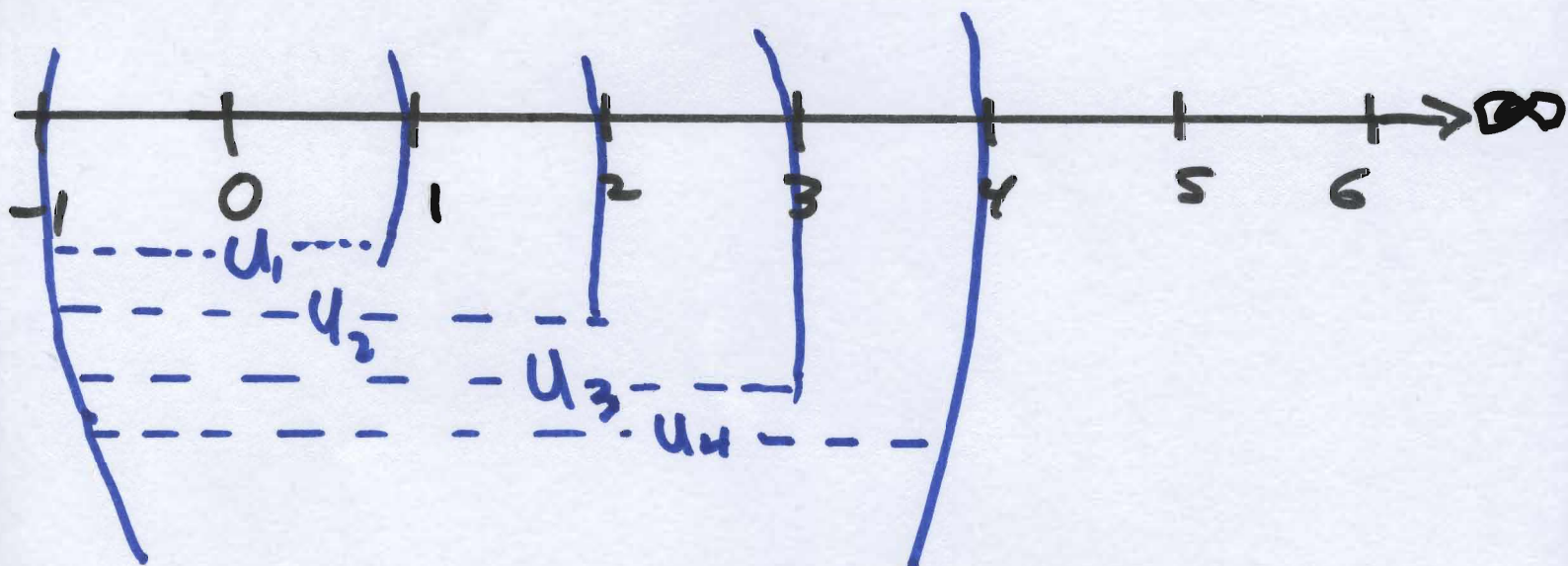
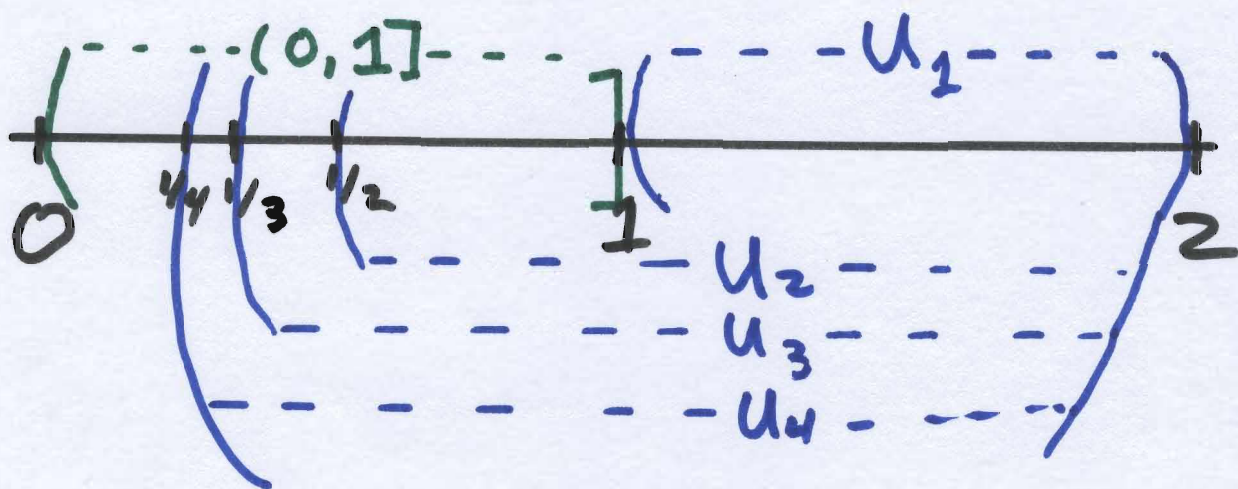
*It is important to understand what this definition does not say. In particular, it does not say “A has a finite open cover;” note that every set is contained in  $X$ , and  $X$  is open, so every set has a cover consisting of exactly one open set. Like the  $\varepsilon$ - $\delta$  definition of continuity, in which you are given an arbitrary  $\varepsilon > 0$  and are challenged to specify an appropriate  $\delta$ , here you are given an arbitrary open cover and challenged to specify a finite subcover of the given open cover.*

*Example:*  $(0, 1]$  is not compact in  $E^1$ . To see this, let

$$\mathcal{U} = \left\{ U_m = \left( \frac{1}{m}, 2 \right) : m \in \mathbf{N} \right\}$$

Then

$$\cup_{m \in \mathbf{N}} U_m = (0, 2) \supset (0, 1]$$



Given any finite subset  $\{U_{m_1}, \dots, U_{m_n}\}$  of  $\mathcal{U}$ , let

$$m = \max\{m_1, \dots, m_n\}$$

Then

$$\cup_{i=1}^n U_{m_i} = U_m = \left(\frac{1}{m}, 2\right) \not\supseteq (0, 1]$$

so  $(0, 1]$  is not compact.

Note that this argument does not work for  $[0, 1]$ . Given an open cover  $\{U_\lambda : \lambda \in \Lambda\}$ , there must be some  $\lambda \in \Lambda$  such that  $0 \in U_\lambda$ , and therefore  $U_\lambda \supseteq [0, \varepsilon)$  for some  $\varepsilon > 0$ , and a finite number of the  $U_m$ 's we used to cover  $(0, 1]$  would cover the interval  $(\varepsilon, 1]$ . This is not a proof that  $[0, 1]$  is compact, since we need to show that *every* open cover has a finite subcover, but it is suggestive, and we will soon see that  $[0, 1]$  is indeed compact.

*Example:*  $[0, \infty)$  is closed but not compact. To see that  $[0, \infty)$  is not compact, let

$$\mathcal{U} = \{U_m = (-1, m) : m \in \mathbf{N}\}$$

Given any finite subset

$$\{U_{m_1}, \dots, U_{m_n}\}$$

of  $\mathcal{U}$ , let

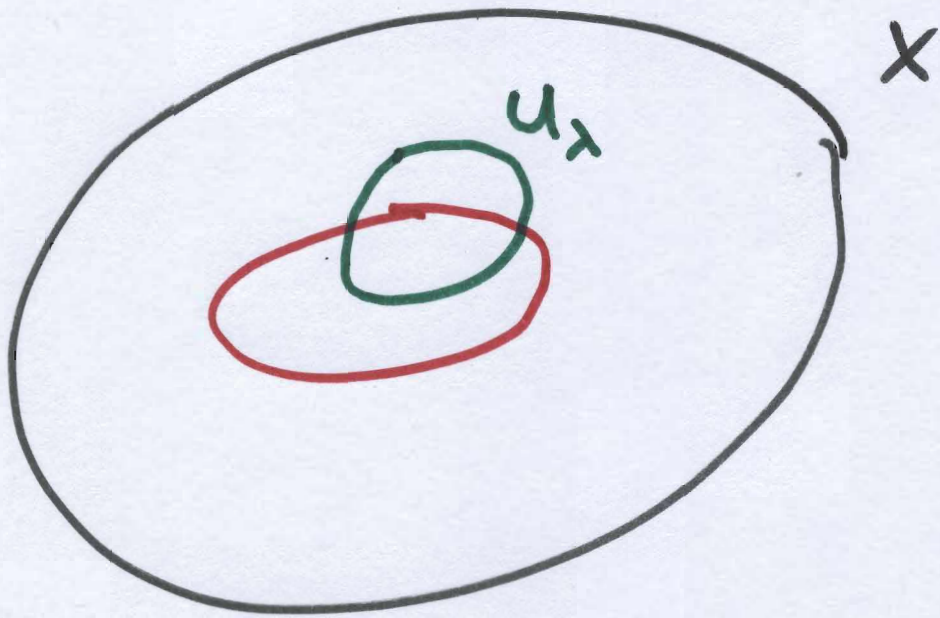
$$m = \max\{m_1, \dots, m_n\}$$

Then

$$U_{m_1} \cup \dots \cup U_{m_n} = (-1, m) \not\supseteq [0, \infty)$$

**Theorem 2 (8.14)** *Every closed subset  $A$  of a compact metric space  $(X, d)$  is compact.*

**Proof:** If you can get past the abstraction (admittedly, a serious hurdle), this is easy. Let  $\{U_\lambda : \lambda \in \Lambda\}$  be an open cover of  $A$ .



$$U_\alpha' = U_\beta \cup (X - A)$$

In order to use the compactness of  $X$ , we need to produce an open cover of  $X$ . There are two ways to do this:

$$U'_\lambda = U_\lambda \cup (X \setminus A)$$

$$\Lambda' = \Lambda \cup \{\lambda_0\}, U_{\lambda_0} = X \setminus A$$

We choose the first path, and let

$$U'_\lambda = U_\lambda \cup (X \setminus A)$$

Since  $A$  is closed,  $X \setminus A$  is open; since  $U_\lambda$  is open, so is  $U'_\lambda$ .

$$\begin{aligned} x \in X &\Rightarrow x \in A \quad \vee \quad x \in X \setminus A \\ &\Rightarrow (\exists \lambda \in \Lambda \ x \in U_\lambda \subseteq U'_\lambda) \quad \vee \quad (\forall \lambda \in \Lambda \ x \in U'_\lambda) \end{aligned}$$

Therefore,  $X \subseteq \cup_{\lambda \in \Lambda} U'_\lambda$ , so  $\{U'_\lambda : \lambda \in \Lambda\}$  is an open cover of  $X$ .

Since  $X$  is compact,

$$\exists \lambda_1, \dots, \lambda_n \in \Lambda \ X \subseteq U'_{\lambda_1} \cup \dots \cup U'_{\lambda_n}$$

so

$$\begin{aligned} a \in A &\Rightarrow a \in X \\ &\Rightarrow a \in U'_{\lambda_i} \text{ for some } i \\ &\Rightarrow a \in U_{\lambda_i} \cup (X \setminus A) \\ &\Rightarrow a \in U_{\lambda_i} \end{aligned}$$

so

$$A \subseteq U_{\lambda_1} \cup \dots \cup U_{\lambda_n}$$

so  $A$  is compact. ■

**Theorem 3 (8.15)** *If  $A$  is a compact subset of the metric space  $(X, d)$ , then  $A$  is closed.*

**Proof:** Suppose  $A$  is not closed. Then  $X \setminus A$  is not open, so we can find a point  $x \in X \setminus A$  such that, for every  $\varepsilon > 0$ ,  $A \cap B_\varepsilon(x) \neq \emptyset$ , and hence  $A \cap B_\varepsilon[x] \neq \emptyset$ . For  $n \in \mathbf{N}$ , let

$$U_n = X \setminus B_{1/n}[x]$$

Each  $U_n$  is open, and

$$\cup_{n \in \mathbf{N}} U_n = X \setminus \{x\} \supseteq A$$

since  $x \notin A$ . Therefore,  $\{U_n : n \in \mathbf{N}\}$  is an open cover for  $A$ . Since  $A$  is compact, there is a finite subcover  $\{U_{n_1}, \dots, U_{n_k}\}$ . Let  $n = \max\{n_1, \dots, n_k\}$ . Then

$$\begin{aligned} U_n &= X \setminus B_{1/n}[x] \\ &\supseteq X \setminus B_{1/n_j}[x] \quad (j = 1, \dots, k) \\ U_n &\supseteq \cup_{j=1}^k U_{n_j} \\ &\supseteq A \end{aligned}$$

But  $A \cap B_{1/n}[x] \neq \emptyset$ , so  $A \not\subseteq X \setminus B_{1/n}[x] = U_n$ , a contradiction which proves that  $A$  is closed. ■

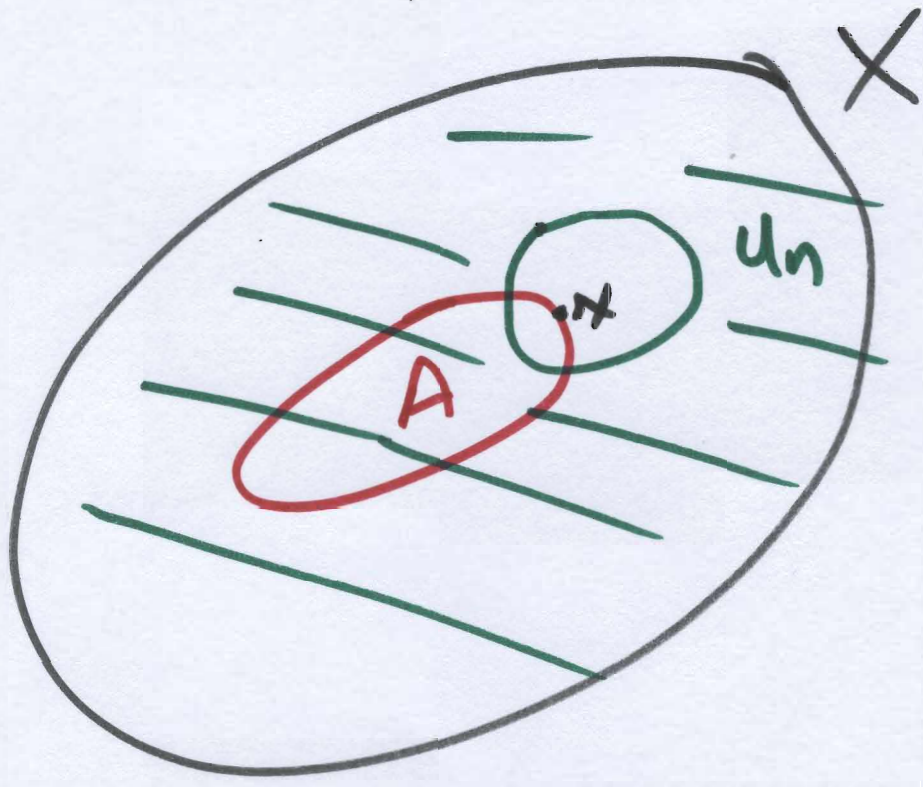
**Definition 4** A set  $A$  in a metric space  $(X, d)$  is *sequentially compact* if every sequence of elements of  $A$  contains a convergent subsequence whose limit lies in  $A$ .

**Theorem 5 (8.5, 8.11)** *A set  $A$  in a metric space  $(X, d)$  is compact if and only if it is sequentially compact.*

**Proof:** Suppose  $A$  is compact. We will show that  $A$  is sequentially compact. If not, we can find a sequence  $\{x_n\}$  of elements of  $A$  such that no subsequence converges to *any* element of  $A$ . Recall that  $a$  is a cluster point of the sequence  $\{x_n\}$  means that

$$\forall \varepsilon > 0 \{n : x_n \in B_\varepsilon(x)\} \text{ is infinite}$$





and this is equivalent to the statement that there is a subsequence  $\{x_{n_k}\}$  converging to  $a$ . Thus, no element of  $a$  can be a cluster point for  $\{x_n\}$ , and hence

$$\forall a \in A \exists \varepsilon_a > 0 \{n : x_n \in B_{\varepsilon_a}(a)\} \text{ is finite} \quad (1)$$

Then

$$\{B_{\varepsilon_a}(a) : a \in A\}$$

is an open cover of  $A$  (if  $A$  is uncountable, it will be an uncountable open cover). Since  $A$  is compact, there is a finite subcover

$$\{B_{\varepsilon_{a_1}}(a_1), \dots, B_{\varepsilon_{a_m}}(a_m)\}$$

Then

$$\begin{aligned} \mathbf{N} &= \{n : x_n \in A\} \\ &\subseteq \{n : x_n \in (B_{\varepsilon_{a_1}}(a_1) \cup \dots \cup B_{\varepsilon_{a_m}}(a_m))\} \\ &= \{n : x_n \in B_{\varepsilon_{a_1}}(a_1)\} \cup \dots \cup \{n : x_n \in B_{\varepsilon_{a_m}}(a_m)\} \end{aligned}$$

so  $\mathbf{N}$  is contained in a finite union of sets, each of which is finite by Equation (1). Thus,  $\mathbf{N}$  must be finite, a contradiction which proves that  $A$  is sequentially compact.

For the converse, see de la Fuente.■

**Definition 6** A set  $A$  in a metric space  $(X, d)$  is *totally bounded* if, for every  $\varepsilon > 0$ ,

$$\exists_{x_1, \dots, x_n \in A} A \subseteq \cup_{i=1}^n B_{\varepsilon}(x_i)$$

(This is the standard definition; de la Fuente's definition is equivalent to this. See the comments in the *Corrections* handout.)

*Example:* Take  $A = [0, 1]$  with the Euclidean metric. Given  $\varepsilon > 0$ , let  $n > \frac{1}{\varepsilon}$ . Then we may take

$$x_1 = \frac{1}{n}, x_2 = \frac{2}{n}, \dots, x_{n-1} = \frac{n-1}{n}$$



*Example:* Consider  $X = [0, 1]$  with the discrete metric

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

$X$  is not totally bounded. To see this, take  $\varepsilon = \frac{1}{2}$ . Then for any  $x$ ,  $B_\varepsilon(x) = \{x\}$ , so given any finite set  $x_1, \dots, x_n$ ,

$$\cup_{i=1}^n B_\varepsilon(x_i) = \{x_1, \dots, x_n\} \not\supseteq [0, 1]$$

However,  $X$  is bounded because  $X = B_2(0)$ .

**Remark 7** Fix  $\varepsilon$  and consider the open cover

$$\mathcal{U}_\varepsilon = \{B_\varepsilon(a) : a \in A\}$$

If  $A$  is compact, then every open cover of  $A$  has a finite subcover; in particular,  $\mathcal{U}_\varepsilon$  must have a finite subcover, but this just says that  $A$  is totally bounded.

**Theorem 8 (8.16)** *Let  $A$  be a subset of a metric space  $(X, d)$ . Then  $A$  is compact if and only if it is complete and totally bounded.*

**Proof:** See de la Fuente.■

**Corollary 9** *Let  $A$  be a subset of a complete metric space  $(X, d)$ . Then  $A$  is compact if and only if it is closed and totally bounded.*

**Theorem 10 (8.19, Heine-Borel)** *If  $A \subseteq E^1$ , then  $A$  is compact if and only if  $A$  is closed and bounded.*

**Proof:** Let  $A$  be a closed, bounded subset of  $\mathbf{R}$ . Then  $A \subseteq [a, b]$  for some interval  $[a, b]$ . Let  $\{x_n\}$  be a sequence of elements of  $[a, b]$ . By the Bolzano-Weierstrass Theorem,  $\{x_n\}$  contains a convergent subsequence

with limit  $x \in \mathbf{R}$ . Since  $[a, b]$  is closed,  $x \in [a, b]$ . Thus, we have shown that  $[a, b]$  is sequentially compact, hence compact.  $A$  is a closed subset of  $[a, b]$ , hence  $A$  is compact.

Conversely, if  $A$  is compact,  $A$  is closed. The argument that showed that  $[0, \infty)$  is not compact is easily adapted to show that compact sets are bounded.■

**Theorem 11 (8.20, Heine-Borel)** *If  $A \subseteq \mathbf{E}^n$ , then  $A$  is compact if and only if  $A$  is closed and bounded.*

**Proof:** See de la Fuente.■

**Theorem 12 (8.21)** *Let  $(X, d)$  and  $(Y, \rho)$  be metric spaces. If  $f : X \rightarrow Y$  is continuous and  $C$  is a compact subset of  $(X, d)$ , then  $f(C)$  is compact in  $(Y, \rho)$ .*

**Proof:** There is a proof in de la Fuente. In Problem 5(a) of Problem Set 3, you are asked to give a proof using directly the open cover definition of compactness. ■

**Corollary 13 (8.22, Extreme Value Theorem)** *Let  $C$  be a compact set in a metric space  $(X, d)$ , and suppose  $f : C \rightarrow \mathbf{R}$  is continuous. Then  $f$  is bounded on  $C$  and attains its minimum and maximum on  $C$ .*

**Proof:**  $f(C)$  is compact by Theorem 8.21, hence closed and bounded. Let  $M = \sup f(C)$ ;  $M < \infty$ . Then there exists  $y_m \in f(C)$  such that

$$M - \frac{1}{m} \leq y_m \leq M$$

so  $M$  is a limit point of  $f(C)$ . Since  $f(C)$  is closed,  $M \in f(C)$ , i.e. there exists  $c \in C$  such that  $f(c) = M = \sup f(C)$ , so  $f$  attains its maximum at  $c$ . The proof for the minimum is similar.■

**Theorem 14 (8.24)** *Let  $(X, d)$  and  $(Y, \rho)$  be metric spaces,  $C$  a compact subset of  $X$ , and  $f : C \rightarrow Y$  continuous. Then  $f$  is uniformly continuous on  $C$ .*

**Proof:** Fix  $\varepsilon > 0$ . We ignore  $X$  and consider  $f$  as defined on the metric space  $(C, d)$ . Given  $c \in C$ , find  $\delta(c) > 0$  such that

$$x \in C, d(x, c) < 2\delta(c) \Rightarrow \rho(f(x), f(c)) < \frac{\varepsilon}{2}$$

Let

$$U_c = B_{\delta(c)}(c)$$

Then

$$\{U_c : c \in C\}$$

is an open cover of  $C$ . Since  $C$  is compact, there is a finite subcover

$$\{U_{c_1}, \dots, U_{c_n}\}$$

Let

$$\delta = \min\{\delta(c_1), \dots, \delta(c_n)\}$$

Given  $x, y \in C$  with  $d(x, y) < \delta$ , note that  $x \in U_{c_i}$  for some  $i \in \{1, \dots, n\}$ , so  $d(x, c_i) < \delta(c_i)$ .

$$\begin{aligned} d(y, c_i) &\leq d(y, x) + d(x, c_i) \\ &< \delta + \delta(c_i) \\ &\leq \delta(c_i) + \delta(c_i) \\ &= 2\delta(c_i) \end{aligned}$$

so

$$\begin{aligned} \rho(f(x), f(y)) &\leq \rho(f(x), f(c_i)) + \rho(f(c_i), f(y)) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

which proves that  $f$  is uniformly continuous. ■