Econ 204 Summer 2009
Problem Set 2
Due in Lecture Tuesday, August 2 2009

1. **Boundary, Exterior and Closure**

   Find the boundary, exterior, and closure of the following sets:

   (a) $\{(x, y) \in \mathbf{R}^2 | x^2 + y^2 > 1\}$

   (b) $\{(x, y) \in \mathbf{R}^2 | x - y = 3\}$

2. **Closed Set**

   Show that $E = \{x \in \mathbf{R}^1 :| x - a |\leq 2\}$ is a closed set where a is a real number.

3. **Intersection of Closed Sets**

   Suppose $\{A_k\}$ is a sequence of non-empty closed sets on $\mathbf{R}^n$ such that $A_1 \supset A_2 \supset A_3 ... \supset A_k \supset$ ...Show that if A_m is bounded for some m, then $\cap_{k=1}^{\infty} A_k \neq \varnothing$.

4. **Uniform Continuity in Euclidean Metric Space**

   $(\mathbf{R}^n, d)$ is the n-dimentional Euclidean metric space. Suppose $E \subset \mathbf{R}^n$ is a nonempty set. Define $d(x, E) = \inf \{d(x, y) : y \in E\}$

   (a) Show that E is a closed set if and only if for any $x \in \mathbf{R}^n$, there exists $y \in E$, such that $d(x, y) = d(x, E)$.

   (b) Define function $f : \mathbf{R}^n \to \mathbf{R}_+$ as $f(x) = d(x, E)$. Show that $f(x)$ is uniformly continuous.

5. **Continuous Function in Euclidean Metric Space**

   $(\mathbf{R}^n, d)$ is the n-dimentional Euclidean metric space. $f : \mathbf{R}^n \to \mathbf{R}^1$ is a function. Show that f is countinuous if and only if for every $c \in \mathbf{R}^1$, A_c and B_c are closed sets where $A_c = \{x \in \mathbf{R}^n : f(x) \geq c\}$ and $B_c = \{x \in \mathbf{R}^n : f(x) \leq c\}$.

6. **Lipschitz Equivalent**

   Theorem 10.8 on page 107 of de la Fuente says that all norms on $\mathbf{R}^n$ are Lipschitz-equivalent to the Euclidean norm. The Theorem is correct, but is the proof correct?

   (a) Suppose $\| \cdot \|$: $(\mathbf{R}^n, d) \to (\mathbf{R}_+, \rho)$ is a norm on $\mathbf{R}^n$. d is the metric generated by the norm, $d(x, y) = \| x - y \|$. ρ is the Euclidean metric. Show that $\| \cdot \|$ is a continuous function. (Hint: Use the triangle inequality.)

   (b) Now consider the Euclidean norm $\| \cdot \|_E$: $\mathbf{R}^n \to \mathbf{R}_+$. The unit circle on $\mathbf{R}^n$ is defined as $C = \{x \in \mathbf{R}^n : \| x \|_E = 1\}$. Show that C is compact. (Hint: Show that C is closed and bounded.)

   (c) Can we use the result of part a and the extreme-value theorem to prove that that $\| \cdot \|$ attains a minimum and a maximum in the set C defined in part b?