Economics 204
Problem Set 4
Due Tuesday, August 11

Exercise 1

State (and check) whether each of the following is a vector space (over $\mathbf{R}$).
a) $S = \{cv : c \in R,\, v = (1, 1, 1)\}$
b) $S = \{(x_1, x_2, x_3) : x_1 + x_2 + x_3 = 0,\, x_1 + 2x_2 = 0\}$
c) $S = \{(x_1, x_2) : x_1 + x_2 = 1\}$
d) $S = \{f : [0, 1] \to [0, 1] : f \text{ continuous}\}$. (first, define $(f + g)(x) := f(x) + g(x)$ and $(cf)(x) = cf(x)$).
If to any one of $(a) - (c)$ you answered "yes", then find the dimension of the space and a Hamel basis for it.

Exercise 2

Let Z, V, W be vector spaces and $g : Z \to V$, $f : V \to W$ be linear transformations and $Z, V,$ and W have dimension n.
a) Show that $Ker(g) \subseteq Ker(f \circ g)$ and thus $\dim \operatorname{Im} g \geq \dim \operatorname{Im}(f \circ g)$, where $\dim \operatorname{Im} g$ indicates the dimension of the image of the map g, and $Ker(h) = \{x \in V : h(x) = 0\}$.
b) Show that f is one-to-one if and only if $Ker(f) = \{0\}$.
c) Let $Z = W = V$. Show that if f, g are autmorphisms of V (i.e. isomorphisms from V to V), then $f \circ g$ is an automorphism of V.

Exercise 3

a) What 2 by 2 matrix represents, with respect to the standard basis, the transformation which rotates every vector in $\mathbf{R}^2$counterclockwise 90 degrees and then projecs the result onto the $x - axis$?
b) What 2 by 2 matrix represents, with respect to the standard basis, projection of $\mathbf{R}^2$onto the $x - axis$ followed by projection onto the $y - axis$?
c) What 3 by 3 matrices represent, with respect to the standard basis, the tranformations that $i)$ project every vector onto the $x - y$ plane; $ii)$ reflect every vector through the $x - y$ plane.

Exercise 4

Let $T : M_{2\times 2} \to M_{2\times 2}$

$$T\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} b_{11} - b_{12} & 0 \\ b_{12} - b_{11} & 0 \end{pmatrix}$$

Determine $Ker(T)$, $\dim Ker(T), rank(T)$. Is T one-to-one, onto, or neither?

Exercise 5

Suppose that square matrices A and B are similar, i.e. $A = P^{-1}BP$, for some invertible matrix P. Find an expression for A^n. Suppose B is diagonal with b_{ii} on it's diagonal. Find an expression for $Tr(A^n)$ and $Det(A^n)$ in terms of b_{ii}.

Exercise 6

Let V, W be vector spaces with bases $\{v_\theta\}_{\theta\in\boldsymbol{\Theta}}$ and $\{w_\gamma\}_{\gamma\in\Gamma}$ and $T : V \to W$ be a linear transformation. Say which of the following statements is true/false, and prove your claim.

a) If $Ker(T) = \{0\}$, then $\{w_\gamma\}_{\gamma\in\Gamma} \subset Span\{Tv_\theta\}_{\theta\in\Theta}$ i.e. the set of vectors $\{w_\gamma\}_{\gamma\in\Gamma}$ is spanned by the set of vectors $\{Tv_\theta\}_{\theta\in\Theta}$.

b) If T is an isomorphism from V to W, then the bases $\{w_\gamma\}_{\gamma\in\Gamma}$ and $\{Tv_\theta\}_{\theta\in\Theta}$ are numerically equivalent, i.e. there exists a bijection between the two.

c) If $\{Tv_\theta\}_{\theta\in\Theta}$ spans W, then $\{v_\theta\}_{\theta\in\boldsymbol{\Theta}}$ and $\{w_\gamma\}_{\gamma\in\Gamma}$ are numerically equivalent.