Problem Set 5
Econ 204
Due Friday, August 14

Exercise 1

Identify which of the following matrices are diagonalizable and provide the diagonalization. If the diagonalization does not exist, prove it.

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 0 \\ -2 & -2 & -1 \end{pmatrix}, \ B = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}, \text{ and } C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

Exercise 2

An $n \times n$ matrix A is called positive semidefinite if for all vectors $x \in \mathbf{R}^n$, $x^T Ax \geq 0$. A is said to be positive definite if for all non-zero vectors $x \in \mathbf{R}^n$, $x^T Ax > 0$. Suppose A is an $n \times n$ positive semidefinite matrix. Is $B^T AB$ positive semidefinite for an arbitrary $n \times m$ matrix B ? Suppose A is positive definite; is $B^T AB$ in this case positive definite? What assumptions (if any) can you make to get these conclusions?

Exercise 3

Let u, v be vectors in R^2.

a) Let $\alpha^* v$, $\alpha^* \in R$ be the vector in direction of vector v that minimizes $\|u - \alpha v\|$, where $\|w\|$ denotes the Euclidean norm of w. What can you conclude about the vectors $z := u - \alpha^* v$ and v. What is the expression for α^*?

b) Let $u = av_1 + bv_2$ where $v_1, v_2 \in R^2$ and $v_1 \cdot v_2 = 0$ and $v_i \cdot v_i = 1$, $i = 1, 2$. Let $\gamma^* v_1$ and $\beta^* v_2$ be vectors as in (a), i.e. γ^* minimizes $\|u - \gamma v_1\|$ and β^* minimizes $\|u - \beta v_2\|$. What are the expressions for γ^* and β^*? How are they related to a and b?

Exercise 4

$f : R^3 \to R^3$ is a vector valued function given by $f(x, y, z) = (x^2, y^2 - 1, z^2)$. Provide sufficient conditions for f to be locally invertible, and find all the points in R^3 where these conditions are satisfied. What is the Jacobian of f^{-1} at such points.

Exercise 5

(a) Write the second-order Taylor expansion for $f(x, y) = 11y^2 + 7x^2 - 10x + (2x + 3)y$ around the point (x_0, y_0).

(b) Consider the following quadratic form: $f(x, y, z) = 4x^2 + 4y^2 + 3z^2 + 4xy$. Let $\vec{x} = (x, y, z)$; for what matrix A do we have $f(\vec{x}) = \vec{x}^T A\vec{x}$?

Find the eigenvalues of A and thus determine the maxima, minima, or saddles of f.

Exercise 6

Let $F : R^3 \to R^1$, $F(x, y, z) = x^2yz^3 - 3$. Consider the set of points in R^3, where $F(x, y, z) = 0$. In which region of R^3 can you locally solve x in terms of y and z? How about if $F(x, y, z) = x^2yz^3 - 3x^{10}$? (If you are applying the Inverse Function Theorem, note that the theorem gives the sufficient conditions for local inversion; however, that will be enough in this excercise to receive full credit, unless you can also find neccessary conditions).

Exercise 7

Use the Implicit Function Theorem to prove the Inverse Function Theorem (i.e. take the assumptions of the InvFT as given, relate those to the assumptions of the ImpFT and use the conclusions of the ImpFT to derive the conclusions of the InvFT).